Discrete Dirac Structures and Variational Discrete Dirac Mechanics

Melvin Leok

Mathematics, University of California, San Diego

Joint work with Tomoki Ohsawa (Michigan)

5th Structured Integrators Workshop
California Institute of Technology, Pasadena, CA, May 8, 2009.

arXiv:0810.0740

Supported in part by NSF DMS-0714223, DMS-0726263, DMS-0747659 (CAREER).

Introduction

Dirac Structures

- Dirac structures can be viewed as simultaneous generalizations of symplectic and Poisson structures.
- Implicit Lagrangian and Hamiltonian systems[1] provide a unified geometric framework for studying degenerate, interconnected, and nonholonomic Lagrangian and Hamiltonian mechanics.

[1] H. Yoshimura, J.E. Marsden, Dirac structures in Lagrangian mechanics. Part I: Implicit Lagrangian systems, *J. of Geometry and Physics*, **57**, 133–156, 2006.

Introduction

Variational Principles

- The Hamilton–Pontryagin principle[2] on the Pontryagin bundle $TQ \oplus T^*Q$, unifies Hamilton's principle, Hamilton's phase space principle, and the Lagrange–d'Alembert principle.
- Provides a variational characterization of implicit Lagrangian and Hamiltonian systems.

[2] H. Yoshimura, J.E. Marsden, Dirac structures in Lagrangian mechanics. Part II: Variational structures, *J. of Geometry and Physics*, **57**, 209–250, 2006.

Introduction

Discrete Dirac Structures

- Continuous Dirac structures are constructed by considering the geometry of symplectic vector fields and their associated Hamiltonians.
- By analogy, we construct discrete Dirac structures by considering the geometry of symplectic maps and their associated generating functions.
- Provides a unified treatment of implicit discrete Lagrangian and Hamiltonian mechanics in the presence of discrete Dirac constraints.

Introduction

Discrete Hamilton–Pontryagin principle

- We define a discrete Hamilton–Pontryagin principle on the discrete Pontryagin bundle $(Q \times Q) \oplus T^*Q$.
- Obtained from the discrete Hamilton's principle by imposing the discrete second-order curve condition using Lagrange multipliers.
- Provides an alternative derivation of implicit discrete Lagrangian and Hamiltonian mechanics.
- In the absence of constraints, implicit discrete Hamiltonian mechanics reduce to the usual definition of discrete Hamiltonian mechanics[3] obtained using duality in the sense of optimization.

[3] S. Lall, M. West, Discrete variational Hamiltonian mechanics, *J. Phys. A* **39**(19), 5509–5519, 2006.

Dirac Structures on Vector Spaces

Properties

- Given a n-dimensional vector space V, consider the pairing $\langle\langle \cdot\, , \,\cdot \rangle\rangle$ on $V \oplus V^*$ given by

$$\langle\langle (v, \alpha), (\tilde{v}, \tilde{\alpha}) \rangle\rangle = \langle \alpha, \tilde{v} \rangle + \langle \tilde{\alpha}, v \rangle,$$

where $\langle \cdot\, , \,\cdot \rangle$ is the natural pairing between covectors and vectors.

- A **Dirac Structure** is a subspace $D \subset V \oplus V^*$, such that

$$D = D^{\perp}.$$

- In particular, $D \subset V \oplus V^*$ is a Dirac structure iff

$$\dim D = n$$

and

$$\langle \alpha, \tilde{v} \rangle + \langle \tilde{\alpha}, v \rangle = 0,$$

for all $(v, \alpha), (\tilde{v}, \tilde{\alpha}) \in D$.

Dirac Structures on Manifolds

Properties

- An **almost Dirac Structure** on a manifold M is a subbundle $D \subset TM \oplus T^*M$ such that $D_q \subset T_qM \oplus T_q^*M$ is a Dirac structure.
- A **Dirac structure** on a manifold is an almost Dirac structure such that

$$\langle \pounds_{X_1}\alpha_2, X_3\rangle + \langle \pounds_{X_2}\alpha_3, X_1\rangle + \langle \pounds_{X_3}\alpha_1, X_2\rangle = 0,$$

for all pairs of vector fields and one-forms

$$(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in D,$$

and where $\pounds_X$ is the Lie derivative along the vector field X.

- This is a generalization of the condition that the symplectic two-form is closed, or that the Poisson bracket satisfies Jacobi's identity.

Dirac Structures on Manifolds

Generalizing Symplectic and Poisson Structures

- Let $M = T^*Q$.
- The graph of the symplectic two-form $\Omega : TM \times TM \longrightarrow \mathbb{R}$, viewed as a map $TM \longrightarrow T^*M$,

$$v_z \mapsto \Omega(v_z, \cdot),$$

  is a Dirac structure.
- Similarly, the graph of the Poisson structure $B : T^*M \times T^*M \longrightarrow \mathbb{R}$, viewed as a map T^*M to $T^{**}M \cong TM$,

$$\alpha_z \mapsto B(\alpha_z, \cdot),$$

  is a Dirac structure.
- Furthermore, if the symplectic form and the Poisson structure are related, they induce the same Dirac structure on $TM \oplus T^*M$.

Motivating Example: Electrical Circuits

Configuration space and constraints

- The **configuration** $q \in E$ of the electrical circuit is given by specifying the current in each branch of the electrical circuit.
- Not all configurations are admissible, due to **Kirchhoff's Current Laws**:

  the sum of currents at a junction is zero.

  This induce a **constraint KCL space** $\Delta \subset TE$.
- Its annihilator space $\Delta^\circ \subset T^*E$ is defined by

$$\Delta_q^\circ = \{e \in T_q^* E \,|\, \langle e, f \rangle = 0 \text{ for all } f \in \Delta_q\},$$

  which can be identified with the set of **branch voltages**, and encodes the **Kirchhoff's Voltage Laws**:

  the sum of voltages about a closed loop is zero.

Motivating Example: Electrical Circuits

Dirac structures and Tellegen's theorem

- Given $\Delta \subset TE$ and $\Delta^\circ \subset T^*E$ which encode the Kirchhoff's current and voltage laws,

$$D_E = \Delta \oplus \Delta^\circ \subset TE \oplus T^*E$$

is a Dirac structure on E.

