Variational Integrators for Electrical Circuits

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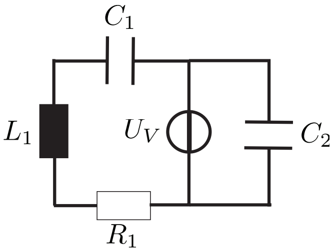
Motivation

Mechanical systems:

- variational integrators well established
- based on discrete variational principle in mechanics

Electrical systems:



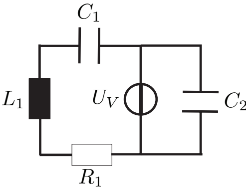
- variational principle required
- Lagrangian formulation with degenerate symplectic form
- constraints
- discrete variational scheme

Future goal: powerful unified variational scheme for the simulation of electromechanical systems

Outline

- basics on circuit modeling
- variational modeling, constraints, and degeneracy
- approaches to handle constraints and degeneracy
- construction of integrators
- standard approach: Modified Nodal Analysis (MNA) and Linear Multistep Methods
- numerical examples
- future work

Circuit modeling



device	linear	nonlinear
resistor	$i_R = G u_R$	$i_R = g(u_R, t)$
capacitor	$i_C = C \frac{d}{dt} u_C$	$i_C = \frac{d}{dt} q_C(u_C, t)$
inductor	$u_L = L \frac{d}{dt} i_L$	$u_L = \frac{d}{dt} \phi_L(i_L, t)$

device	independent	controlled
voltage source	$u_V = v(t)$	$u_V = v(u_{ctrl}, i_{ctrl}, t)$
current source	$i_I = i(t)$	$i_I = i(u_{ctrl}, i_{ctrl}, t)$

characteristic equations for basic elements

$+$

Kirchoff constraint laws

Kirchoff constraint laws

Network topology

- graph consisting of n branches, $m+1$ nodes
- incidence matrix $K \in \mathbb{R}^{n,m}$ describes brach-node connection

$$K_{ij} = \begin{cases} -1 & \text{branch } i \text{ connected inward to node } j \\ +1 & \text{branch } i \text{ connected outward to node } j \\ 0 & \text{otherwise} \end{cases}$$

$$K = [K_L^T \quad K_C^T \quad K_R^T \quad K_V^T \quad K_I^T]^T, \quad K_J \in \mathbb{R}^{n_J,m},\ J \in \{L, C, R, V, I\}$$

Constraints

current law (KCL)	$K^T \cdot I(t) = 0$	$I(t)$:	branch currents
voltage law (KVL)	$K \cdot u(t) = U(t)$	$u(t)$:	node voltages
		$U(t)$:	branch voltages

Variational formulation

mechanical system	**electrical circuit**	(linear relations)
configuration	charge	$q \in Q \subset \mathbb{R}^n$
velocity	current	$v \in T_q Q \subset \mathbb{R}^n$
momentum	flux	$p \in T_q^* Q \subset \mathbb{R}^n$
kinetic energy	magnetic energy	$\frac{1}{2} v^T L v \in \mathbb{R}$
potential energy	electrical energy	$\frac{1}{2} q^T C q \in \mathbb{R}$
friction	dissipation	$R \cdot v \in \mathbb{R}^n$
external force	external sources	$\mathcal{E} \in \mathbb{R}^n$

with $L = \mathrm{diag}(L_1, \ldots, L_n), \quad C = \mathrm{diag}(\frac{1}{C_1}, \ldots, \frac{1}{C_n}),$
$R = \mathrm{diag}(R_1, \ldots, R_n), \quad \mathcal{E} = (\epsilon_1, \ldots, \epsilon_n)$

Variational formulation

- Lagrangian $\mathcal{L}(q,v) = \frac{1}{2}v^T L v - \frac{1}{2}q^T C q$ on TQ
- forces $f = R \cdot v + \mathcal{E} \in T_q^* Q$
- distribution $\Delta_Q(q) = \{v \in T_q Q \,|\, K^T v = 0\} = \mathcal{N}(K^T)$
- annihilator $\Delta_Q^0(q) = \{w \in T_q^* Q \,|\, \langle w, v\rangle = 0 \,\forall v \in \Delta_Q(q)\} = \mathcal{R}(K)$
- Legendre transformation $\mathbb{F}\mathcal{L}(q,v) = (q, \partial\mathcal{L}/\partial v)$

