Goal: develop and understand spatially- and temporally-adaptive variational integrators for mechanical PDEs that do not require a constant number of degrees of freedom in the material domain. This problem is different from (but related to) r-adaptive or asynchronous variational integrators, which adapt DOFs in either the material or time domain but always maintain a constant number of material DOFs. In general, allowing a variable number of material DOFs allows the material domain to be remeshed in an arbitrary way over the course of the simulation. For the course project I will focus on a highly simplified model of 1D elastic body dynamics. My overall approach is a direct discretization of Hamilton’s principle on a spacetime mesh. Rather than trying to adapt this mesh to meet some performance criterion, I will simply assume that a “good” mesh is known a priori (for now). The figure below is a cartoon of spacetime meshes for the 1D elasticity problem using different variational schemes; the types of problems I am interested are highlighted in the bottom row.

1 Lagrangian

Consider a 1D elastic body immersed in $\mathbb{R}^3$ with reference configuration $\mathcal{B} = [0, 1]$ and uniform mass density $\rho = 2$. The immersion $q : \mathcal{B} \rightarrow \mathbb{R}^3$ gives the configuration of the body at time $t$. The total kinetic energy at time $t$ can therefore be written as

$$K_t = \int_{\mathcal{B}} \rho \frac{1}{2} \left\| \frac{\partial q_t(X)}{\partial t} \right\|^2 dV.$$ 

We use a simple elastic potential

$$U_t = \frac{1}{2} \int_{\mathcal{B}} \rho \left( \left\| \frac{\partial q_t(X)}{\partial X} \right\|^2 - 1 \right) dV$$

where the leading factor $1/2$ is merely for convenience. The resulting Lagrangian is then simply

$$L_t = V_{\mathcal{B}} + \int_{\mathcal{B}} \left\| \nabla q_t \right\|^2 dV.$$
where in the final expression \( V_B \) is the volume of the body, \( \nabla \) represents partial derivatives with respect to both space and time, and \( \| \cdot \| \) is the usual Lorentzian metric on \( \mathbb{R}^{1,1} \). The trajectory of \( q \) in time is given by the extrema of the corresponding action integral, i.e., solutions of

\[
\delta S = \delta \int_{t_0}^{t_f} L dt = 0
\]

where there is zero variation in \( q \) at \( t_0 \) and \( t_f \).

## 2 Discrete Lagrangian

Consider a spacetime domain with coordinates \( X \) and \( t \) along the material and temporal axes, respectively. Let \( V, E, \) and \( F \) be the sets of vertices, edges, and faces in a simplicial mesh of this domain. For the moment, assume that we are only interested in configurations in \( \mathbb{R} \) (instead of \( \mathbb{R}^3 \)). We can then describe the discrete configuration of the body via the map \( q_D : V \to \mathbb{R} \). Suppose that we interpolate \( \hat{q} \) piecewise linearly over each simplex to get a map \( \hat{q} : B \to \mathbb{R}^3 \). We can then express a discrete action \( S_D \) as the sum over each simplex of the action of the interpolated configuration:

\[
S_D = \sum_{f \in F} \int_f L.
\]

Note that the integral over \( f \) is actually an integral over space and time. In fact, this integral is very straightforward to compute. Let \( A_{qX}, A_{qt}, \) and \( A_{Xt} \) be the projected areas of \( f \) along the \( t, X, \) and \( q \) axes, respectively. It is easy to show that on \( f \)

\[
\frac{\partial \hat{q}}{\partial t} = \frac{A_{qX}}{A_{Xt}}
\]

and

\[
\frac{\partial \hat{q}}{\partial X} = \frac{A_{qt}}{A_{Xt}},
\]

hence integrating these quantities over \( f \) (which has area \( A_{Xt} \) in the material/time domain) gives us

\[
S_D(f) = A_{qt}^2 - A_{qX}^2 + A_{Xt},
\]

where the final term accounts for \( V_B \) in the continuous Lagrangian.
3 Numerical Implementation

Our discrete action is quadratic in $q$, hence the system expressing extremization of this action with respect to $q$ is linear. From here we can proceed in a number of ways – for instance, we can specify the values of $q$ at $t_0$ and $t_f$ and solve for the remaining values of $q$ such that the action is extremized. The remainder of this project entails implementing the system described above and analyzing the behavior of the resulting solutions.