Geometry of Configurational Forces

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Abstract: In this paper we study the geometry of configurational forces. We first review the efforts in understanding and formulating configurational forces in the framework of continuum mechanics. A brief formulation of continuum mechanics from a geometric point of view is given. We then give a geometric treatment of configurational forces. In the end, configurational forces are studied in spacetime and multisymplectic continuum mechanics.

Contents

1 Introduction

Driving (configurational, material, etc.) force in continuum mechanics has an old history and many researchers have studied it from different points of view. Understanding configurational forces and their balance is important in formulating the evolution of defects in a solid in the setting of continuum mechanics. Driving force in continuum mechanics was introduced by Eshelby [4, 5, 6] and was elaborated by many other researchers (see [1, 9, 12, 18] and references therein).

Understanding geometric aspects of continuum mechanics is very important in having a consistent theory. A good example of the lack of geometric insight in the literature of continuum mechanics is the existence of different objective stress rates and the belief that some stress rates are more objective than others. Indeed all the objective stress rates are different associated tensors of a Lie derivative (see Marsden and Hughes [14] for more details). In addition to giving a deeper theoretical understanding of continuum mechanics, the geometric approach enables one to develop more efficient and consistent numerical techniques (see Lew et al. [13] and references therein). We believe that a geometric study of configurational forces is missing. This paper aims to fill this gap.

This paper is organized as follows. Section 2 reviews some of the important contributions in the literature of configurational forces. In the course of this review some comparisons and observations are made. Geometry of continuum mechanics is reviewed in Section 3. Variational and Hamiltonian structures of continuum mechanics are briefly explained in Section 4. In Section 5, we study configurational forces from a geometric point of view. Spacetime formulation of configurational forces is given in Section 6. Section 7 studies configurational forces in the framework of multisymplectic continuum mechanics. Finally, conclusions are given in Section 8.

2 Configurational Forces and Their History

The idea of driving force in continuum mechanics goes back to Eshelby [4, 5, 6]. Understanding driving force is important in developing evolution laws for movement of defects. Dislocations, vacancies, interfaces, cavities, cracks, etc. are examples of defects. Driving force on these defects causes climb and glide of dislocations, diffusion of point defects, migration of interfaces, changing the shape of cavities and propagation of cracks, to mention a few examples. Eshelby defines the force on a defect as the generalized force corresponding to position of the defect, which is thought of as a generalized displacement. Eshelby studied inhomogeneities in elastostatic systems by considering explicit dependence of the elastic energy density on position. Suppose the elastic energy density has explicit dependence on X (position of material points in the undeformed configuration), i.e.,

$$W = W(\varphi, \mathbf{F}, X) \tag{2.0.1}$$

where φ and $\mathbf{F}$ are deformation mapping and deformation gradient, respectively [1]. Eshelby considers an open neighborhood Ω of an isolated defect and shows that,

$$\mathbf{F}^{\text{defect}} = \int_\Omega \left(\frac{\partial W}{\partial X}\right)_{\text{explicit}} dV = \int_\Omega \text{Div}\mathbf{E}\, dV = \int_{\partial\Omega} \mathbf{E}\hat{\mathbf{N}}\, dA \tag{2.0.2}$$

where,

$$\mathbf{E} = W\mathbf{I} - \mathbf{F}^T\mathbf{P} \tag{2.0.3}$$

is Eshelby's energy-momentum tensor, $\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}$ is the first Piola-Kirchhoff stress tensor and $(\frac{\partial}{\partial X})_{\text{explicit}}$ is differentiation with respect to X when all other independent variables are fixed. It turns out that for a crack (thought of as a defect) this is nothing but J-integral [20].

Knowles [12] and Knowles and Abarayaneh [1, 2] chose a different point of departure. Suppose there is a surface of discontinuity $S(t)$ moving in a continuum. They look at the rate of dissipation due to this surface of discontinuity. They observe that the rate of dissipation may be written as a surface integral. The integrand is proportional to normal velocity of the moving surface and what is paired with the normal velocity is nothing but the normal component of the jump in Eshelby's energy-momentum tensor. Knowles [12] looks at the rate of change of elastic energy in a domain $\mathcal{D}$ that intersects with the surface of deformation gradient discontinuity $\mathfrak{S}$ (in the reference configuration),

$$\dot{U}(t) = \frac{d}{dt}\int_{\mathcal{D}} W(\varphi, \mathbf{F}, X) dV, \quad \mathcal{D} \cap \mathfrak{S} \neq \emptyset \tag{2.0.4}$$

