Discrete Mechanics and Optimization

Jerrold E. Marsden

Control and Dynamical Systems, Caltech, (marsden@cds.caltech.edu)

Objectives

Study flow geometry of unsteady systems in terms of ***Lagrangian Coherent Structures*** (LCS), in a manner analogous studying the geometry of autonomous and periodic systems in terms of invariant manifolds. Several works have shown the usefulness of ***Finite-Time Lyapunov Exponent*** (FTLE) plots for locating LCS. Goals of this poster are to:

- Provide a precise definition of Lagrangian Coherent Structures
- Overview the Lagrangian properties of such structures
- Demonstrate ideas on a range of applications

Notation and Definitions

Given a time-dependent velocity field $\mathbf{v}(\mathbf{x}, t)$ defined over a domain $D \in \mathbb{R}^2$ or $\mathbb{R}^3$, let $\mathbf{x}(t; t_0, \mathbf{x}_0)$ denote the trajectory that passes through $\mathbf{x}_0$ at time t_0, with the associated *flow map* $\phi^t_{t_0} : \mathbf{x}_0 \mapsto \phi^t_{t_0}(\mathbf{x}_0) = \mathbf{x}(t; t_0, \mathbf{x}_0)$, where t is fixed.
Let $\delta\mathbf{x}(t_0)$ be an infinitesimal perturbation to the point $\mathbf{x}_0$ at time t_0. After a time interval T, where $-\infty < T < \infty$, this perturbation becomes

$$\delta\mathbf{x}(t) = \frac{d\phi^t_{t_0}(\mathbf{x}_0)}{d\mathbf{x_0}}\delta\mathbf{x}(t_0) ,$$

where $t = T + t_0$ and terms of $\mathcal{O}(\|\delta\mathbf{x}(t_0)\|^2)$ have been dropped. Using the L_2-norm, the magnitude of the perturbation is given by

$$\|\delta\mathbf{x}(t)\| = \sqrt{\left\langle \delta\mathbf{x}(t_0), \frac{d\phi^t_{t_0}(\mathbf{x_0})}{d\mathbf{x_0}}^* \frac{d\phi^t_{t_0}(\mathbf{x_0})}{d\mathbf{x_0}}\delta\mathbf{x}(t_0)\right\rangle}$$

where the symmetric matrix

$$\Delta = \frac{d\phi^t_{t_0}(\mathbf{x_0})}{d\mathbf{x_0}}^* \frac{d\phi^t_{t_0}(\mathbf{x_0})}{d\mathbf{x_0}}$$

is a *finite-time* version of the (right) Cauchy-Green deformation tensor studied in continuum mechanics .
Maximum stretching occurs when $\delta\mathbf{x}(t_0)$ is chosen such that it is aligned with the maximum eigenvalue direction of Δ. That is, if $\lambda_{\max}(\Delta)$ is the maximum eigenvalue of Δ, thought of as an operator, then

$$\max_{\delta\mathbf{x}(t_0)} \|\delta\mathbf{x}(t)\| = \sqrt{\lambda_{\max}(\Delta)}\, \left\|\overline{\delta\mathbf{x}}(t_0)\right\| \tag{1}$$

where $\overline{\delta\mathbf{x}}(t_0)$ is aligned with the eigenvector associated with $\lambda_{\max}(\Delta)$. Then, Eq. (1) can be recast as

$$\max_{\delta\mathbf{x}(t_0)} \|\delta\mathbf{x}(t)\| = e^{\sigma^{t_0}_T(\mathbf{x_0})|T|}\, \left\|\overline{\delta\mathbf{x}}(0)\right\| ,$$

where

$$\sigma^{t_0}_T(\mathbf{x_0}) = \frac{1}{|T|}\ln\sqrt{\lambda_{\max}(\Delta)} = \frac{1}{|T|}\ln\left\|\frac{d\phi^t_{t_0}(\mathbf{x_0})}{d\mathbf{x_0}}\right\| \tag{2}$$

is referred to as the ***Finite-Time Lyapunov Exponent*** at point x_0 and time t_0, with integration time T.

Flux Across an LCS

Notice

$$\frac{dL}{dt} = \frac{\partial L}{\partial \mathbf{x}}\cdot\frac{d\mathbf{x}}{dt} + \frac{\partial L}{\partial \mathbf{x_q}}\cdot\frac{d\mathbf{x}_q}{dt} .$$

However,

$$\frac{\partial L}{\partial \mathbf{x_q}} = \frac{\mathbf{x}_q - \mathbf{x}}{L} = -\nabla L ,$$

and so

$$\frac{dL}{dt} = \nabla L \cdot \left(\frac{d\mathbf{x}}{dt} - \frac{d\mathbf{x}_q}{dt}\right) . \tag{3}$$

On the LCS, the two points $\mathbf{x}$ and $\mathbf{x}_q$ are equal; however, we think of $\mathbf{x}$ as being a *Lagrangian*, or material, point while $\mathbf{x}_q$ is viewed as a point which moves with the LCS. The right-hand side of Eq. (3) represents the difference in the velocity of the two points, projected in the direction normal to the LCS, which is precisely what contributes to particles crossing the LCS. Therefore, the total flux across the LCS is given by

$$\Phi(t) = \int_{\text{LCS}} \frac{dL}{dt}\, ds , \tag{4}$$

where the integral is taken over the length of the LCS. Although dL/dt is not *directly* obtainable, the following Theorem provides an estimate for dL/dt based on quantities defining the FTLE field.