- Since $D = D^\perp$, we have that for each $(f, e) \in D_E$,

$$\langle e, f \rangle = 0.$$

This is a statement of **Tellegen's theorem**, which is an important result in the network theory of circuits.

Motivating Example: Electrical Circuits

Lagrangian for LC-circuits

- **Dirac's theory of constraints** was concerned with degenerate Lagrangians where the set of **primary constraints**, the image $P \subset T^*Q$ of the Legendre transformation, is not the whole space.
- The **magnetic energy** is given by
$$T(f) = \sum \frac{1}{2} L_i f_{L_i}^2.$$
- The **electric potential energy** is
$$V(q) = \sum \frac{1}{2} \frac{q_{C_i}^2}{C_i}.$$
- The **Lagrangian** of the LC circuit is given by
$$L(q, f) = T(f) - V(q).$$

Variational Principles

Continuous Hamilton–Pontryagin principle

Pontryagin bundle and Hamilton–Pontryagin principle

- Consider the **Pontryagin bundle** $TQ \oplus T^*Q$, which has local coordinates (q, v, p).
- The **Hamilton–Pontryagin principle** is given by

$$\delta \int [L(q, v) - p(v - \dot{q})] = 0,$$

where we impose the second-order curve condition, $v = \dot{q}$ using Lagrange multipliers p.

Continuous Hamilton–Pontryagin principle

Implicit Lagrangian systems

- Taking variations in q, v, and p yield

$$\begin{aligned}
\delta \int [L(q,v) - p(v-\dot{q})]dt \\
&= \int \left[\frac{\partial L}{\partial q}\delta q + \left(\frac{\partial L}{\partial v} - p \right) \delta v - (v - \dot{q})\delta p + p\delta\dot{q} \right] dt \\
&= \int \left[\left(\frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left(\frac{\partial L}{\partial v} - p \right) \delta v - (v - \dot{q})\delta p \right] dt
\end{aligned}$$

  where we used integration by parts, and the fact that the variation δq vanishes at the endpoints.

- This recovers the **implicit Euler–Lagrange equations**,

$$\dot{p} = \frac{\partial L}{\partial q}, \qquad p = \frac{\partial L}{\partial v}, \qquad v = \dot{q}.$$

Continuous Hamilton–Pontryagin principle

Hamilton's phase space principle

- By taking variations with respect to v, we obtain the **Legendre transform**,

$$\frac{\partial L}{\partial v}(q,v) - p = 0.$$

- The **Hamiltonian**, $H : T^*Q \to \mathbb{R}$, is defined to be,

$$H(q,p) = \operatorname*{ext}_v \Big(pv - L(q,v)\Big) = pv - L(q,v)\big|_{p=\partial L/\partial v(q,v)}\,.$$

- The Hamilton–Pontryagin principle reduces to,

$$\delta \int [p\dot{q} - H(q,p)] = 0,$$

which is the **Hamilton's principle in phase space**.

Continuous Hamilton–Pontryagin principle

Lagrange–d'Alembert–Pontryagin principle

- Consider a constraint distribution $\Delta_Q \subset TQ$.
- The **Lagrange–d'Alembert–Pontryagin principle** is given by

$$\delta \int L(q, v) - p(v - \dot{q}) dt = 0,$$

for fixed endpoints, and variations $(\delta q, \delta v, \delta p)$ of $(q, v, p) \in TQ \oplus T^*Q$, such that $(\delta q, \delta v) \in (T\tau_Q)^{-1}(\Delta_Q)$, where $\tau_Q : TQ \to Q$.

Discrete Hamilton–Pontryagin principle

Discrete Pontryagin bundle and Hamilton–Pontryagin principle

- Consider the **discrete Pontryagin bundle** $(Q \times Q) \oplus T^*Q$, which has local coordinates (q_k^0, q_k^1, p_k).
- The **discrete Hamilton–Pontryagin principle** is given by

$$\delta \sum \left[L_d(q_k^0, q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0) \right] = 0,$$

where we impose the second-order curve condition, $q_k^1 = q_{k+1}^0$ using Lagrange multipliers p_{k+1}

Discrete Hamilton–Pontryagin principle

Implicit discrete Lagrangian systems

- Taking variations in q_k^0, q_k^1, and p_k yield

$$\delta \sum \left[L_d(q_k^0, q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0) \right] \\ = \sum \Big\{ [D_1 L_d(q_k^0, q_k^1) + p_k] \delta q_k^0 \\ - [q_k^1 - q_{k+1}^0] \delta p_{k+1} + [D_2 L_d(q_k^0, q_k^1) - p_{k+1}] \delta q_k^1 \Big\} .$$

- This recovers the **implicit discrete Euler–Lagrange equations**,

$$p_k = -D_1 L_d(q_k^0, q_k^1), \qquad p_{k+1} = D_2 L_d(q_k^0, q_k^1), \qquad q_k^1 = q_{k+1}^0 .$$

Discrete Hamilton–Pontryagin principle

Discrete Hamilton's phase space principle

- By taking variations with respect to q_k^1, we obtain the **discrete Legendre transform**,

$$D_2 L_d(q_k^0, q_k^1) - p_{k+1} = 0$$

- The **discrete Hamiltonian**, $H_{d+} : \mathcal{H}_+ \to \mathbb{R}$, is defined to be,

$$\begin{aligned} H_{d+}(q_k^0, p_{k+1}) &= \operatorname*{ext}_{q_k^1} \; p_{k+1} q_k^1 - L_d(q_k^0, q_k^1) \\ &= p_{k+1} q_k^1 - L_d(q_k^0, q_k^1) \Big|_{p_{k+1} = D_2 L_d(q_k^0, q_k^1)} . \end{aligned}$$

- The discrete Hamilton–Pontryagin principle reduces to,

$$\delta \sum [p_{k+1} q_{k+1} - H_{d+}(q_k, p_{k+1})] = 0,$$

which is the **discrete Hamilton's principle in phase space**.