  if $\mathbb{F}\mathcal{L}$ is not invertible $\Rightarrow$ degenerate system

- Lagrange-d'Alembert-Pontryagin Principle on $TQ \bigoplus T^*Q$ gives implicit Euler-Lagrange equations

$$\delta \int_0^T \left(\mathcal{L}(q,v) + \langle p, \dot{q} - v\rangle\right) dt + \int_0^T f \cdot \delta q dt = 0, \; \delta q \in \Delta_Q(q)$$

$$\Rightarrow \quad \begin{array}{rcll} \dot{q} &=& v & \text{charge conservation} \quad (\text{n } eq.) \\ \dot{p} &=& \frac{\partial\mathcal{L}}{\partial q} + f + K\cdot\lambda & \text{KVL with node voltages } \lambda \in \mathbb{R}^m \quad (\text{n } eq.) \\ 0 &=& \frac{\partial\mathcal{L}}{\partial v} - p & \text{flux conservation} \quad (\text{n } eq.) \\ 0 &=& K^T \cdot v & \text{KCL} \quad (\text{m } eq.) \end{array}$$

Literature overview

Lagrangian / Hamiltonian formulation of circuits

- MacFarlane (1967): tree structures (capacitor charges or inductor fluxes as generalized coordinates)
- Chua, McPherson (1973): mixed set of coordinates
- Chua, McPherson and Milic, Novak: enlarge applicability of Lagrangian methods to electrical networks

Degenerate Lagrangian / Hamiltonian

- Dirac, Bergmann (1956): construction of submanifolds
- Gotay, Nester (1979): Lagrangian viewpoint
- van der Schaft (1995): implicit Hamiltonian systems
- Yoshimura, Marsden (2006): implicit Lagrangian systems
- Leok, Ohsawa (2009): discrete version

Reduction

- integrable distribution $\Delta_Q(q) = \{v \in T_qQ \,|\, K^T v = 0\} = \mathcal{N}(K^T)$
- configuration submanifold $\mathcal{C} = \{q \in Q \,|\, K^T(q - q_0) = 0\}$
- $\exists K_2 \in \mathbb{R}^{n,n-m}$ with $\mathcal{R}(K) \perp \mathcal{R}(K_2)$:
  $q = K_2\tilde{q} + \alpha \quad \tilde{q} \in \tilde{Q} \subset \mathbb{R}^{n-m} \quad \alpha = 0$ for consistent q_0
  $v = K_2\tilde{v} \quad \tilde{v} \in T_{\tilde{q}}\tilde{Q} \subset \mathbb{R}^{n-m}$
- constrained Lagrangian $\mathcal{L}^c(\tilde{q}, \tilde{v}) := \mathcal{L}(K_2\tilde{q}, K_2\tilde{v})$
- constrained Legendre transformation $\mathbb{F}\mathcal{L}^c(\tilde{q}, \tilde{v}) = (\tilde{q}, \partial\mathcal{L}^c/\partial\tilde{v})$
- forces $\tilde{f} = K_2^T f$, momenta $\tilde{p} = K_2^T p \in T^*_{\tilde{q}}\tilde{Q} \subset \mathbb{R}^{n-m}$

- reduced L-d'A-Pontryagin Principle on $T\tilde{Q} \bigoplus T^*\tilde{Q}$

$$\delta \int_0^T \left(\mathcal{L}^c(\tilde{q}, \tilde{v}) + \langle \tilde{p}, \dot{\tilde{q}} - \tilde{v}\rangle\right) dt + \int_0^T \tilde{f} \cdot \delta\tilde{q} dt = 0$$

$$\Rightarrow \quad \begin{aligned} \dot{\tilde{q}} &= \tilde{v} \\ \dot{\tilde{p}} &= \frac{\partial\mathcal{L}^c}{\partial\tilde{q}} + \tilde{f} \\ 0 &= \frac{\partial\mathcal{L}^c}{\partial\tilde{v}} - \tilde{p} \end{aligned}$$

Is $\frac{\partial\mathcal{L}^c}{\partial\tilde{v}}$ invertible?