He shows that,

$$\dot{U}(t) = \int_{\partial\mathcal{D}} \mathbf{P}\hat{\mathbf{N}} \cdot \mathbf{v} dA - \int_{\mathfrak{s}_t} [\![\mathbf{E}]\!]\hat{\mathbf{N}} \cdot \mathbf{V} dA \tag{2.0.5}$$

where $\mathfrak{s}_t = \varphi_t^{-1}(S(t))$ and $S(t)$ is the surface of discontinuity in the deformed configuration at time t. Note that this surface is evolved in the reference configuration, i.e., at any moment t the deformed configuration is the motion of a new reference configuration. Even with this point of view which is different from that of Eshelby's original idea, Eshelby's energy-momentum tensor $\mathbf{E}$ shows up. Here $[\![g]\!] = (g)^+ - (g)^{-1}$, where $(g)^+$ and $(g)^-$ are outer and inner traces of g on the surface of discontinuity. It should be noted that $[\![\mathbf{E}]\!]\hat{\mathbf{N}}$ can be thought of as a force per unit area associated with the material surface of discontinuity.

Abeyaratne and Knowles [1] consider the inertial effects and show that the driving force on a surface of discontinuity of deformation gradient has the following form,

$$f = \hat{\mathbf{n}} \cdot [\![\rho\psi\mathbf{1} - \mathbf{F}^T\mathbf{T} + \frac{1}{2}\rho V_n^2\mathbf{C}]\!]\hat{\mathbf{n}} \tag{2.0.6}$$

where ψ is the free energy, $\mathbf{1}$ is the identity tensor, $\mathbf{F}$ is the deformation gradient, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{C}$ is the right Cauchy-Green deformation tensor and V_n is the normal velocity of the surface of discontinuity [2]. Abeyaratne and Knowles [1, 2] introduce kinetic relations for evolution of surfaces of discontinuity. Their kinetic relations have the following form,

$$V_n(x,t) = \Lambda(f(x,t)), \quad x \in S(t) \tag{2.0.7}$$

[1]All the notations used in this paper are explained in Section 3.

[2] $V_n = \tilde{\mathbf{V}} \cdot \hat{\mathbf{n}}$, where $\tilde{\mathbf{V}} = \tilde{\mathbf{V}}(x,t)$ is the velocity of the point x on the moving surface of discontinuity $S(t)$.

where Λ is some function. The above equation closes the system of partial differential equations governing an elastodynamic problem with moving surfaces of discontinuity.

Gurtin [9, 10] studies configurational forces from a more abstract point of view. He believes that one should view configurational forces as basic primitive objects rather than variational constructs. This would imply that configurational forces will have their own balance laws. Gurtin introduces referential control volumes (RCV) and writes the balance laws for them. This is a natural thing to do for a continuum with defects. Movement of a defect is independent of its elastic motion and this requires a larger configuration space. This will become clearer when we give a geometric picture of configurational forces. Boundary of an RCV evolves with the reference configuration. Gurtin considers a local parametrization of the boundary of an RCV, $\partial\mathcal{R}(t)$ and defines its velocity by,

$$\mathrm{v}(X,t) = \frac{\partial}{\partial t}\hat{\mathrm{X}}(u_1, u_2, t) \tag{2.0.8}$$

where $\mathrm{X} = \hat{\mathrm{X}}(u_1, u_2, t)$ is a local chart for $\partial\mathcal{R}(t)$. The normal component of this velocity V_n is intrinsic while the component that lies in the tangent space of $\partial\mathcal{R}(t)$ depends on the parametrization. [3] When a surface of discontinuity moves, there is some removal and addition of matter in any fixed part of the continuum. Gurtin calls this *accretion*. Accretion is independent of deformation map φ and hence can be thought of as an independent kinematical process. This leads us to expect to have an independent system of configurational forces.[4] Introducing a configurational stress tensor $\mathbf{P}^{\text{config.}}$ and imposing the invariance of 'working' with respect to reparametrizations of $\partial\mathcal{R}(t)$, leads to the following balance equation,

