Eulerian Variational Integrators
for Incompressible Fluids

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Computational Context

Incompressible, (Inviscid) Fluids $\frac{du}{dt} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + f$, $\nabla \cdot u = 0$

- Lagrangian impractical
  - meshes get entangled, needing constant remeshing
  - particles need constant reseeding
- Eulerian far from flawless
  - numerical viscosity's a plague
    - large timesteps induce huge viscosity
  - multiple tools to combat dissipation in practice
    - vorticity confinement: reinjecting vorticity
    - vortex particles or other Lagrangian devices

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Theoretical Context

Variational Integrators for Discrete Mechanics
- capture proper dynamics nicely
  - [West et al., Hairer, ...]
- as well as energy decay for dissipative systems
- Alas, mostly Lagrangian treatment
  - Eulerian treatment missing

Quest: Discrete, Eulerian Fluid Mechanics
- Lie group of volume preserving diffeomorphisms
- motion = geodesic on this group
  - [Lin, Newcomb, Bretherton, Marsden et al.]

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Spatial Discretization

Simplicial Meshes
- or regular grid
  - sometimes simpler to explain (or too simple...)
- over any domain
  - topology

Assumption:
"good" meshes
- cells C_i well shaped

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Configuration Space (I)

Discrete diffeomorphism
- let's turn to Koopman for inspiration
  - if g vol.-preserv. diffeom. then U_g unitary operator on L_2

$$U_g : L_2 \to L_2, \quad \phi(x) \mapsto \phi(g^{-1}(x)).$$

- discrete configuration q
  - must preserve constants: $\forall j, \sum_i q_{ij} = 1$
  - must preserve L_2 inner product of functions

$$q^T \Omega q = \Omega \qquad \Omega = \begin{pmatrix} |C_1| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |C_N| \end{pmatrix}$$

  - forming a finite dimensional Lie group
    - regular grids: q is an orthogonal, (doubly) stochastic matrix

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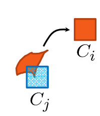
Configuration Space (II)

Two comments:
- q pushes PWC functions forward
  - discrete functions constant per cell C_i

$$q(t)\phi_0 \approx \phi_0(g_t^{-1}(x)) = \phi(t)$$

- from a continuous g how to get q?
  - discretize continuous map:

$$[\![\mathbf{g}]\!]_{ij} \equiv \frac{\mathrm{mes}(g^{-1}(C_i) \cap C_j)}{\mathrm{mes}(C_j)}.$$



  - then do polar decomposition: $\mathbf{g} = qs$.
    - remove symmetric part

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Eulerian Velocity $v_t(x) = \dot{g}_t(g_t^{-1}(x))$

Assuming continuous time for now, define:

$$A(t) = \dot{q}(t)q^{-1}(t)$$

- Lie algebra of previous Lie group
  - inherited properties:
    - Preservation of volume $A^T\Omega + \Omega A = 0$
    - Preservation of mass $\sum_i A_{ij} = 0$

A is not div-free velocity per se...

$$\dot{\phi}(t) = A(t)\phi(t) \quad q(t)\phi_0 \approx \phi_0(g_t^{-1}(x)) = \phi(t)$$

- A is thus a (negated) discrete Lie derivative $\mathcal{L}_v$

Eulerian Velocity (II)

Two comments

- convergence of A induces convergence of q
  - see Dmitry's thesis
- commutator of matrices for Lie bracket
  - important for Lin constraints

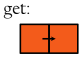
$$A \to u,\ B \to v \Rightarrow [A, B] \to -[u, v]$$

Constraints on Velocity

Discrete velocity non sparse...

- norm of vector fields thus containing many terms
- coeffs of A not intuitive, except for adjacent cells!
  - directional transfer densities (per second) from C_i to C_j
  - through Stokes and divergence-freeness, we get:

$$\Omega_i A_{ij} = -v(x_{ij}) \cdot \vec{n}_{ij}\, S_{ij}$$



  - link to existing numerical schemes (Harlow-Welsh, Arakawa, ...)
    - one flux per face— common staggered grids setup
- Ouch... commutator not satisfying constraints
  - **non holonomic** constraints (NHC)
  - will not be that bad....