Discrete Lagrange–d'Alembert–Pontryagin principle

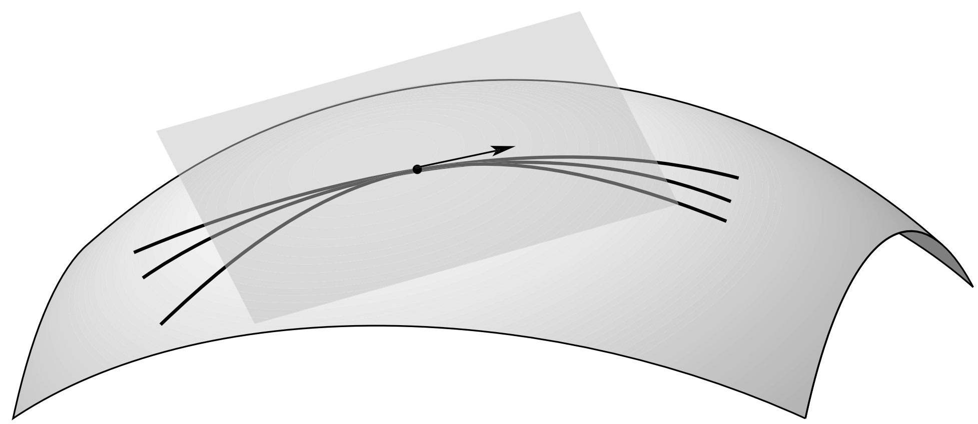
Continuous constraints from discrete constraints

- Given $\Delta_Q^d \subset Q \times Q$, consider compatible curves on Q,

$$\mathcal{C}_{\Delta_Q^d} := \{\varphi \in C^\infty([-1,1], Q) \,|\, \exists \epsilon > 0, \\ \forall \tau \in (0,\epsilon), (\varphi(-\tau), \varphi(0)), (\varphi(0), \varphi(\tau)) \in \Delta_Q^d\}.$$

- Identify $v_q \in T_q Q$ with $[\varphi]$, the **equivalence class of curves** where $\varphi(0) = q$, and $D\varphi(0) = v$, and define $\Delta_Q \subset TQ$,

$$\varphi \in \mathcal{C}_{\Delta_Q^d} \implies [\varphi] \in \Delta_Q.$$



Discrete Lagrange–d'Alembert–Pontryagin principle

Discrete Lagrange–d'Alembert–Pontryagin principle

- The **Discrete Lagrange–d'Alembert–Pontryagin principle** is given by

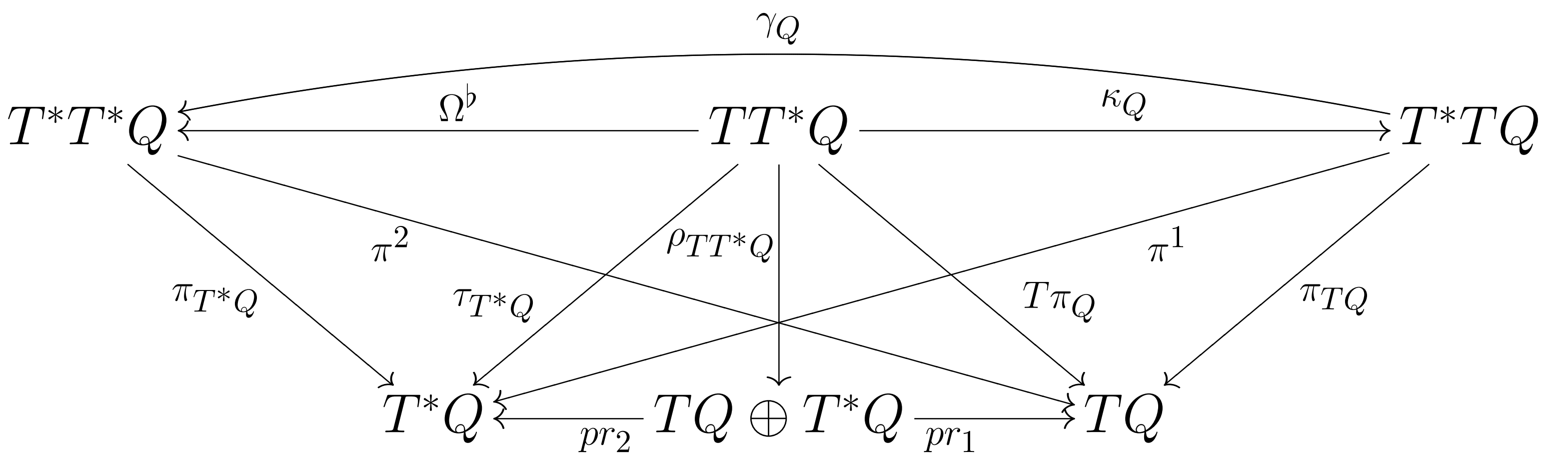
$$\delta \sum \left[L_d(q_k^0, q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0) \right] = 0,$$

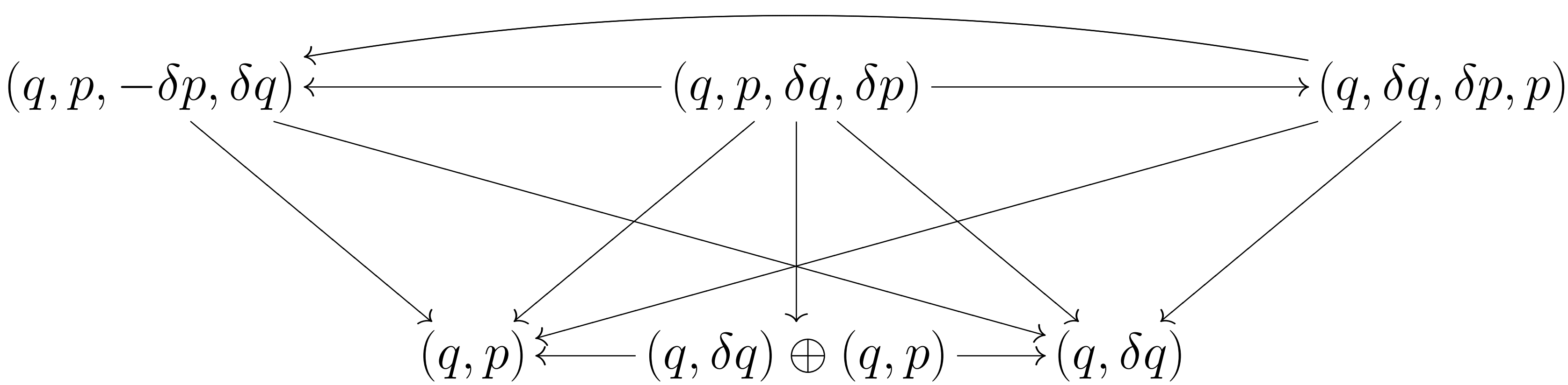
for fixed endpoints q_0^0 and q_N^0, and variations $(\delta q_k^0, \delta q_k^1, \delta p_k)$ of $(q_k^0, q_k^1, p_k) \in (Q \times Q) \oplus T^*Q$ such that $\delta q_k^0 \in \Delta_Q(q_k^0)$, $\delta q_k^1 \in \Delta_Q(q_k^1)$, and $(q_k^0, q_k^1) \in \Delta_Q^d$.