$$\mathbf{P}^{\text{config.}} + \mathbf{F}^T\mathbf{P} = \pi\mathbf{I} \tag{2.0.9}$$

where π is a bulk tension, which is work conjugate to volume change of $\mathcal{R}(t)$ due to accretion. Then, invariance of entropy inequality with respect to reparametrizations of $\partial\mathcal{R}(t)$ implies that $\pi = \Psi$, the free energy density. This and Eq. (2.0.9) yield,

$$\mathbf{P}^{\text{config.}} = \Psi\mathbf{I} - \mathbf{F}^T\mathbf{P} = \mathbf{E} \tag{2.0.10}$$

i.e., the configurational stress tensor is nothing but Eshelby's energy-momentum tensor. Gurtin defines a configurational body force field (internal configurational force) $\mathbf{B}^{\text{config.}}$, which does not contribute to 'working' or entropy inequality as material is being removed or added only through the boundary of an RCV. A balance of configurational forces is postulated which states,[5]

$$\int_{\partial\mathcal{R}(t)} \mathbf{P}^{\text{config.}}\hat{\mathbf{N}}dA + \int_{\mathcal{R}(t)} \mathbf{B}^{\text{config.}}dV = 0 \tag{2.0.11}$$

Or,

$$\text{Div}\mathbf{P}^{\text{config.}} + \mathbf{B}^{\text{config.}} = 0 \tag{2.0.12}$$

[3] This suggests that there is a momentum map related to this symmetry. What is it?

[4] This statement can be made more rigorous and will become clearer in our geometric treatment of configurational forces.

[5] Think about a momentum balance of configurational forces and its importance and implications. Specific examples in fracture mechanics are called M- and L-integrals. Are there any applications for these?

Now consider a nonhomogeneous elastic solid, i.e., $\Psi = \Psi(\mathbf{F}, X)$. It can be easily shown that for such a material,

$$\mathbf{B}^{\text{config.}} = -\left(\frac{\partial \Psi}{\partial X}\bigg|_{\text{expl.}} + \mathbf{F}^T\mathbf{B}\right) \tag{2.0.13}$$

This is now related to Eshelby's idea when $\mathbf{B} = \mathbf{0}$,[6]

$$\mathbf{F}^{\text{defect}} = \int_{\Omega} \frac{\partial \Psi}{\partial X}\bigg|_{\text{expl.}} dV = \int_{\Omega} -\mathbf{B}^{\text{config.}} dV = \int_{\partial\Omega} \mathbf{E}\hat{\mathbf{N}} dA \tag{2.0.15}$$

Gurtin also considers surfaces of discontinuity of deformation gradient (phase boundaries). Suppose there are two phases separated by a smooth surface $S(t)$. Gurtin defines a surface configurational stress $\mathbf{P}^{\text{surf.}}$ and an internal configurational surface body force $\mathbf{B}^{\text{surf.}}$. The following configurational balance law is postulated,

$$\int_{\partial\mathcal{R}(t)} \mathbf{P}^{\text{config.}}\hat{\mathbf{N}} dA + \int_{\mathcal{R}(t)} \mathbf{B}^{\text{config.}} dV + \int_{\partial\mathcal{R}(t)\cap S(t)} \mathbf{P}^{\text{surf.}}\tilde{\mathbf{N}} ds + \int_{\mathcal{R}(t)\cap S(t)} \mathbf{B}^{\text{surf.}} dA = 0 \tag{2.0.16}$$

where $\tilde{\mathbf{N}}$ is the unit formal to $\partial\mathcal{R}(t) \cap S(t)$.

Maugin and Trimarco [17] and Maugin [18] write the balance of linear momentum in Lagrangina coordinates and introduce a *pseudomomentum*.[7] Gurevich and Thellung [8] use a similar idea to obtain a form of conservation of *quasimomentum*, but without giving a clear explanation of why they multiply the balance of linear momentum by the deformation gradient. Here we summarize the work of Maugin and Trimarco. Let us define the following two linear momentum densities,[8]

$$\mathbf{l} = \mathbf{l}(x,t) = \rho(x,t)\mathbf{v}(x,t), \quad \mathbf{l}_0 = \mathbf{l}_0(x,t) = \rho_0(x,t)\mathbf{v}(x,t) \tag{2.0.17}$$