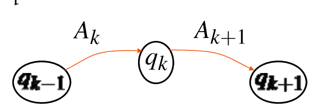
Variational Picture At a Glance

Lagrangian $L(q,\dot{q}) \approx \frac{1}{2} \ll A^\flat, A \gg$

- on NHC subspace, sum of fluxes squared

Extremization of action

- Lagrange-d'Alembert's principle
  - to deal with non-holonomic constraints
- DEL equation links two consecutive 's



Discrete Euler Lagrange Eqs

"Right" variational principle (no vakonomics):

$$\delta \int_0^1 L_h(q,\dot{q})dt = 0 \text{ with } \begin{cases} \delta q \in S_q \\ \delta q(0) = \delta q(1) = 0. \end{cases}$$

- Lagrangian: $L(A) = \frac{1}{2} \ll A^\flat, A \gg \equiv \frac{1}{4}\mathbf{Tr}(\Omega A^\flat A^T)$
  - induced variation of A "special"

    $$\delta A = \dot{B} - [A, B] \quad \text{with } B = \delta q\, q^{-1}$$

    - B is like a test function
  - after integration by parts

    $$\mathbf{Tr}((2\dot{A}^\flat\Omega + [A^\flat\Omega, A])B) = 0$$

- Final DEL:

$$\dot{A}_{ij} + \frac{1}{2}\frac{S_{ij}}{\Omega_i h_{ij}}\left(\frac{1}{\Omega_j}[A^\flat\Omega, A]_{ij} + p_i - p_j\right) = 0$$

Flat Operator

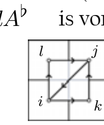
Transform a vector field into a 1-form

- we will restrict ourselves to the NHC
- cannot be trivial (no dynamics otherwise....)
- general idea: $<< A^\flat, B >> = \int (A, B)dV \quad \forall B$
  - must paired not only with NHC and [NHC,NHC]

From A_{ij} values

- define $A^\flat_{ij} = \Omega_i A_{ij} h_{ij}/S_{ij}$, when $j \in N(i)$ (think dual 1-form)
- define 2-away terms such that $dA^\flat$ is vorticity
  - see Dmitry's thesis

Example on regular grid:



Numerics

Update as function of fluxes only:

$$\frac{F^{n+1}-F^n}{h}+\frac{1}{2}\mathsf{Adv}(F^n)+\frac{1}{2}\mathsf{Adv}(F^{n+1}) = -\star_2 d_2^t P^{n+\frac{1}{2}}$$

- this time, space and time both discrete

Newton's steps

$$R(F,P) := \begin{pmatrix} \frac{1}{h}(F-F^n)+\frac{1}{2}\mathsf{Adv}(F^n)+\frac{1}{2}\mathsf{Adv}(F)+\star_2^{-1}d_2^t P \\ d_2 F \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{h}\mathrm{Id}+\frac{1}{2}\frac{\partial \mathsf{Adv}}{\partial F} & \star_2^{-1}d_2^t \\ d_2 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \delta F \\ \delta P \end{pmatrix} = -R(F^{n+1},P^{n+\frac{1}{2}})$$

- saddle point problem
- Schur complement and approximated Jacobian

Note about Extensions

Energy preserving schemes

- change trapezoidal time update to midpoint

$$F^{n+1} = F^n - h\left[\mathsf{Adv}\left(\bar{F}^{n+\frac{1}{2}}\right) - \star_2^{-1}d_2^t P^{n+\frac{1}{2}}\right]$$

- w/ single linear solve [Simo & Armero '94]

$$\frac{1}{h}(F^{n+1}-F^n)+\mathsf{Adv}(F^{n+1},\omega^n) = -\star_2^{-1}d_2^t P^{n+\frac{1}{2}}$$
$$d_2 F^{n+1} = 0.$$

Navier-Stokes equations

$$F^{n+1} = F^n - h\left[\mathsf{Adv}\left(F^{n+\frac{1}{2}}\right) - \star_2^{-1}d_2^t P^{n+\frac{1}{2}} - \nu d_1 \star_1^{-1} d_1^t \star_2 F^{n+\frac{1}{2}}\right]$$