Dirac Structures

Continuous Dirac Mechanics

The Big Diagram





Continuous Dirac Mechanics

Dirac Structures and Constraints

- A constraint distribution $\Delta_Q \subset TQ$ induces a **Dirac structure** on T^*Q,

$$D_{\Delta_Q}(z) := \Big\{ (v_z, \alpha_z) \in T_zT^*Q \times T_z^*T^*Q \;\Big|\; v_z \in \Delta_{T^*Q}(z),\; \alpha_z - \Omega^\flat(v_z) \in \Delta^\circ_{T^*Q}(z) \Big\}$$

where $\Delta_{T^*Q} := (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q$.

- Holonomic and nonholonomic constraints, as well as constraints arising from interconnections can be incorporated into the Dirac structure.

Continuous Dirac Mechanics

Implicit Lagrangian Systems

- Let $\gamma_Q := \Omega^\flat \circ (\kappa_Q)^{-1} : T^*TQ \to T^*T^*Q$.
- Given a Lagrangian $L : TQ \to \mathbb{R}$, define $\mathfrak{D}L := \gamma_Q \circ dL$.
- An **implicit Lagrangian system** (L, Δ_Q, X) is,

$$(X, \mathfrak{D}L) \in D_{\Delta_Q},$$

where $X \in \mathfrak{X}(T^*Q)$.

- This gives the **implicit Euler–Lagrange equations**,

$$\dot{q} = v \in \Delta_Q(q), \qquad p = \frac{\partial L}{\partial v}, \qquad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^\circ(q).$$

- In the special case $\Delta_Q = TQ$, we obtain,

$$\dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v}.$$

Continuous Dirac Mechanics

Implicit Hamiltonian Systems

- Given a Hamiltonian $H : T^*Q \to \mathbb{R}$, an **implicit Hamiltonian system** (H, Δ_Q, X) is,

$$(X, dH) \in D_{\Delta_Q},$$

  which gives the **implicit Hamilton's equations**,

$$\dot{q} = \frac{\partial H}{\partial p} \in \Delta_Q(q), \qquad \dot{p} + \frac{\partial H}{\partial q} \in \Delta_Q^\circ(q).$$

- In the special case $\Delta_Q = TQ$, we recover the standard Hamilton's equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

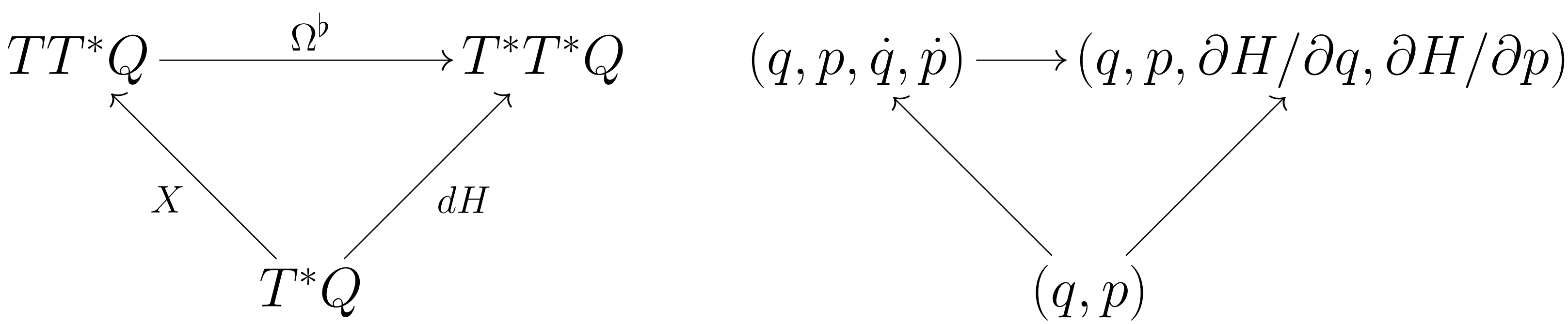
The Geometry of Symplectic Flows

Hamiltonian Flows and the $\Omega^\flat$ map

- The flow F_X of a vector field $X \in \mathfrak{X}(T^*Q)$ is symplectic if locally,

$$i_X\Omega = dH, \qquad \text{for some function } H : T^*Q \to \mathbb{R}.$$

- We require that the following diagram commutes,



- This gives rise to the map $\Omega^\flat : TT^*Q \to T^*T^*Q$,

$$\Omega^\flat : (q, p, \delta q, \delta p) \mapsto (q, p, -\delta p, \delta q).$$