It can be shown that equilibrium equations are equivalent to,[9]

$$\text{div}(\mathbf{T} - \mathbf{v} \otimes \mathbf{l}) - \frac{\partial \mathbf{l}}{\partial t} = \mathbf{0} \tag{2.0.18}$$

We now rewrite (2.0.18) in terms of the first Piola-Kirchhoff stress tensor $\mathbf{P} = J\mathbf{F}^{-1}\mathbf{T}$,

$$\text{div}\left(J^{-1}\mathbf{F}\mathbf{P} - \frac{\rho}{\rho_0}\mathbf{v} \otimes \mathbf{l}_0\right) - \frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\mathbf{l}_0\right) = \mathbf{0} \tag{2.0.19}$$

[6]$\mathbf{F}^T\mathbf{B}$ has a nice geometric interpretation. $\mathbf{B}$ is body force per unit undeformed volume but defined on the tangent space of the deformed configuration. $\mathbf{B}$ acts on virtual displacements in the deformed configuration. Thus,

$$\langle \mathbf{B}, \delta\mathbf{w} \rangle = \langle \mathbf{B}, \mathbf{F}\delta\mathbf{W} \rangle = \langle \mathbf{F}^T\mathbf{B}, \delta\mathbf{W} \rangle \tag{2.0.14}$$

This means that $\mathbf{F}^T\mathbf{B}$ is the equivalent body force in the material space. This might not be written well but I think the idea is interesting.

[7]This paper is not cited in many other related papers. This shows that people do not see the connection between different works. This is something that the present paper should address.

[8]What is the meaning and significance of the second one?

[9]What is the motivation behind this? Is it possible to start from Div $\mathbf{P} + \mathbf{B} = \mathbf{0}$? This will be done later.

Using conservation of mass and the fact that $\nabla_X = \mathbf{F}\nabla$ this simplifies to,

$$\mathrm{Div}\mathbf{P} - \frac{\partial \mathbf{l}_0}{\partial t} = 0 \tag{2.0.20}$$

In components,

$$\frac{\partial P^{iJ}}{\partial X^J} - \frac{\partial}{\partial t}(\rho_0 v^i) = 0 \tag{2.0.21}$$

It is seen that these equations are still in spacial coordinates. Maugin and Trimarco then try to rewrite these equations completely in material coordinates.[10] We know that,

$$\frac{\partial \mathbf{F}^T}{\partial t} = \mathrm{Grad}\,\mathbf{v}, \quad \mathrm{Div}(W\mathbf{I}) = \frac{\partial W}{\partial X} + (\mathrm{Grad}\mathbf{F}^T)\cdot\frac{\partial W}{\partial \mathbf{F}} \tag{2.0.22}$$

Also note that,

$$\begin{aligned} \mathbf{F}^T\frac{\partial \mathbf{l}_0}{\partial t} &= \frac{\partial}{\partial t}(\mathbf{F}^T\mathbf{l}_0) - (\mathrm{Grad}\,\mathbf{v})\mathbf{l}_0 = \frac{\partial}{\partial t}(\mathbf{F}^T\mathbf{l}_0) - \mathrm{Div}\Big(\frac{1}{2}\rho_0|\mathbf{v}|^2\Big) + \frac{1}{2}|\mathbf{v}|^2\mathrm{Grad}\,\rho_0 \\ \mathbf{F}^T\mathrm{Div}\mathbf{P} &= \mathrm{Div}(\mathbf{F}^T\mathbf{P}) - \mathrm{Div}(W\mathbf{I}) + \frac{\partial W}{\partial X} \end{aligned} \tag{2.0.23}$$

Multiplying (2.0.20) by $\mathbf{F}^T$ from left and substituting from (2.0.23), we will have, [11]

$$\frac{\partial}{\partial t}(\rho_0\mathbf{F}^T\mathbf{v}) + \mathrm{Div}\Big[\Big(W - \frac{1}{2}\rho_0|\mathbf{v}|^2\Big)\mathbf{I} - \mathbf{F}^T\mathbf{P}\Big] + \frac{1}{2}|\mathbf{v}|^2\mathrm{Grad}\,\rho_0 - \frac{\partial W}{\partial X} = 0 \tag{2.0.24}$$

