Discrete Mechanics and Optimization

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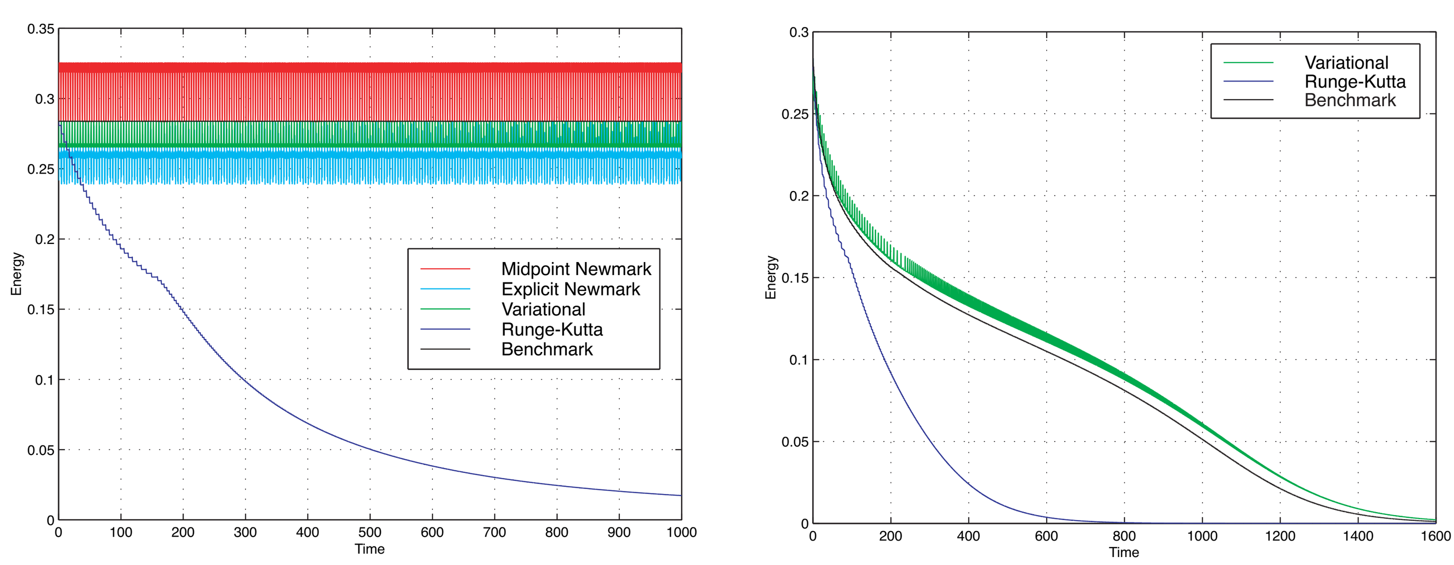
Objectives

- Develop an easy to use optimal control method for mechanical systems
- Apply it to interesting examples, such as to fleets of vehicles.
- Eventually couple the method to fluid optimization problems

Methodology

DMOC (Discrete Mechanics and Optimal Control) deals with the optimal control of mechanical systems, such as the deployment of vehicles, be they spacecraft, hovercraft, or other mechanical systems. DMOC has its basis in discrete mechanics and associated variational integrators, which have a good respect for the energy budget—even in the presence of external forces, such as control forces. The figure shows a variational integration computation for a particle moving in the plane under a rotationally symmetric polynomial potential and without friction (left) and with frictional forces (right). The variational integrator clearly has remarkable respect for the energy budget.



The Problem: A mechanical system with configuration space Q is to move, during a time interval $[0, T]$, from a state $(q^0, \dot{q}^0)$ to a state $(q^T, \dot{q}^T)$ along some curve $q(t) \in Q$ (to be found) and under the influence of a force f (also to be found), such that a given *cost functional*

$$J(q, f) = \int_0^T C(q(t), \dot{q}(t), f(t))\, dt \tag{1}$$

is minimized. At the same time, the motion $q(t)$ of the system is to satisfy $F = ma$; that is, the *Lagrange–d'Alembert principle* holds:

$$\delta \int_0^T L(q(t), \dot{q}(t))\, dt + \int_0^T f(t) \cdot \delta q(t)\, dt = 0 \tag{2}$$

for all variations δq with $\delta q(0) = \delta q(T) = 0$, where $L : TQ \to \mathbb{R}$ is the Lagrangian of the mechanical system.

Discretization: The DMOC method is to *exploit the variational structure and the equations of discrete mechanics directly*. This is in contrast to other methods such as *shooting*, *multiple shooting*, or *collocation*, which rely on a direct integration or a fulfillment at certain grid points of the associated ordinary differential equations (i.e., the Euler-Lagrange equations). Using a global discretization of the states and the controls one directly obtains, via the *discrete Lagrange–d'Alembert principle*, equality constraints for the resulting finite dimensional nonlinear optimization problem, which is solved by standard methods—in our case SQP: *sequential quadratic programming*.