Upwind Advection

- add symmetric part to A to get upwind advection

Connections to Prior Work

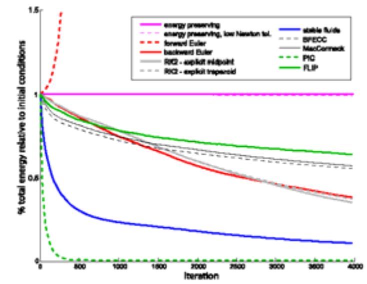
DEC-like Calculus Behind the Scenes

- discrete forms consistent with setup
  - 0-form: discrete PWC functions
  - 1-form: antisymmetric NxN matrices
    - pairs naturally with vector fields A
  - k-forms: tensor of order (k+1) antisym. w.r.t. 2 last indices
- an extension of DEC
  - $d, *, \wedge, \flat, \sharp, i_X, \mathcal{L}_X$

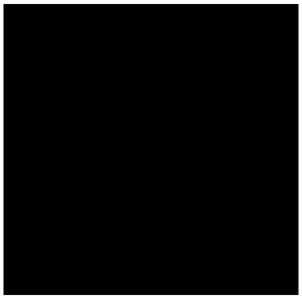
Discrete Kelvin's theorem

- DEL defines advection of curves (dual 1-chain)
  - vorticity transport (circulation preserved along the flow)
    - in Eulerian sense now, unlike our previous integrator

A Few Curves

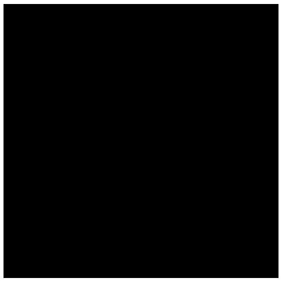


2D Obstacle Course (I)



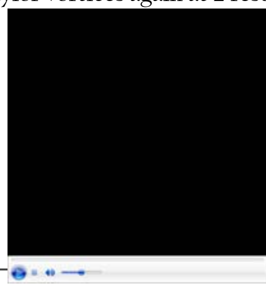
2D Obstacle Course (II)

Taylor vortices at critical distance

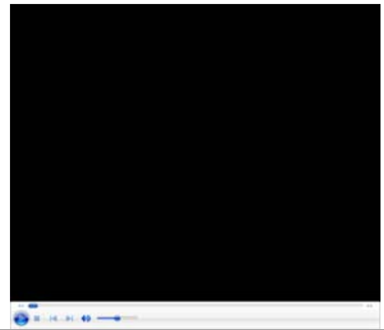


Symmetry and Robustness

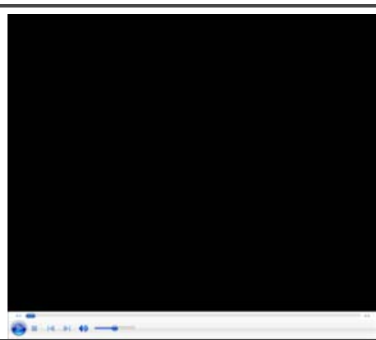
Teapot-shaped Taylor vortices….
then taylor vortices again at 2 resolutions



3D Movie

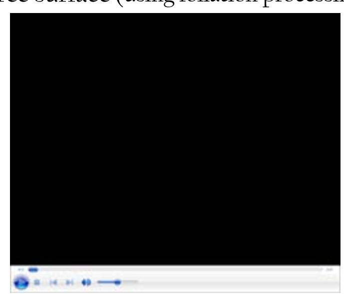


Another Movie



Pipe Dream…. Fully variational?

With free surface (using foliation processing)



Future Work

More analysis needed

- energy cascading, symplecticity, subscale models,…
  - shedding light on the numerical blow-up problem?
  - see recent work of Tom Hou

Optimal Transport

- measures instead of functions
  - use in image processing (registration)

Magnetohydrodynamics

- mixing E&M with discrete forms and fluids
  - SURFer Evan Gawlik

and more….