Now define,

$$\begin{aligned} \mathcal{E} &= W(\mathbf{F},X) - \frac{1}{2}\rho_0|\mathbf{v}|^2 \\ \mathcal{P} &= -\mathbf{F}^T\mathbf{l}_0 \\ \mathbf{E} &= \mathcal{E}\mathbf{I} - \mathbf{F}^T\mathbf{P} \\ \mathbf{f}^{\text{inhom.}} &= -\frac{\partial \mathcal{E}}{\partial X} \end{aligned}$$

Then, (2.0.24) can be written as,

$$\mathrm{Div}\mathbf{E} = \mathbf{f}^{\text{inhom.}} + \frac{\partial \mathcal{P}}{\partial t} \tag{2.0.25}$$

Now we give a geometric interpretation of what Maugin and Trimarco did. The quantity defined in the left hand side of Eq.(2.0.20) is a vector-valued 1-form on the deformed configuration [12], i.e.,

$$\alpha = \mathrm{Div}\mathbf{P} - \frac{\partial \mathbf{l}_0}{\partial t} \in T_x^*S \tag{2.0.26}$$

[10] They do not explain how and in what sense the balance of linear momentum is rewritten in the material space. We will make this clear.

[11] They do not explain why they left multiply by $\mathbf{F}^T$ and, for example, not by $\mathbf{F}^{-1}$.

[12] See Section 3 for notations used here.

Balance of linear momentum states that,

$$\alpha = 0 \tag{2.0.27}$$

which is equivalent to,

$$\varphi^*(\alpha) = \mathbf{F}^T\alpha = 0 \tag{2.0.28}$$

Rogula [21] mentions a connection between balance of configurational forces and Noether's theorem. For example, if reference configuration is invariant with respect to translations, a conservation law follows. However, he does not follow this in more detail and starts from a definition of *quasi-momentum*. Huang and Batra [11] have a short discussion on this. Nelson [19] studies the consequence of invariance of the Lagrangian with respect to translations in the reference configuration. He shows that the conserved quantity is the *canonical crystal momentum*, which is nothing but the pull-back of the linear momentum. In this paper, we will investigate all the relevant invariance groups and the corresponding momentum maps.

Observations:

- Configurational forces are due to change of reference configuration during deformation. However, these forces cannot be independent of the deformation. The same solid behaves differently under the action of external forces regarding the motion of defects. For example, a crack does not propagate unless the external forces are 'large' enough as the stress intensity factor is proportional to the external forces in linear elastic fracture mechanics.

- Goal: Similar to what exists for balance laws in the physical space (deformed configuration), it should be possible to systematically postulate balance laws for evolution of reference configuration.

- Having local balance of linear and angular momentum, left multiplication by $\mathbf{F}^T$ gives us the corresponding balance law in the material space. Does this mean balance laws in the material space are not independent of those in the physical space? I think the answer is no. Balance of configurational forces is locally related to balance of linear momentum through $\mathbf{F}^T$. But globally they are independent. Gurtin [9] starts from the global balance laws. This is something that needs more thinking.

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3 Geometry of Continuum Mechanics

Here,we first review geometry of continuum mechanics.[13] For more details refer to Marsden and Hughes [14].

[13] This review is not complete.

3.1 Preliminaries from Manifold Theory

This brief review follows Marsden, Ratiu and Abraham [16], Marsden and Hughes [14] and Marsden and Ratiu [15].

Definition 3.1.1 ***(Tangent Space)*** *Suppose M is an n-manifold and $p \in M$. The tangent space to M at p is the set of all equivalent curves at p. The tangent space of M at p is denoted T_pM and can be identified with $\mathbb{R}^n$. The tangent bundle of M is the disjoint union of all the tangent spaces,*

$$TM = \bigsqcup_{p \in M} T_pM \tag{3.1.1}$$

Tangent bundle consists of pairs $(p, \mathbf{v})$ of base points and tangent vectors at p.