The Geometry of Symplectic Flows

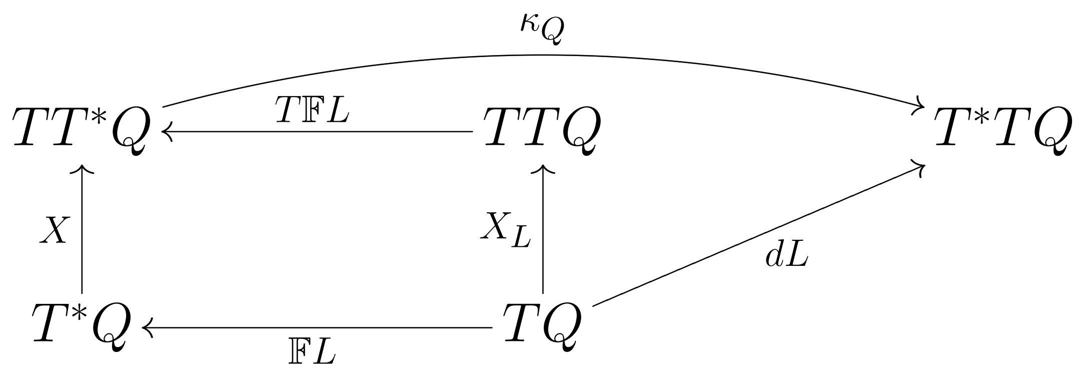
Lagrangian Flows and the κ_Q map

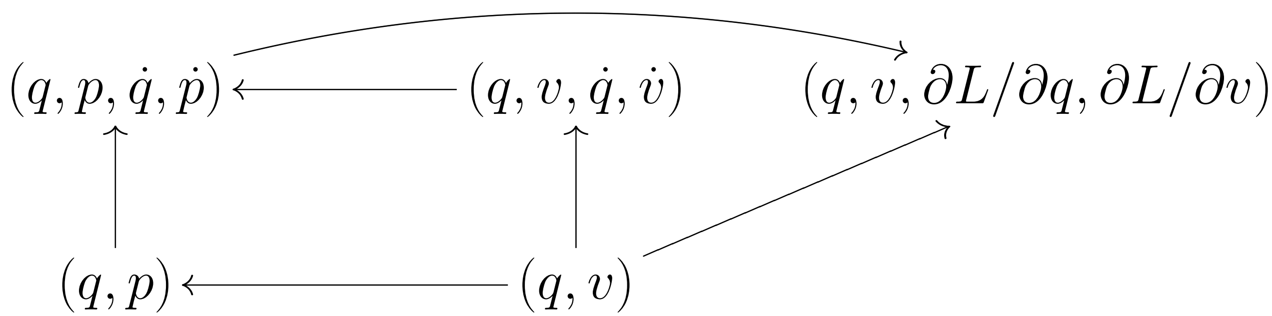
- The second-order vector field $X_L \in \mathfrak{X}(TQ)$ preserves the Lagrangian symplectic form if, $\pounds_{X_L}\Omega_L = 0$.
- Consider the Lagrange one-form, given by,
$$\Theta_L = (\mathbb{F}L)^*\Theta = \frac{\partial L}{\partial v} dq.$$
- Since $\pounds_{X_L}\Theta_L$ is closed, by the Poincaré lemma, we have a local function $L : TQ \to \mathbb{R}$ such that,
$$\pounds_{X_L}\Theta_L = dL,$$
which is the intrinsic Euler–Lagrange equation.
- In terms of the $\mathbb{F}L$-related vector field $X \in \mathfrak{X}(T^*Q)$, we have,
$$p = \frac{\partial L}{\partial v}, \qquad \dot{q} = v, \qquad \dot{p} = \frac{\partial L}{\partial q}.$$

The Geometry of Symplectic Flows

Lagrangian Flows and the κ_Q map

- We require that the following diagram commutes,





- This gives rise to the map κ_Q,

$$\kappa_Q : (q, p, \delta q, \delta p) \mapsto (q, \delta q, \delta p, p).$$

The Geometry of Symplectic Maps

Generating Functions

- The Lagrangian and the Hamiltonian induce a Lagrangian and Hamiltonian vector field.
- In discrete time, the analogue would be the generating functions of a symplectic map.
- In particular, a discrete Lagrangian is a Type I generating function, and discrete Hamiltonians are Type II or III generating functions.

The Geometry of Generating Functions

■ Generating Functions of Type I and the κ_Q^d map

- The flow F on T^*Q is symplectic iff there exists $S_1 : Q \times Q \longrightarrow \mathbb{R}$,

$$(i_F^{Q\times Q})^* \Theta_{T^*Q\times T^*Q} = dS_1.$$

which gives

$$p_0 = -D_1 S_1, \qquad p_1 = D_2 S_1.$$

- We require that the following diagram commutes,

$$\begin{array}{ccc} T^*Q \times T^*Q & \xrightarrow{\kappa_Q^d} & T^*(Q\times Q) \\ & {\scriptstyle i_F^{Q\times Q}}\nwarrow \quad \nearrow{\scriptstyle dS_1} & \\ & Q \times Q & \end{array} \qquad \begin{array}{ccc} ((q_0,p_0),(q_1,p_1)) & \rightarrow & (q_0,q_1,D_1S_1,D_2S_1) \\ & \nwarrow \quad \nearrow & \\ & (q_0,q_1) & \end{array}$$

- This gives rise to a map $\kappa_Q^d : T^*Q \times T^*Q \rightarrow T^*(Q \times Q)$

$$\kappa_Q^d : ((q_0,p_0),(q_1,p_1)) \mapsto (q_0,q_1,-p_0,p_1).$$

The Geometry of Generating Functions

Generating Functions of Type II

- Consider $\mathcal{H}_+$, whose local coordinates are (q_0, p_1).
- Then the flow F on T^*Q is symplectic if and only if there exists $S_2 : \mathcal{H}_+ \longrightarrow \mathbb{R}$ such that