Cancellation of degeneracy by constraints

- linear RCLV circuit $\mathbb{F}\mathcal{L}^c(\tilde{q},\tilde{v}) = (\tilde{q}, K_2^T L K_2 \tilde{v})$
- constrained Lagrangian $\mathcal{L}^c$ is non-degenerate if $\mathcal{R}(K_2) \cap \mathcal{N}(L) = \{0\}$
- it holds $\mathcal{N}(L) = \{v \in T_q Q \,|\, v_L = 0\}$
- with $\mathcal{R}(K_2) \perp \mathcal{R}(K)$ and $\mathcal{R}(K) \perp \mathcal{N}(K^T)$ we have

  $$\mathcal{R}(K_2) = \mathcal{N}(K^T) = \{v \in T_q Q \,|\, K_L^T v_L + K_C^T v_C + K_R^T v_R + K_V^T v_V = 0\}$$

- and thus

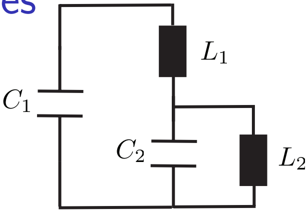
  $$\mathcal{R}(K_2) \cap \mathcal{N}(L) = \{v \in T_q Q \,|\, [K_C^T \; K_R^T \; K_V^T] \cdot [v_C \; v_R \; v_V]^T = 0, \; v_L = 0\}$$

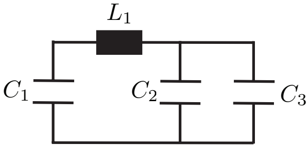
- it follows that for

  $$\mathcal{N}([K_C^T \; K_R^T \; K_V^T]) = \{0\}$$

  the constrained Lagrangian $\mathcal{L}^c$ is non-degenerate

Examples





$$K_L^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; K_C^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\mathcal{N}(K_C^T) = \{0\}$$

$$\Downarrow$$

non-degenerate constrained Lagrangian

$$K_L^T = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \; K_C^T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\mathcal{N}(K_C^T) \neq \{0\}$$

$$\Downarrow$$

degenerate constrained Lagrangian

- non-degeneracy iff number of capacitors = number of independent constraints involving capacitors
- equivalent statements, when resistors and voltage sources are included

Projection for degenerate systems

- identify Dirac constraints: denote by $\mathbb{F}\mathcal{L}^c_M : T\tilde{Q} \to M$ the restriction of $\mathbb{F}\mathcal{L}^c : T\tilde{Q} \mapsto T^*\tilde{Q}$ to its image
- primary submanifold $M = \{(\tilde{q}, \tilde{p}) \in T^*\tilde{Q} \,|\, \phi(\tilde{q}, \tilde{p}) = 0 \in \mathbb{R}^l\}$
  l: degree of degeneracy
- consistency with dynamics leads to more constraints (motion constrained to lie in M)
- Dirac-Bergmann: $M \to M_2 \to \cdots \to M_k \subset T^*\tilde{Q}$
- Gotay-Nester: $N \to N_2 \to \cdots \to N_k \subset T\tilde{Q}$
- knowledge about existence of solutions if constrained algorithm converges
- consistent equations of motion on N_k
- useful for circuit modeling? Can we find a regular Lagrangian on the final constraint manifold N_k?

Example: linear RLCV circuit

$$\begin{aligned}\dot{\tilde{q}} &= \tilde{v} \\ K_2^T L K_2 \dot{\tilde{v}} &= -K_2^T C \tilde{q} - K_2^T R \tilde{v} + \tilde{\mathcal{E}}\end{aligned} \quad \Longrightarrow \quad \begin{aligned}&\phi(\tilde{q}, \tilde{v}, \tilde{\mathcal{E}}) = 0, \text{ linear case:} \\ &P^T \tilde{q} + U^T \tilde{v} + V^T \tilde{\mathcal{E}} = 0\end{aligned}$$

case 1: no resistors and no external sources

- holonomic constraint defines constraint manifolds $\mathcal{C}_2 = \{\tilde{q} \in \tilde{Q} \mid P^T \tilde{q} = 0\}$, $T_{\tilde{q}}\mathcal{C}_2 = \{\tilde{v} \in T_{\tilde{q}}\tilde{Q} \mid P^T \tilde{v} = 0\}$
- reparametrizations $\tilde{q} = P_2 \hat{q}$, $\hat{q} \in \hat{Q} \subset \mathbb{R}^{n-m-l}$ and $\tilde{v} = P_2 \hat{v}$, $\hat{v} \in T_{\hat{q}}\hat{Q} \subset \mathbb{R}^{n-m-l}$
- constrained Lagrangian $\mathcal{L}^{c_2}(\hat{q}, \hat{v}) := \mathcal{L}(K_2 P_2 \hat{q}, K_2 P_2 \hat{v})$