Definition 3.1.2 ***(Directional Derivative)*** *Suppose $\mathcal{B}$ is an n-manifold and $f : \mathcal{B} \to \mathbb{R}$ is C^1. Let $\mathbf{V}_X = (X, \mathbf{V}) \in T_X\mathcal{B}$. $\mathbf{V}_X[f]$ denotes the derivative of f at X in the direction of $\mathbf{V}_X$, i.e., $\mathbf{V}_X[f] = \mathrm{D}f(X) \cdot \mathbf{V}$. In a local chart $\{X^I\}$,*

$$\mathbf{V}_X[f] = \frac{\partial f}{\partial X^I} V^I \tag{3.1.2}$$

Definition 3.1.3 ***(Tangent Map)*** *Suppose $\mathcal{B}$ and $\mathcal{S}$ are manifolds and $\varphi : \mathcal{B} \to \mathcal{S}$ is C^1. The tangent map of φ is defined in any local chart as,*

$$T\varphi : T\mathcal{B} \to T\mathcal{S}, \quad T\varphi(X, \mathbf{V}) = (\varphi(X), \mathrm{D}\varphi(X) \cdot \mathbf{V}) \tag{3.1.3}$$

Definition 3.1.4 ***(Vector Field)*** *A vector field on a manifold M is a mapping $\mathbf{v} : M \to TM$ such that,*

$$\mathbf{v}(p) \in T_pM \quad \forall\, p \in M \tag{3.1.4}$$

Definition 3.1.5 ***(Pull-Back and Push-Forward of Scalar Functions)*** *Let $\varphi : \mathcal{B} \to \mathcal{S}$ be a map of manifolds and $f : \mathcal{S} \to \mathbb{R}$. The pull-back of f by φ is defined by,*

$$\varphi^* f = f \circ \varphi \tag{3.1.5}$$

If $g : \mathcal{B} \to \mathbb{R}$, the push-forward of g by φ is defined by,

$$\varphi_* g = g \circ \varphi^{-1} \tag{3.1.6}$$

Note that for push-forward φ is required to be invertible.

Definition 3.1.6 ***(Pull-Back and Push-Forward of Vector Fields)*** *If $\mathbf{Y}$ is a vector field on $\mathcal{B}$ and $\varphi : \mathcal{B} \to \mathcal{S}$ is a C^1 diffeomorphism, then $\varphi_*\mathbf{Y} = T\varphi \circ \mathbf{Y} \circ \varphi^{-1}$ is a vector field on $\varphi(\mathcal{B})$ and is called the push-forward of $\mathbf{Y}$ by φ.*

If $\mathbf{y}$ is a vector field on $\varphi(\mathcal{B})$ and φ is C^1, $\varphi^*\mathbf{y} = T(\varphi^{-1}) \circ \mathbf{y} \circ \varphi$ is a vector field on $\mathcal{B}$ and is called the pull-back of $\mathbf{y}$ by φ.

Definition 3.1.7 *(One-forms) Let M be an n-manifold and $p \in M$. A one-form at p is a linear map $\alpha_p : T_pM \longrightarrow \mathbb{R}$. The vector space of one-forms at p is denoted by T_p^*M. Cotangent bundle of M is the disjoint union of these sets, i.e.,*

$$T^*M = \bigsqcup_{p \in M} T_p^*M \tag{3.1.7}$$

A one-form on M is a map $\alpha : M \longrightarrow T^*M$ such that,

$$\alpha_p = \alpha(p) \in T_p^*M \quad \forall\, p \in M \tag{3.1.8}$$

Definition 3.1.8 *If $\varphi : \mathcal{B} \longrightarrow \mathcal{S}$ is a C^1 mapping and $\beta \in T^*\mathcal{S}$, then the one-form on $\mathcal{B}$ defined as,*

$$(\varphi^*\beta)_X \cdot \mathbf{V}_X = \beta_{\varphi(X)} \cdot (T\varphi \cdot \mathbf{V}_X) \quad \forall\, X \in \mathcal{B},\ \mathbf{V}_X \in T_X\mathcal{B} \tag{3.1.9}$$

is called the pull-back of β by φ. If φ is a C^1 diffeomorphism, the push-forward of a one-form α on $\mathcal{B}$ is defined by $\varphi_*\alpha = (\varphi^{-1})^*\alpha$.