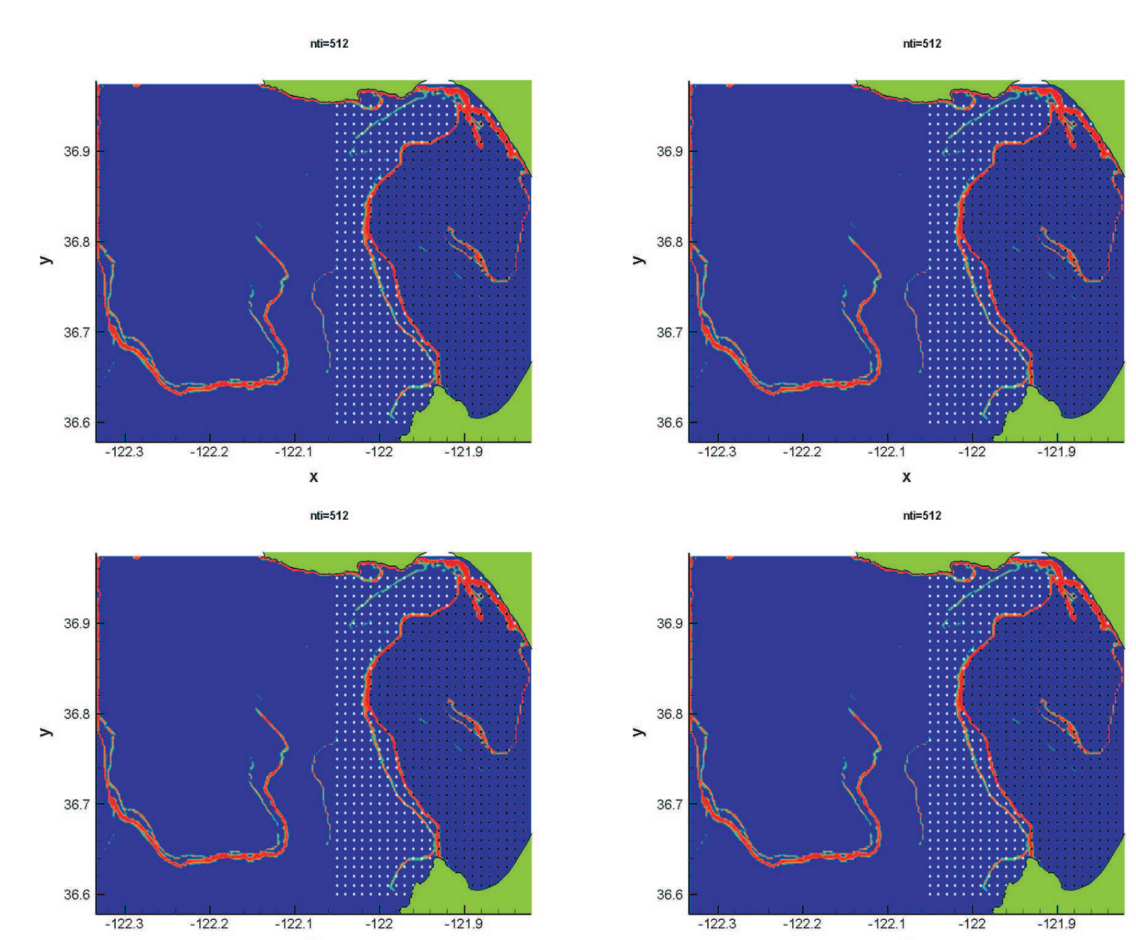
Theorem: For $L = 0$, we have

$$\frac{dL}{dt} = \underbrace{\frac{\langle \hat{\mathbf{t}}, \nabla\sigma\rangle}{\langle \hat{\mathbf{n}}, \Sigma\hat{\mathbf{n}}\rangle}}_{A} \underbrace{\left\langle \hat{\mathbf{t}}, \frac{\partial \hat{\mathbf{n}}}{\partial t} - J\hat{\mathbf{n}}\right\rangle}_{B} + \underbrace{\mathcal{O}\left(\frac{1}{|T|}\right)}_{C} \tag{5}$$

where J is the Jacobian (derivative) of the velocity field, and $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ are unit vectors respectively tangent and normal to the LCS.