case 2: with resistors and external sources

- affine non-holonomic constraint defines constraint manifold $\mathcal{P}_2 = \{\tilde{v} \in T_{\tilde{q}}\tilde{Q} \mid P^T \tilde{q} + U^T \tilde{v} + V^T \tilde{\mathcal{E}} = 0\}$
- reparametrization $\tilde{v} = U_2 \hat{v} + \gamma(\tilde{q}, \tilde{\mathcal{E}})$, $\hat{v} \in \mathbb{R}^{n-m-l}$, $\tilde{q} \in \mathbb{R}^{n-m}$
- constrained Lagrangian $\mathcal{L}^{c_2}(\tilde{q}, \hat{v}) := \mathcal{L}(K_2 \tilde{q}, K_2 U_2 \hat{v} + K_2 \gamma(\tilde{q}, \tilde{\mathcal{E}}))$

The constrained Variational Integrator

- same reduction steps can be performed on the discrete level to construct a discrete constrained Lagrangian
- discrete variational principle for discrete constrained Lagrangian
- case 1:

$$\delta \sum_{k=0}^{N-1} \left(\mathcal{L}^{c_2}(\hat{q}_k, \hat{v}_k) + \left\langle \hat{p}_k, \frac{\hat{q}_k - \hat{q}_{k-1}}{h} - \hat{v}_k \right\rangle \right) + \sum_{k=0}^{N-1} \hat{f}_k \cdot \delta \hat{q}_k dt = 0$$

  with variations $\delta \hat{q}_k,\ \delta \hat{v}_k,\ \delta \hat{p}_k$

- case 2:

$$\delta \sum_{k=0}^{N-1} \left(\mathcal{L}^{c_2}(\tilde{q}_k, \hat{v}_k) + \left\langle \tilde{p}_k, \frac{\tilde{q}_k - \tilde{q}_{k-1}}{h} - (U_2 \hat{v}_k + \gamma(\tilde{q}_k, \tilde{\mathcal{E}}_k) \right\rangle \right) + \sum_{k=0}^{N-1} \tilde{f}_k \cdot \delta \tilde{q}_k dt = 0$$

with variations $\delta \tilde{q}_k \in \mathcal{P}_2,\ \delta \hat{v}_k,\ \delta \tilde{p}_k$

- gives well-defined scheme without any constraints

The constrained Variational Integrator

- straight forward for linear systems
- loss of physical meaning of generalized coordinates
- for nonlinear systems the additional constraints are hard to identify
- no global reparametrization
- are there other ways to deal with the degeneracy?

Keep degeneracy

- Variational Principle results into DAE system

$$\begin{aligned} \dot{q} &= v \\ \dot{p} &= \frac{\partial \mathcal{L}}{\partial q} + f + K \cdot \lambda \\ 0 &= \frac{\partial \mathcal{L}}{\partial v} - p \\ 0 &= K^T \cdot v \end{aligned} \quad \Rightarrow \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \dot{\begin{pmatrix} q \\ v \\ p \\ \lambda \end{pmatrix}} = F(q, v, p, \lambda, f)$$

- for a linear circuit (no external sources) we have

$$\underbrace{\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{E} \cdot \underbrace{\dot{\begin{pmatrix} q \\ v \\ p \\ \lambda \end{pmatrix}}}_{\dot{x}} = \underbrace{\begin{pmatrix} 0 & I & 0 & 0 \\ -C & -R & 0 & K \\ 0 & L & -I & 0 \\ 0 & K^T & 0 & 0 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} q \\ v \\ p \\ \lambda \end{pmatrix}}_{x}$$

Existence and uniqueness for linear DAE

Definition: The pair (E, A) of square matrices is *regular*, if $\exists \mu \in \mathbb{R}$ s.t. $\det(\mu E - A) \neq 0$.

Theorem: Let the pair (E, A) of square matrices be regular. Then it holds

1. The DAE system $E \cdot \dot{x}(t) = A \cdot x(t)$ is solvable.
2. The set of consistent initial conditions x_0 is nonempty.
3. Every initial value problem with consistent initial conditions is uniquely solvable.