The discrete Lagrange–d'Alembert principle (i.e., *forced discrete Euler–Lagrange equations*) have the form

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f^+_{k-1} + f^-_k = 0,$$

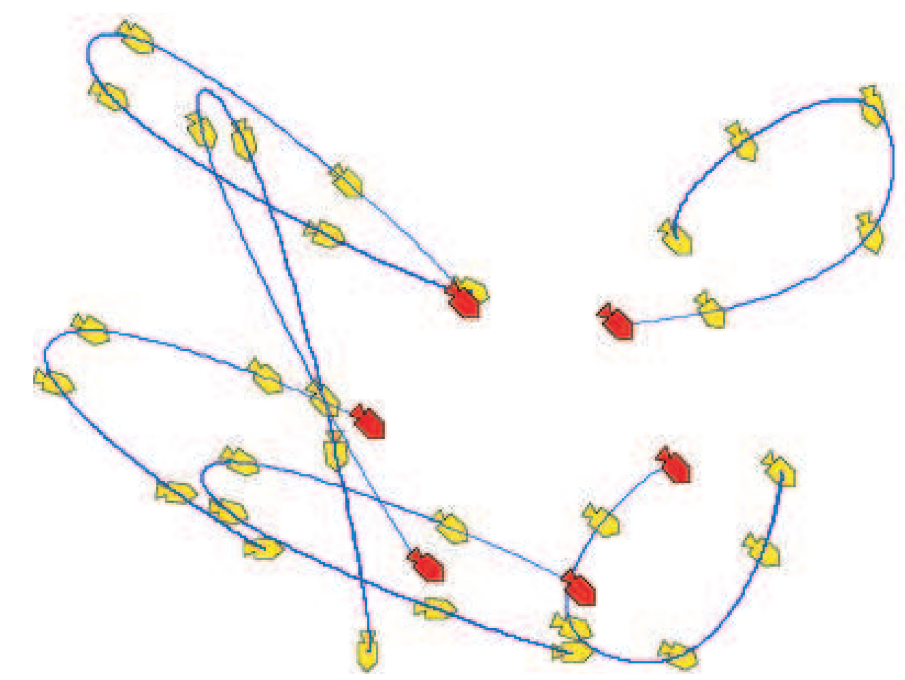
where $k = 1, \ldots, N-1$, the f_k are discrete forces, and $L_d(q, q')$ is a discrete Lagrangian. Likewise, after choosing a *discrete cost function* $C_d(q, q', f, f')$, we get the *discrete cost functional* to be minimized subject to the discrete Lagrange–d'Alembert equations:

$$J_d(q_d, f_d) = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, f_k, f_{k+1}).$$

This is an optimization problem that can be directly fed to SQP.