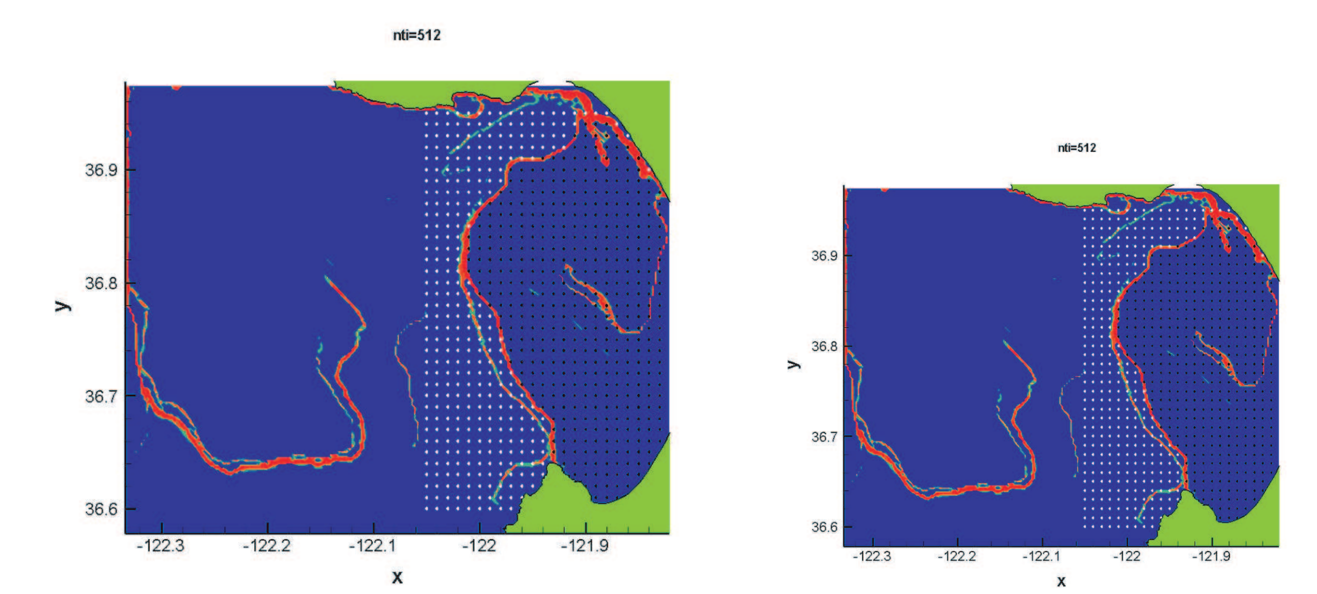
Turbulent Vortex Flow

Direct Numerical Simulations of fully compressible Navier-Stokes equations were used to generate the flow field of turbulent vortex rings with disturbances added. As with the laminar vortex ring experiment discussed in the previous column, capturing the extent and propagation of the vortex over time is very difficult with traditional methods (i.e. streamlines, vorticity plots, etc.). However, LCS are able to nicely capture the flow geometry of the vortex dynamics. Below we show the FTLE fields at two time instances for the turbulent vortex data, with $T > 0$ (rLCS) for plots on the left and $T < 0$ (aLCS) for plots on the right.



Current and planned work

- LCS is currently being applied in the ***Adaptive Sampling And Prediction*** (ASAP) program, whose purpose is to research how to deploy, direct and utilize pseudo-autonomous vehicles most efficiently to sample the ocean, assimilate the data into numerical models in real or near-real time, and predict future conditions with minimal error.
- We are currently investigating a number of interesting bio-fluid systems such as flow around a jellyfish and cardiovascular flow.



- We are also involved in applying these technique for studying a range of other applications, including polar vortex dynamics in the atmosphere, micro-mixing devices, and flow around flapping wings.

Acknowledgements

This work was partially supported by ONR grants N00014-97-10071 and N0001495-0257. Shawn Shadden was supported by an NSF Fellowship. We extend our sincerest thanks to the following collaborators for providing flow data for the examples shown in this poster: Tim Colonius for the turbulent vortex data, John Dabiri for the laminar vortex ring propagation and jellyfish data, Jeff Eldredge for the GLAS-II airfoil data, Mory Gharib for the image of the ventricle simulator, Jeff Paduan for the Monterey Bay data, and Arthur Mariano, Edward Ryan and Nick Shay for the Florida data.

Suggested Reading

http://www.cds.caltech.edu/∼shawn/LCS-tutorial/contents.html

G. Haller, 2001. Distinguished material surfaces and coherent structures in 3d fluid flows. *Physica D*, 149, 248–277.

S.C. Shadden, F. Lekien, and J.E. Marsden, 2005. Definition and Properties of Lagrangian Coherent Structures. *Physica D*, submitted.
http://www.lekien.com/∼francois/papers/qLCS/

S.C. Shadden, J.O. Dabiri, and J.E. Marsden, 2005. Lagrangian analysis of entrained and detrained fluid in vortex rings. *J. Fluid Mech.*, submitted.