(Kunkel, Mehrmann: *Differential-Algebraic equations*)

Existence and uniqueness for linear DAE

Lagrangian system:

- for $\mu = 1$ we have

$$\mu E - A = E - A = \begin{pmatrix} I & -I & 0 & 0 \\ C & R & I & -K \\ 0 & -L & I & 0 \\ 0 & -K^T & 0 & 0 \end{pmatrix}$$

- $\det(E - A) \neq 0 \Leftrightarrow \mathcal{N}(K^T) \cap \mathcal{N}(L + C + R) = \{0\}$
- a linear RLC circuit has a unique solution (it holds $\mathcal{N}(L + C + R) = \{0\}$)
- a linear RLCV circuit has a unique solution if $\mathcal{N}(K_V^T) = \{0\}$

Variational Integrator I

- discrete Lagrange-d'Alembert-Pontryagin Principle

$$\delta \sum_{k=0}^{N-1} \left(\mathcal{L}(q_k, v_k) + \left\langle p_k, \frac{q_k - q_{k-1}}{h} - v_k \right\rangle \right) + \sum_{k=0}^{N-1} f_k \cdot \delta q_k dt = 0$$

$$\delta q_k \in \Delta_Q(q_k) \Rightarrow \quad \begin{aligned} q_{k+1} &= q_k + h v_{k+1} \\ p_{k+1} &= p_k - hCq_k - hRv_k + K \cdot \lambda_k \\ 0 &= Lv_{k+1} - p_{k+1} \\ 0 &= K^T \cdot v_{k+1} \end{aligned}$$

$$\underbrace{\begin{pmatrix} I & -hI & 0 & 0 \\ 0 & 0 & I & -K \\ 0 & -L & I & 0 \\ 0 & -K^T & 0 & 0 \end{pmatrix}}_{E_d} \cdot \underbrace{\begin{pmatrix} q_{k+1} \\ v_{k+1} \\ p_{k+1} \\ \lambda_k \end{pmatrix}}_{x_{k+1}} = \underbrace{\begin{pmatrix} I & 0 & 0 & 0 \\ -hC & -hR & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A_d} \cdot \underbrace{\begin{pmatrix} q_k \\ v_k \\ p_k \\ \lambda_{k-1} \end{pmatrix}}_{x_k}$$

- unique discrete solution exists if $\mathcal{N}(K^T) \cap \mathcal{N}(L) = \{0\}$
- integrator applicable if KCL cancels degeneracy

Variational Integrator II

- discrete Lagrange-d'Alembert-Pontryagin Principle

$$\delta \sum_{k=0}^{N-1} \left(\mathcal{L}(q_k, v_k) + \left\langle p_k, \frac{q_{k+1} - q_k}{h} - v_k \right\rangle \right) + \sum_{k=0}^{N-1} f_k \cdot \delta q_k dt = 0$$

$$\delta q_k \in \Delta_Q(q) \Rightarrow \quad \begin{array}{rcl} q_{k+1} & = & q_k + h v_k \\ p_{k+1} & = & p_k - hC q_{k+1} - hR v_{k+1} + K \cdot \lambda_{k+1} \\ 0 & = & L v_{k+1} - p_{k+1} \\ 0 & = & K^T \cdot v_{k+1} \end{array}$$

$$\underbrace{\begin{pmatrix} I & 0 & 0 & 0 \\ hC & hR & I & -K \\ 0 & -L & I & 0 \\ 0 & -K^T & 0 & 0 \end{pmatrix}}_{E_d} \cdot \underbrace{\begin{pmatrix} q_{k+1} \\ v_{k+1} \\ p_{k+1} \\ \lambda_{k+1} \end{pmatrix}}_{x_{k+1}} = \underbrace{\begin{pmatrix} I & hI & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A_d} \cdot \underbrace{\begin{pmatrix} q_k \\ v_k \\ p_k \\ \lambda_k \end{pmatrix}}_{x_k}$$

- unique discrete solution exists if $\mathcal{N}(K^T) \cap \mathcal{N}(L + hR) = \{0\}$