$$(i_F^{\mathcal{H}_+})^* \Theta^{(2)}_{T^*Q \times T^*Q} = dS_2,$$

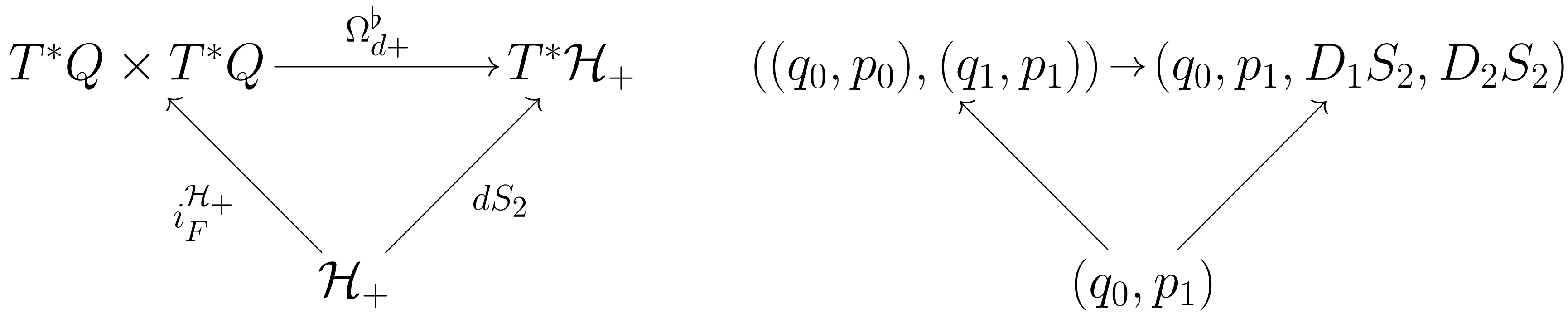
which gives

$$p_0 = D_1 S_2, \qquad q_1 = D_2 S_2.$$

The Geometry of Generating Functions

Generating Functions of Type II and the $\Omega^\flat_{d+}$ map

- We require that the following diagram commutes,

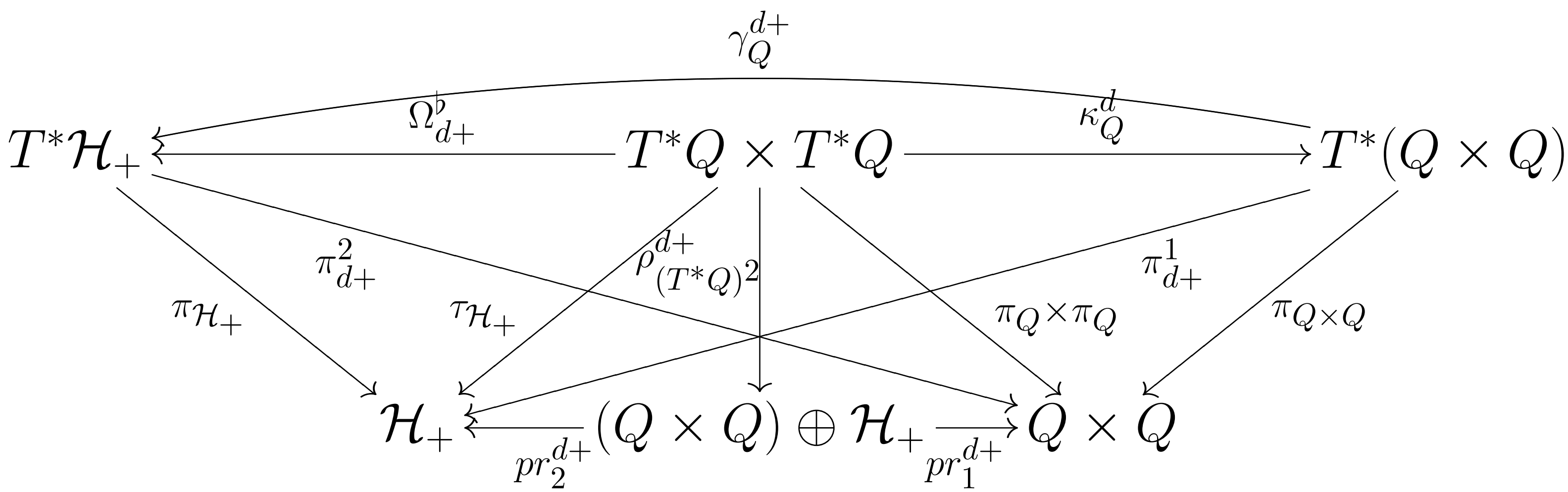


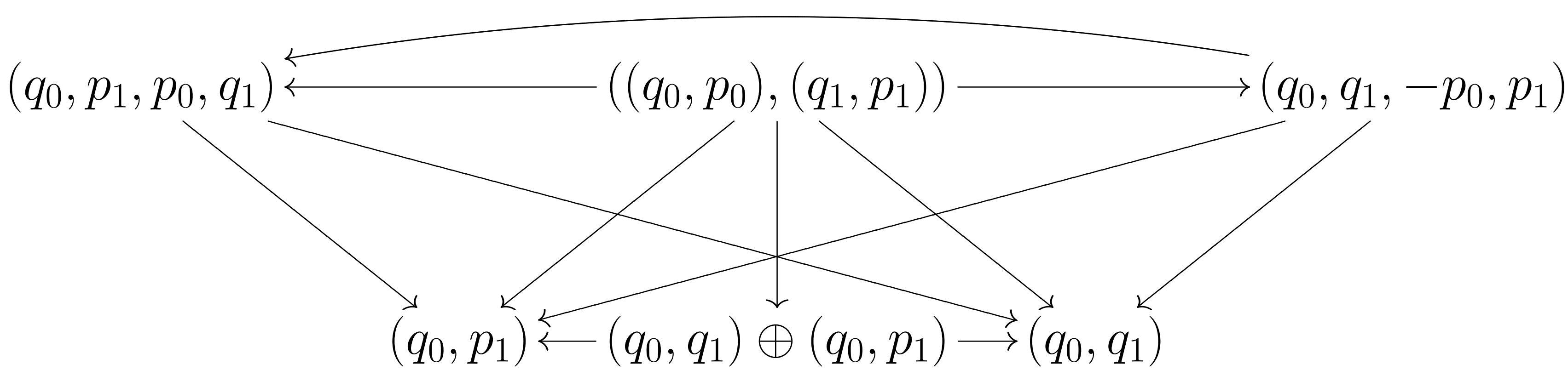
- This gives rise to a map $\Omega^\flat_{d+} : T^*Q \times T^*Q \to T^*\mathcal{H}_+$

$$\Omega^\flat_{d+} : ((q_0, p_0), (q_1, p_1)) \mapsto (q_0, p_1, p_0, q_1).$$

(+)-Discrete Dirac Mechanics

The Big Diagram





(+)-Discrete Dirac Mechanics

Discrete Dirac Structures and Discrete Constraints

- A discrete constraint distribution $\Delta_Q^d \subset Q \times Q$ induces a continuous constraint distribution $\Delta_Q \subset TQ$.
- These two distributions yield a **discrete Dirac structure**,

$$D_{\Delta_Q}^{d+}(z) := \Big\{ ((z, z^1), \alpha_{z_+}) \in (\{z\} \times T^*Q) \times T^*_{z_+}\mathcal{H}_+ \,\Big|\, \left(z, z^1\right) \in \Delta_{T^*Q}^d,\ \alpha_{z_+} - \Omega_{d+}^\flat\left((z, z^1)\right) \in \Delta_{\mathcal{H}_+}^\circ \Big\},$$

where

$$\Delta_{T^*Q}^d := (\pi_Q \times \pi_Q)^{-1}(\Delta_Q^d) \subset T^*Q \times T^*Q,$$

$$\Delta_{\mathcal{H}_+}^\circ := \left(\Omega_{d+}^\flat\right)\left(\Delta_Q^\circ \times \Delta_Q^\circ\right) \subset T^*\mathcal{H}_+.$$