Definition 3.1.9 *A type $\begin{pmatrix} p \\ q \end{pmatrix}$ tensor at $x \in \mathcal{B}$ is a multilinear map,*

$$T : \underbrace{T_X^*\mathcal{B} \times ... \times T_X^*\mathcal{B}}_{p\ copies} \times \underbrace{T_X\mathcal{B} \times ... \times T_X\mathcal{B}}_{q\ copies} \rightarrow \mathbb{R} \tag{3.1.10}$$

T is said to be contravariant of order p and covariant of order q. In a local coordinate chart,

$$T(\alpha^1, ..., \alpha^p, \mathbf{V}_1, ..., \mathbf{V}_q) = T^{\,i_1...i_p}{}_{j_1...j_q} \alpha^1_{i_1} ... \alpha^p_{i_p} V_1^{j_1} ... V_q^{j_q} \tag{3.1.11}$$

where, $\alpha^k \in T_X^*\mathcal{B}$ and $\mathbf{V}^k \in T_X\mathcal{B}$.

Definition 3.1.10 *Suppose $\varphi : \mathcal{B} \longrightarrow \mathcal{S}$ is a regular map and $\mathbf{T}$ is a tensor of type $\begin{pmatrix} p \\ q \end{pmatrix}$. Push-forward of $\mathbf{T}$ by φ is denoted $\varphi_*\mathbf{T}$ and is a $\begin{pmatrix} p \\ q \end{pmatrix}$-tensor on $\varphi(\mathcal{B})$ defined by,*

$$(\varphi_*\mathbf{T})(x)(\alpha^1, ..., \alpha^p, \mathbf{v}_1, ..., \mathbf{v}_q) = \mathbf{T}(X)(\varphi_*\alpha^1, ..., \varphi_*\alpha^p, \varphi_*\mathbf{v}_1, ..., \varphi_*\mathbf{v}_q) \tag{3.1.12}$$

where, $\alpha^k \in T_x^*\mathcal{S}, \mathbf{v}_k \in T_x\mathcal{S}, X = \varphi^{-1}(x), \varphi^*(\alpha^k) \cdot \mathbf{v}_l = \alpha^k \cdot (T\varphi \cdot \mathbf{v}_l)$ and $\varphi^*(\mathbf{v}_l) = T(\varphi^{-1})\mathbf{v}_l$. Similarly, pull-back of a tensor $\mathbf{t}$ defined on $\varphi(\mathcal{B})$ is given by $\varphi^*\mathbf{t} = (\varphi^{-1})_*\mathbf{t}$. In the setting of continuum mechanics push-forward and pull-back of tensors will have the following forms,

$$\begin{aligned}
(\varphi_*\mathbf{T})^{i_1...i_p}{}_{j_1...j_q}(x) &= F^{i_1}{}_{I_1}(X)...F^{i_p}{}_{I_p}(X)\ T^{I_1...I_p}{}_{J_1...J_q}\ (F^{-1})^{J_1}{}_{j_1}(x)...(F^{-1})^{J_q}{}_{j_q}(x) \\
(\varphi^*\mathbf{t})^{I_1...I_p}{}_{J_1...J_q}(X) &= (F^{-1})^{I_1}{}_{i_1}(x)...(F^{-1})^{I_p}{}_{i_p}(x)\ t^{i_1...i_p}{}_{j_1...j_q}\ F^{j_1}{}_{J_1}(X)...F^{j_q}{}_{J_q}(X)
\end{aligned}$$

Definition 3.1.11 *(Two-Point Tensor) A two-point tensor* $\mathbf{T}$ *of type* $\begin{pmatrix} q & q' \\ p & p' \end{pmatrix}$ *at* $x \in \mathcal{B}$ *over a map* $\varphi : \mathcal{B} \to \mathcal{S}$ *is a multilinear map,*

$$T : \underbrace{T_X^*\mathcal{B} \times \ldots \times T_X^*\mathcal{B}}_{p\ copies} \times \underbrace{T_X\mathcal{B} \times \ldots \times T_X\mathcal{B}}_{q\ copies}$$
$$\underbrace{T_x^*\mathcal{S} \times \ldots \times T_x^*\mathcal{S}}_{p\ copies} \times \underbrace{T_x\mathcal{S} \times \ldots \times T_x\mathcal{S}}_{q\ copies} \to \mathbb{R} \tag{3.1.13}$$

where $x = \varphi(X)$*. Deformation gradient is actually a two-point tensor,*

$$\mathbf{F}(X) : T_x^*\mathcal{S} \times T_X\mathcal{B} \to \mathbb{R}$$
$$(\alpha, \mathbf{V}) \mapsto \alpha(T_X\varphi \cdot \mathbf{V})$$