Variational Integrator III – higher order

- discrete Lagrange-d'Alembert-Pontryagin Principle of second order resulting into implicit midpoint rule[1]

$$
\begin{aligned}
q_{k+1} &= q_k + h v_{k+\frac{1}{2}} \\
p_{k+1} &= p_k - hC\frac{q_k+q_{k+1}}{2} - hR\frac{v_k+v_{k+1}}{2} + K\cdot\lambda_{k+\frac{1}{2}} \\
0 &= L v_{k+\frac{1}{2}} - \frac{p_k+p_{k+1}}{2} \\
0 &= K^T \cdot v_{k+\frac{1}{2}}
\end{aligned}
$$

$$
\underbrace{\begin{pmatrix} I & -hI & 0 & 0 \\ \frac{h}{2}C & hR & I & -K \\ 0 & -L & \frac{I}{2} & 0 \\ 0 & -K^T & 0 & 0 \end{pmatrix}}_{E_d} \cdot \underbrace{\begin{pmatrix} q_{k+1} \\ v_{k+\frac{1}{2}} \\ p_{k+1} \\ \lambda_{k+\frac{1}{2}} \end{pmatrix}}_{x_{k+1}} = \underbrace{\begin{pmatrix} I & 0 & 0 & 0 \\ -\frac{h}{2}C & 0 & I & 0 \\ 0 & 0 & -\frac{I}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A_d} \cdot \underbrace{\begin{pmatrix} q_k \\ v_{k-\frac{1}{2}} \\ p_k \\ \lambda_{k-\frac{1}{2}} \end{pmatrix}}_{x_k}
$$

- unique solution exists if $\mathcal{N}(K^T) \cap \mathcal{N}(L + hC + hR) = \{0\}$
- integrator applicable if continuous system has unique solution

[1] cf. Nawaf's thesis: Variational Partitioned RK method

Nonlinear case

- e.g. Variational Integrator II

$$0 = \begin{pmatrix} -q_{k+1} + q_k + hv_k \\ -p_{k+1} + p_k + hD_1\mathcal{L}(q_{k+1}, v_{k+1}) + f_{k+1} + K \cdot \lambda_{k+1} \\ D_2\mathcal{L}(q_{k+1}, v_{k+1}) - p_{k+1} \\ K^T \cdot v_{k+1} \end{pmatrix} = F(x_k, x_{k+1})$$

- find x_{k+1} such that $F(x_k, x_{k+1}) = 0$ for given x_k
- Jacobian of F w.r.t. x_{k+1} has to be regular (e.g. Newton scheme)
- applicability dependent on update rule
- trade off: "the more implicit the better but the more expensive"
- what is "state of the art" (e.g. simulation software SPICE)?

Modified Nodal Analysis (MNA) (charge-flux)

- apply KCL to every node except ground
- insert representation for the branch current of resistors, capacitors and current sources
- add representation for inductors and voltage sources explicitely to the system

$$\begin{aligned} K_C^T C\dot{q}(K_C u, t) + K_R^T g(K_R u, t) + K_L^T v_L + K_V^T v_V + K_I^T v_I(Ku, \dot{q}, v_L, v_V, t) &= 0 \\ \dot{\phi}(v_L, t) - K_L u &= 0 \\ v_V(Ku, \dot{q}_C, v_L, v_V, t) - K_V u &= 0 \end{aligned}$$

- linear relations

$$\begin{aligned} K_C^T C K_C \dot{u} + K_R^T G K_R u + K_L^T v_L + K_V^T v_V + K_I^T v_I(Ku, \dot{q}, v_L, v_V, t) &= 0 \\ L\dot{v}_L - K_L u &= 0 \\ v_V(Ku, \dot{q}_C, v_L, v_V, t) - K_V u &= 0 \end{aligned}$$

- DAE for node voltages u, inductor currents v_L

Numerical integration schemes

- DAE system $\quad A[d(x(t),t)]' + b(x(t),t) = 0$
- conventional approach: Implicit linear multi-step formulas

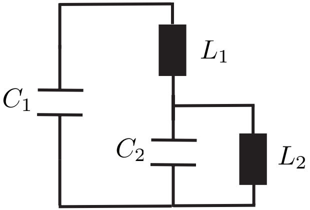
$$\sum_{i=0}^{k} \alpha_i d(x_{n+1-i}, t_{n+1-i}) = h \sum_{i=0}^{k} \beta_i d'_{n+1-i}$$