(+)-Discrete Dirac Mechanics

Implicit Discrete Lagrangian Systems

- Let $\gamma_Q^{d+} := \Omega_{d+}^\flat \circ (\kappa_Q^d)^{-1} : T^*(Q \times Q) \to T^*\mathcal{H}_+$.
- Given a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$, define $\mathfrak{D}^+ L_d := \gamma_Q^{d+} \circ dL_d$.
- An **implicit discrete Lagrangian system** is given by

$$\left(X_d^k, \mathfrak{D}^+ L_d(q_k^0, q_k^1)\right) \in D_{\Delta_Q}^{d+},$$

where $X_d^k = ((q_k^0, p_k^0), (q_{k+1}^0, p_{k+1}^0)) \in T^*Q \times T^*Q$.

- This gives the **implicit discrete Euler–Lagrange equations**,

$$p_{k+1}^0 = D_2 L_d(q_k^0, q_k^1) \in \Delta_Q^\circ(q_k^1), \qquad p_k^0 + D_1 L_d(q_k^0, q_k^1) \in \Delta_Q^\circ(q_k^0),$$

$$q_k^1 = q_{k+1}^0, \qquad (q_k^0, q_{k+1}^0) \in \Delta_Q^d.$$

(+)-Discrete Dirac Mechanics

Implicit Discrete Hamiltonian Systems

- Given a discrete Hamiltonian $H_{d+} : \mathcal{H}_+ \to \mathbb{R}$, an **implicit discrete Hamiltonian system** $(H_{d+}, \Delta_Q^d, X_d)$ is,

$$\left(X_d^k, dH_{d+}(q_k^0, p_k^1)\right) \in D_{\Delta_Q}^{d+},$$

which gives the **implicit discrete Hamilton's equations**,

$$p_k^0 - D_1 H_{d+}(q_k^0, p_k^1) \in \Delta_Q^\circ(q_k^0), \qquad q_{k+1}^0 = D_2 H_{d+}(q_k^0, p_k^1),$$

$$p_k^1 - p_{k+1}^0 \in \Delta_Q^\circ(q_k^1), \qquad (q_k^0, q_{k+1}^0) \in \Delta_Q^d,$$

Extensions to Groupoids, Algebroids, and Field Theories

Generalization to Lie Groupoids

Discrete Dirac Mechanics on Lie Groupoids

- Provides a category which is closed under reduction, with a view towards developing **discrete reduction by stages**.
- Given an element g of a groupoid Γ, and the set S_g of admissible sequences $g_a, \ldots, g_b$ with values in Γ, such that,

$$g_a \cdots g_b = g.$$

- Then, **Hamilton's principle on groupoids**[4], is given by

$$\delta \mathcal{L}(g) = \delta \sum_{j=a}^{b} L(g_j) = 0.$$

[4] A. Weinstein, Lagrangian mechanics and groupoids, *Fields Inst. Comm.* **7** (1996) 207–231.

Generalization to Lie Groupoids

Discrete Dirac Mechanics on Lie Groupoids

- The condition,

$$g_a \cdots g_b = g,$$

encodes both the fixed endpoint condition, and the second-order curve condition.

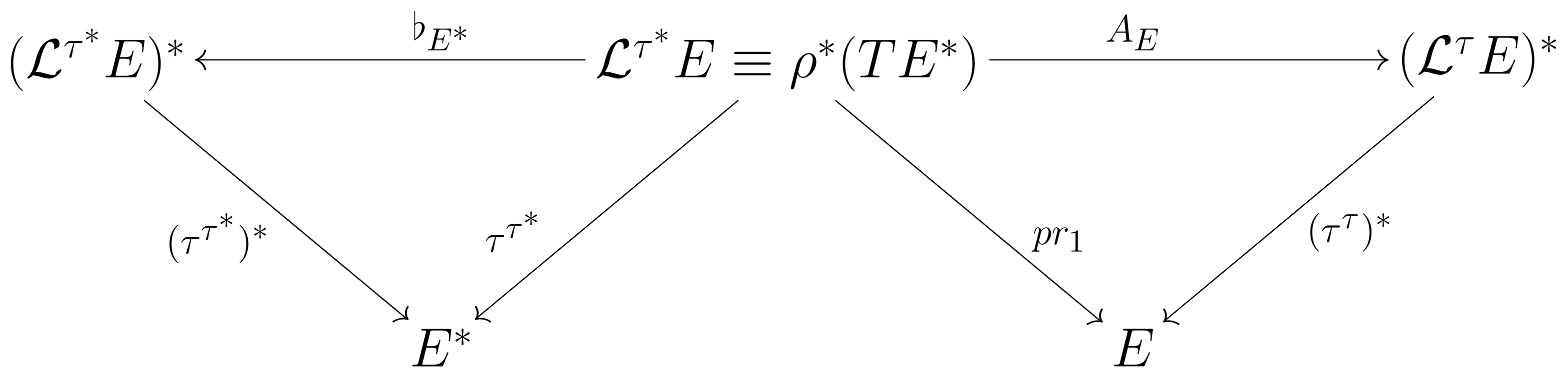
- The second-order curve condition can also be explicitly stated in terms of the source and target maps,

