V&V MURI Overview
Caltech, October 2008

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Goals

- Specification, design, and certification
- Coherent view and computational tools for assessment of performance and uncertainty
- Efficiency (both theoretical and practical)
- Continuous/discrete unification

\[ \langle Q, \Sigma, \delta, S_0, F \rangle \]
\[ \delta : Q \times \Sigma \rightarrow Q \]
How to reason about dynamics?

- Reduction from transitions/dynamics to propositions
  - Vector fields to inequalities via Lyapunov/dissipation: LMIs, SOS
  - Automata to satisfiability: theorem proving, bounded model checking
- Systematize and unify transition from dynamics to algebra
- Develop suitable computational techniques
Personnel at MIT

- Grad students
  - Amir Ali Ahmadi
  - Parikshit Shah
  - Noah Stein (joint w/Prof. Asu Ozdaglar)
  - Ozan Candogan

- Postdocs
  - Danielle Tarraf
    (MIT -> Caltech -> now at Johns Hopkins)
Topics

- Convex approaches to analysis, synthesis and decentralization
- Nash and correlated equilibria. Stochastic games
- Partial orders and decentralized control
- Non-monotonic Lyapunov functions
- SOS techniques and extensions
Adversaries and game theory

- Interesting per se, but also necessary to address robustness
- SOS techniques not just for optimization, but also for games

- Earlier results for semialgebraic games:
  - Two-player, zero-sum, polynomial payoffs
  - Optimal strategies and payoff computed via SOS
  - Extends (with changes) to multiplayer setting
- We can extend to stochastic games

Zero-sum stochastic continuous games

- Two competing players, state-dependent payoffs
- Discounted, infinite game
- Generalizes Markov Decision Processes (MDPs)
- Finite number of states, continuous actions
- Control action affects both immediate payoff and transition probabilities.
- Find Shapley value and optimal strategies

Stochastic continuous games

- single controller assumption yields convexity
- exploit explicit description of moment spaces
- convex optimization – SOS and SDP
- extend techniques from the static case

\[ q_i(x, y), \quad p_{ij}(x, y) \]

Partial orders and decentralized control

- What is a suitable mathematical language and tools to reason about information flow?
- Refined notions of causality: non-determinism, branching time, concurrency, n-D, etc.
- Abstract away continuous/discrete distinction
- What decision-making structures make analysis and synthesis possible?

Propose: partially ordered sets (posets), incidence algebras, and Galois connections

**Definition 1.** A poset $\mathcal{P} = (P, \leq)$ is a set $P$ along with a binary relation $\leq$ which satisfies for all $a, b, c \in P$:

1. $a \leq a$ (reflexivity)
2. $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry)
3. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

**Definition 2.** The set of functions $f : \mathcal{P} \times \mathcal{P} \to \mathbb{Q}$ with the property that $f(x, y) = 0$ whenever $x \not\leq y$ is called the incidence algebra $\mathcal{I}$. 
Posets and incidence algebras

- Posets can be used to model the spatial and/or temporal dependence among subsystems.
- Incidence algebras describe order-preserving maps (e.g., for linearly ordered sets, lower triangular matrices).
- *Galois connections* can be used to describe order-preserving maps between different posets.

\[
\begin{bmatrix}
* & * & * \\
0 & * & 0 \\
0 & 0 & *
\end{bmatrix}
\begin{bmatrix}
* & * & * \\
0 & * & 0 \\
0 & 0 & *
\end{bmatrix}
= 
\begin{bmatrix}
* & * & * \\
0 & * & 0 \\
0 & 0 & *
\end{bmatrix}
\]
Results:

- Unifies most of previous formulations (e.g., partially nested control)
- Poset framework automatically yields formulations that are *quadratically invariant*
- Thus, amenable to *convex optimization*
- Coordinate-free interpretation, via structural matrix algebras and the associated lattice of invariant subspaces
- Galois connections provide a natural way of modeling communications-constrained control

Non-monotonic Lyapunov functions

Lyapunov’s direct method plays a central role in the analysis and control of dynamical systems

- Proving stability
- Synthesis via control Lyapunov functions
- Performance (e.g., rate of convergence analysis)
- Robustness and uncertainty

Why require a monotonic decrease?