- BDF method: $\beta_1 = \cdots = \beta_k = 0, \quad \alpha_0 = 1$

k	β_0	α_1	α_2	scheme
1	1	-1		$d(x_{n+1}) - d(x_n) = hd'_{n+1}$ (implicit Euler)
2	$\frac{2}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$d(x_{n+1}) - \frac{4}{3}d(x_n) + \frac{1}{3}d(x_{n-1}) = h\frac{2}{3}d'_{n+1}$

- advantage: low computational cost (1 function evaluation per step) compared to RK methods
- disadvantage: bad stability properties
  (Voigt: *General linear methods for integrated circuit design*, PhD thesis, 2006)

Example I

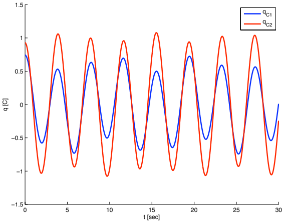


$$L = \begin{pmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_L^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

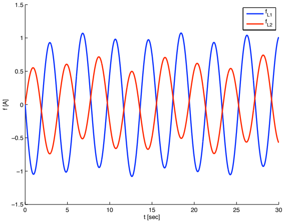
$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \end{pmatrix}, \quad K_C^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

- KCL: $[K_L^T \quad K_C^T] \cdot [v_L \quad v_C]^T = 0$
- $\mathcal{N}(K_C^T) = \{0\} \Rightarrow$ non-degenerate constrained Lagrangian
- $\mathcal{N}(K_C^T) = \{0\} \Rightarrow$ all variational integrators are applicable

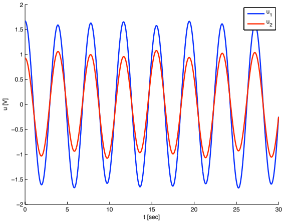
Example I – results



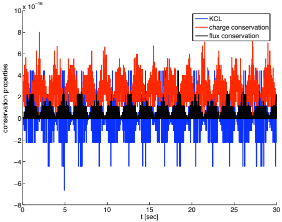
charges on capacitors



currents on inductors

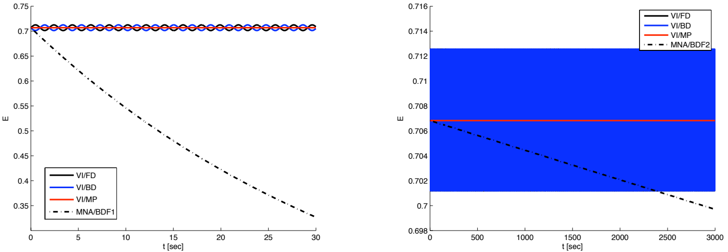


voltages on nodes



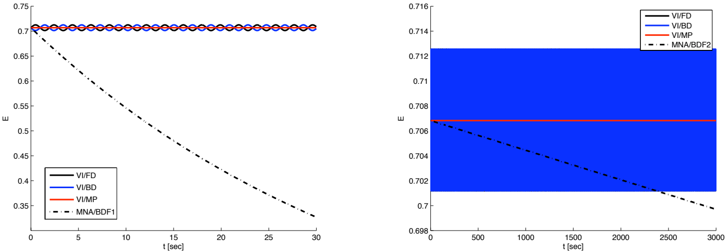
conserved quantities

Example I – results



Energy plots: Comparison with BDF1 (left) and BDF2 (right) ($h = 0.01$)

Example I – results

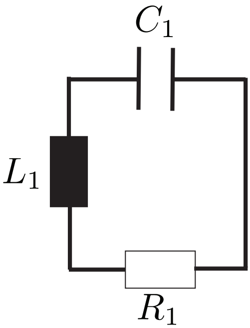


Energy plots: Comparison with BDF1 (left) and BDF2 (right) ($h = 0.01$)

- Hairer, Lubich: multi-step methods are in general not symplectic (Tang (1993): *The underlying one-step method of a linear multistep method cannot be symplectic.*)
- same discrete solutions for full, reduced and projected system, BUT condition numbers

full system	reduced system	projected system
$\mathcal{O}(h^{-2})$	$\mathcal{O}(h^{-1})$	a

Example II



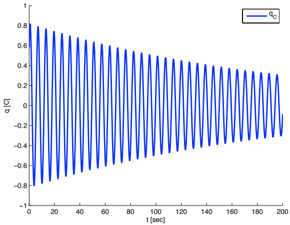
$$L = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; K_L^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; K_C^T = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