$$\alpha(g_{k+1}) = \beta(g_k).$$

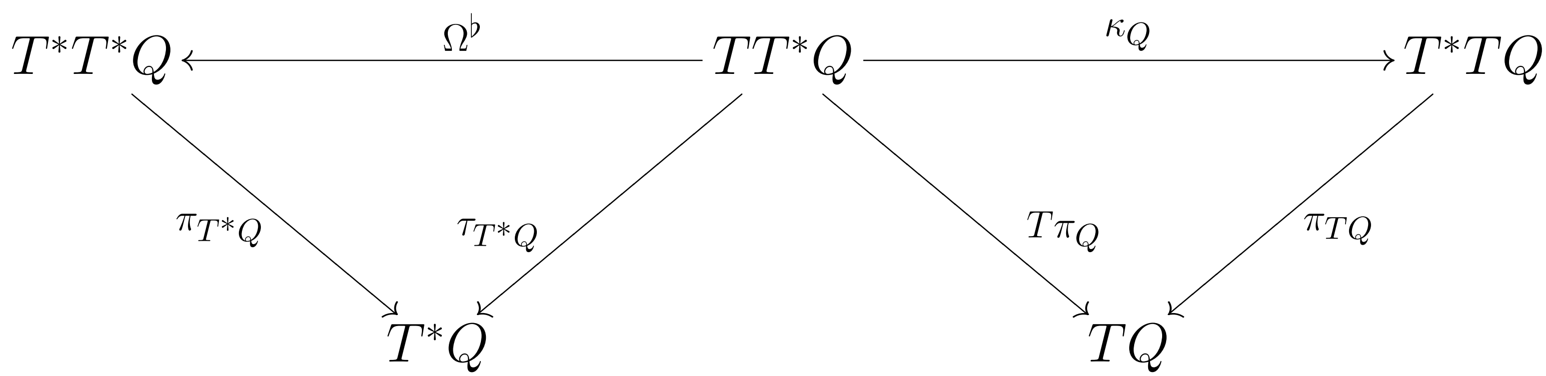
- Alternatively, one can consider the groupoid analogue of the Tulczyjew's triple on Lie algebroids, in order to construct a groupoid analogue of a Dirac structure.

Connections to Mechanics on Lie algebroids

Tulczyjew's triple on Lie algebroids



Tulczyjew's triple on tangent bundles



Connections to Mechanics on Lie algebroids

Dirac Mechanics on Lie algebroids

- Introduce the Lie algebroid analogue of the Pontryagin bundle,

$$E \oplus E^*.$$

- Construct the Lie algebroid analogue of the Dirac structure by using the two vector bundle isomorphisms,

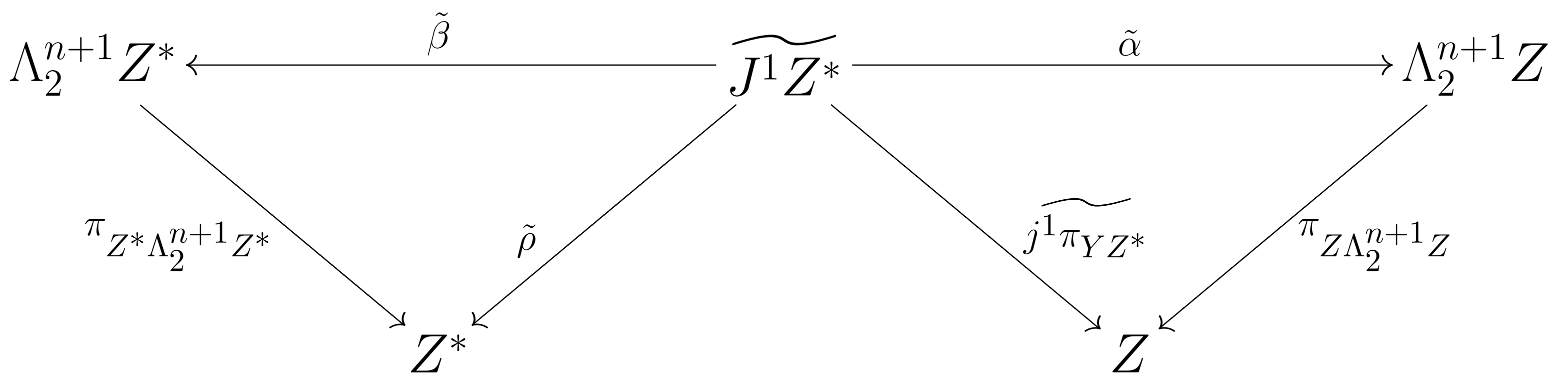
$$A_E : \rho^*(TE^*) \to (\mathcal{L}^\tau E)^*$$
$$\flat_{E^*} : \mathcal{L}^{\tau^*} E \to (\mathcal{L}^{\tau^*} E)^*$$

- Generalizes Dirac mechanics to Lie algebroids, thereby unifying Lagrangian and Hamiltonian mechanics on Lie algebroids.
- Interesting to consider the Lie groupoid analogue of the Tulczyjew's triple, viewed as a generalization of discrete Dirac mechanics.

Connections to Multisymplectic Classical Field Theories

Tulczyjew's triple in classical field theories

- Bundle $\pi_{XY} : Y \to X$.
- Lagrangian density $\mathbb{L} : Z \to \Lambda^{n+1}X$, for first-order field theories $Z = J^1Y$.
- We have the following Tulczyjew's triple,



- Provides a means of developing multisymplectic Dirac mechanics for classical field theories.

Conclusion

Discrete Dirac Structures

- We have constructed a discrete analogue of a Dirac structure by considering the geometry of generating functions of symplectic maps.
- Unifies geometric integrators for degenerate, interconnected, and nonholonomic Lagrangian and Hamiltonian systems.
- Provides a characterization of the discrete geometric structure associated with Hamilton–Pontryagin integrators.

Discrete Hamilton–Pontryagin principle

- Provides a unified discrete variational principle that recovers both the discrete Hamilton's principle, and the discrete Hamilton's phase space principle.
- Is sufficiently general to characterize all near to identity Dirac maps.

Conclusion

Current Work and Future Directions

- Discrete Dirac structures are intimately related to the geometry of Lagrangian submanifolds and the Hamilton–Jacobi equation.
- Derive the Dirac analogue of the Hamilton–Jacobi equation, with nonholonomic Hamilton–Jacobi theory as a special case.
- Continuous and discrete Dirac mechanics on Lie algebroids and Lie groupoids.
- Continuous and discrete multisymplectic Dirac mechanics.

Poster Advertisements

Discrete Hamilton–Jacobi Theory (*Tomoki Ohsawa*)

- Provides a discrete analogue of Hamilton–Jacobi theory in the context of discrete Hamiltonian systems.
- Can be viewed as a composition theorem for discrete Hamiltonians.

The Hamilton's Principle in Phase Space (*Jingjing Zhang*)

- Provides a characterization of Hamiltonian variational integrators that does not rely on going through the Lagrangian side.
- Potential applications to degenerate Hamiltonian systems.

Questions?

M. Leok, T. Ohsawa, Discrete Dirac Structures and Variational Discrete Dirac Mechanics, arXiv:0810.0740