Simpler Lyapunov functions (e.g. polynomials of lower degree) can decrease in a non-monotonic fashion along trajectories.
If you can find 
\[ V^1, V^2 : \mathbb{R}^n \rightarrow \mathbb{R} \]

\[ V^2 > 0, \quad V^1 + V^2 > 0, \quad V^1(0) + 2V^2(0) = 0, \]

such that

\[ \left(V_{k+2}^2 - V_k^2\right) + \left(V_{k+1}^1 - V_k^1\right) < 0, \]

then \( V^1 \rightarrow 0, \quad V^2 \rightarrow 0, \) which implies \( x \rightarrow 0. \)

- State space mapped to more than one Lyapunov function
- Improvements in different steps measured according different functions
- Convex parametrization, can use SOS to search for candidate functions
- Generally “simpler” (e.g., lower degree) than if monotonicity is required
Related progress

- Guaranteed bounds on joint spectral radius via SOS (w/Ali Jadbabaie, UPenn)
- Code for SDP relaxations QP + Branch/Bound
  - Parallel, runs under MPI
  - Fully portable code (uses CSDP solver)
  - Written by Sha Hu (S.M. student)
- Ongoing work: SOS on lattices and semigroups (w/Rekha Thomas, UW)
  - Characterization of “theta bodies” of polynomial ideals (arXiv:0809.3480)
Related outside developments

- Incorporation of SOS methods in HOL Light theorem prover (*hol.sosa*, John Harrison, Intel)
- Ongoing collaboration with Henry Cohn (Microsoft Research) on computation of bounds on density of lattice packings via SOS methods
- Sum of squares package for Macaulay 2 (*SOS.m2*), a software for commutative algebra and algebraic geometry (H. Peyrl, ETH Zurich)
Where things are going

- Dynamics on string and graph grammars
- Sparsity and proofs (L1 and nuclear norms), connections to compressed sensing
- Structure, structure, structure: graphical models + BDDs
- Rewrite and extend SOSTOOLS. Python-based? Interface w/CVX?
Goal: efficient tests

- Can we transition between two states, using only moves from a given finite set? (word problem for finite semi-Thue systems, generally undecidable)
- Direct applications to graph grammars, infinite graph reachability, Petri nets, etc.
- What are the obstructions to reachability?

D. Tarraf and P.A. Parrilo “Commutative relaxations of word problems,” CDC2007
Reachability and word problems

- String grammars: finite alphabet and production rules
- Relaxations: commutative and/or symmetric versions
- Algebraic reformulation in terms of ideal membership and nonnegativity (cf. Mayr-Meyer)
- Convexity enables duality-based considerations

D. Tarraf and P.A. Parrilo “Commutative relaxations of word problems,” CDC2007
Reachability and word problems

- Results:
  - Characterization in terms of polynomial identities and nonnegativity constraints
  - Yields a hierarchy of linear programming (LP) conditions
  - Zero-to-all reachability equivalent to finitely many point-to-point problems
  - Progress towards higher-order relaxations, that do not rely on commutativity assumptions
Related resources

- Papers, tutorials, etc.
  - www.mit.edu/~parrilo
  - www.hot.caltech.edu/math.html

- Software: SOSTOOLS
  - www.mit.edu/~parrilo/sostools
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- What are the obstructions to reachability?

Approach: symbolic-numeric
- Relaxations: commutative and/or symmetric versions
- Algebraic reformulation in terms of ideal membership and nonnegativity
- Convexity enables duality-based considerations

Results to date
- Characterization in terms of polynomial identities and nonnegativity constraints
- Yields a hierarchy of linear programming (LP) conditions
- Zero-to-all reachability equivalent to finitely many point-to-point problems
- Progress towards higher-order relaxations, that do not rely on commutativity assumptions

Analysis via Non-monotonic Lyapunov Functions
Ahmadi, Parrilo (MIT)

Goal: stability and performance
- Traditional Lyapunov-based analysis relies on monotone invariants (e.g., energy)
- This often forces descriptions requiring high algebraic complexity
- Is it possible to relax the monotonicity assumption?

Approach: convexity-based
- Require nonnegativity of linear combinations of time derivatives
- Algebraic reformulation in terms of polynomial nonnegativity
- Yields tractable conditions, verifiable by convex optimization

Results to date
- Convexity-based conditions, checkable by SOS/semidefinite programming
- Easy to apply, more powerful than standard conditions
- Connections with other techniques (e.g., vector Lyapunov functions)
- Many extensions to discrete/continuous/hybrid/switched, etc.

Goal: understand information flow

- A new framework to reason about information flow in terms of partially ordered sets (posets).
- What are the structures amenable to decentralized control design?

Approach: incidence algebras

- Posets and incidence algebras
- Abstract flow of information, generalize notions of causality
- Yields convexity of the underlying control problems. Relations with quadratic invariance.

Results to date

- Generalizes sequential and partially nested structures (e.g., leader-follower)
- Convex characterization of poset-preserving controllers, via Youla
- Captures the right level of abstraction, rich algebraic and combinatorial tools
- Extensions to more complicated situations, via Galois connections

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