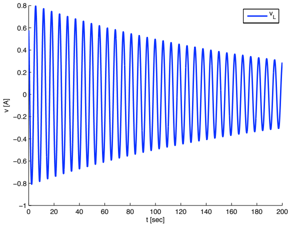
$$R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_1 \end{pmatrix}, \; K_R^T = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

- $\mathcal{N}([K_C^T \; K_R^T]) = \{0\} \Rightarrow$ non-degenerate constrained Lagrangian
- constrained Lagrangian $\mathcal{L} = \frac{1}{2} L_1 v^2 - \frac{1}{2} \frac{q^2}{C_1}$, dissipation $R_1 \cdot v$
  $\Rightarrow$ harmonic oscillator

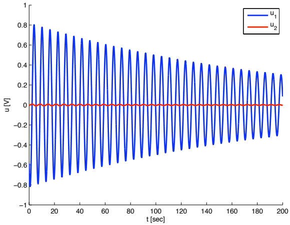
Example II – results



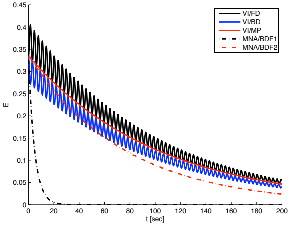
charge on capacitor



current on inductor

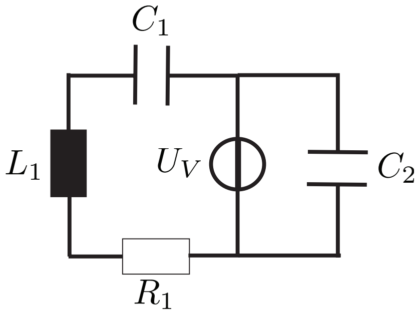


voltages on nodes



energy decay (h=0.2)

Example III



voltage source $U_V(t) = \sin t$

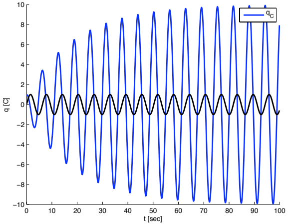
resistor $R_1 = 0.1$

$$K_L^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad K_C^T = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

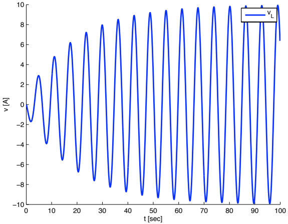
$$K_R^T = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad K_V^T = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

- $\mathcal{N}([K_C^T \; K_R^T \; K_V^T)] \neq \{0\} \Rightarrow$ degenerate constrained Lagrangian and VI I not applicable
- $\mathcal{N}([K_C^T \; K_V^T)] \neq \{0\} \Rightarrow$ VI II not applicable
- $\mathcal{N}(K_V^T) = \{0\} \Rightarrow$ VI III applicable

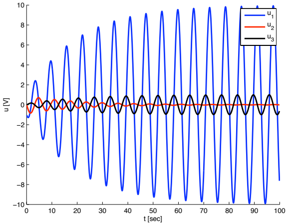
Example III – results



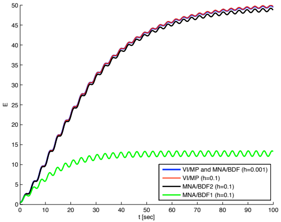
charges on capacitors



currents on inductor



voltages on nodes



energy

Conclusions

- discrete variational approach provides symplectic integrator with good energy behavior
- different approaches to treat degeneracy:
- reduction to final constraint manifold
+ minimal number of variables (reduces cpu time for large circuits)
+ condition number of scheme independent on step size
- difficult for nonlinear systems
- no real physical meaning of generalized coordinates
- keep degeneracy
+ applicable to nonlinear systems
+ easily scalable
+ full information about charges, currents, node voltages
- huge number of variables
- condition becomes worse for small step sizes

Future work

Open questions:

- determine degree of degeneracy dependent on circuit topology
- relation between degree of degeneracy and index of DAE systems

Construction of integrators:

- can we construct multistep variational integrators that are symplectic to be computational more efficient?
- better way to find generalized coordinates, e.g. exploit graph structure (Todd)
- identify generate and degenerate sub-circuits and apply different integrators
- parallelize simulation of sub-circuits

Future applications:

- application to noisy circuits (work in progress with Houman and Molei)
- application to electromechanical systems