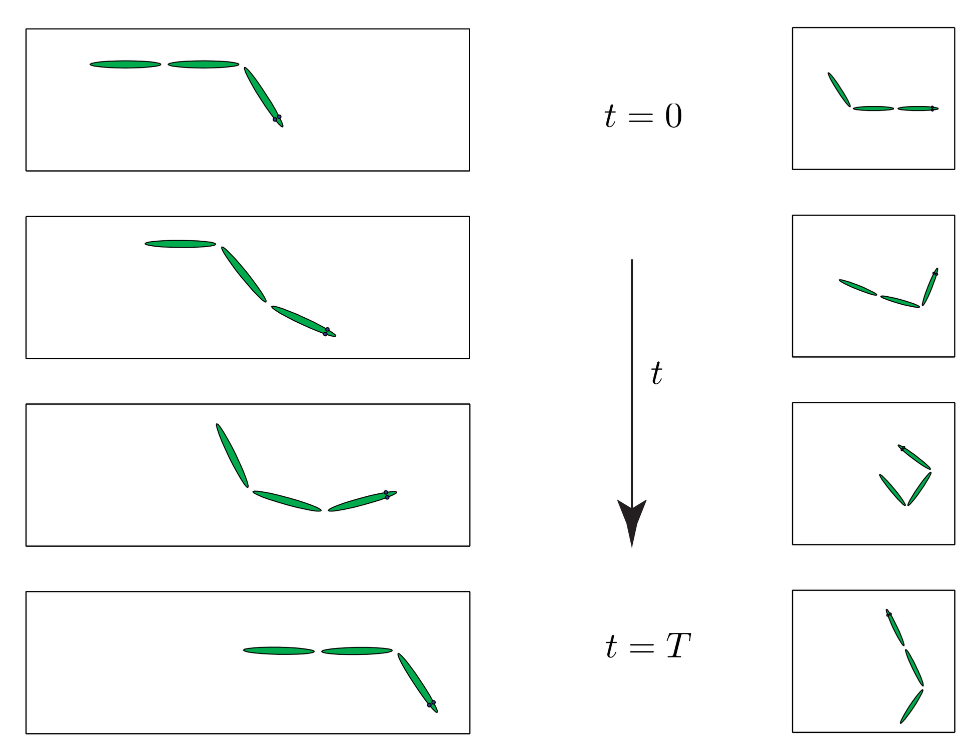
Optimal Hovercraft Formation

Six hovercraft start in given locations and must assume a hexagonal formation in an optimal way. The cost function is the total square of the L^2 norm of the control forces.



Optimal Swimming

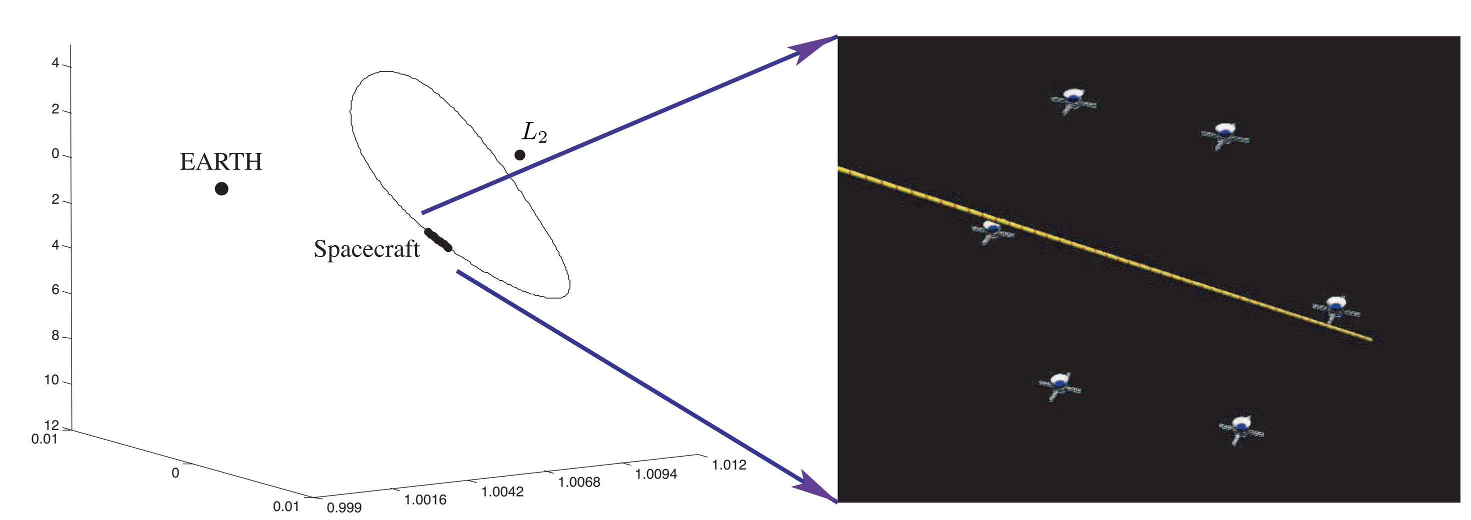
Finding a gait (cyclic shape changes) that optimally move a fish swimming in ideal potential flow from one position to another in a specified time. Shown is a forward swimming gait and a turning gait. This problem involves some ideas from geometric mechanics to separate the equations for the shape changes (including torques at the joints) from the locomotion dynamics in the group SE(2).



(a) (b)

Optimal Spacecraft Reconfiguration

The six spacecraft start off in a line in a periodic orbit around the Sun-Earth L_2 point. They must then assume a hexagonal configuration in an optimal way.



Current and planned work

- DMOC is currently being parallelized so that it will scale well when large numbers of vehicles are involved. There is an inner loop that is done in parallel on each vehicle. There is an outer loop that gathers the resulting information and then performs the optimization.
- Systematic comparison with other methods
- Backward error analysis for systems with control forces.
- Low thrust missions are another interesting potential application, given the good way that the method respects the energy budget.
- In future we will work with the software GAIO (Global Analysis of Invariant Objects) and other global methods to develop techniques for distinguishing local from global minima.
- In future we plan to link these techniques to fluid optimization problems (for example, optimal shape design) using the LANS-α techniques in Mohseni's talk. Include the use of the computational tool *Lagrangian Coherent Structures* as well to identify recirculation zones in unsteady flows.

Acknowledgements

This work was partially supported by AFOSR FA9550-05-1-0343, a Max Planck Research award, Caltech and the University of Paderborn.

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