Definition 3.1.12 *Let* $\mathbf{w} : \mathcal{U} \to T\mathcal{S}$ *be a vector field, where* $\mathcal{U} \subset \mathcal{S}$ *is open. A curve* $\mathbf{c} : I \to \mathcal{S}$*, where* I *is an open interval, is an integral curve of* $\mathbf{w}$ *if*

$$\frac{d\mathbf{c}}{dt}(r) = \mathbf{w}(\mathbf{c}(r)) \quad \forall\, r \in I \tag{3.1.14}$$

If $\mathbf{w}$ *depends on time variable explicitly, i.e,* $\mathbf{w} : \mathcal{U} \times (-\varepsilon, \varepsilon) \to T\mathcal{S}$*, an integral curve is defined by,*

$$\frac{d\mathbf{c}}{dt} = \mathbf{w}(\mathbf{c}(t), t) \tag{3.1.15}$$

Definition 3.1.13 *Let* $\mathbf{w} : \mathcal{S} \times I \to T\mathcal{S}$ *be a vector field. The collection of maps* $F_{t,s}$ *such that for each* s *and* x*,* $t \mapsto F_{t,s}(x)$ *is an integral curve of* $\mathbf{w}$ *and* $F_{s,s}(x) = x$ *is called the flow of* $\mathbf{w}$*.*

Definition 3.1.14 *(Lie Derivative) Let* $\mathbf{w}$ *be a* C^1 *vector field on* $\mathcal{S}$ *and* $F_{t,s}$ *be its flow. Suppose* $\mathbf{t}$ *is a* C^1 *tensor field on* $\mathcal{S}$*. Lie derivative of* $\mathbf{t}$ *with respect to* $\mathbf{w}$ *is defined by,*

$$\mathbf{L}_{\mathbf{w}}\mathbf{t} = \frac{d}{dt}\left(F_{t,s}^*\mathbf{t}\right)\Big|_{t=s} \tag{3.1.16}$$

If we hold t *fixed in* $\mathbf{t}$ *then,*

$$\mathfrak{L}_{\mathbf{w}}\mathbf{t} = \frac{d}{dt}\left(F_{t,s}^*\mathbf{t}\right)\Big|_{t=s} \tag{3.1.17}$$

which is the autonomous Lie derivative. Hence,

$$\mathbf{L}_{\mathbf{w}}\mathbf{t} = \frac{\partial}{\partial t}\mathbf{t} + \mathfrak{L}_{\mathbf{w}}\mathbf{t} \tag{3.1.18}$$

It turns out that all objective stress rates are Lie derivatives.

Proposition 3.1.15 *If* $\mathbf{W}$ *is a vector field on* M*, then,*

$$\mathfrak{L}_{\mathbf{W}}\mathrm{d}V = (Div\ \mathbf{W})\mathrm{d}V \tag{3.1.19}$$

where $\mathrm{d}V$ *is a volume form on* M*.*

Proposition 3.1.16 *Let $\hat{\mathbf{N}}$ be the unit outward normal to $\partial\mathcal{P}(\mathcal{P} \subset \mathcal{B})$ and $\mathbf{V}$ a vector field on $\mathcal{P}$, then on $\partial\mathcal{P}$, $\mathbf{V} \cdot \hat{\mathbf{N}}dA = i_{\mathbf{V}}dV$, where dA is the area element on $\partial\mathcal{P}$ and $i_{\mathbf{V}}dV$ is contraction of the volume element by $\mathbf{V}$.*

Definition 3.1.17 *Let $\mathbf{v}$ be a vector field on $\mathcal{S}$ and $\varphi : \mathcal{B} \longrightarrow \mathcal{S}$ a regular and orientation preserving C^1 map. The Piola transform of $\mathbf{v}$ is,*

$$\mathbf{V} = J\varphi^*\mathbf{v} \tag{3.1.20}$$

where J is the Jacobian of φ.

Proposition 3.1.18 $\mathbf{W}$ *is the Piola transform of $\mathbf{w}$ if and only if $\varphi^*(i_{\mathbf{w}}dv) = i_{\mathbf{W}}dV$.*

Theorem 3.1.19 ***(Piola Identity)*** *If $\mathbf{Y}$ is the Piola transform of $\mathbf{y}$, then,*

$$Div\,\mathbf{Y} = J(div\,\mathbf{y}) \circ \varphi \tag{3.1.21}$$

Theorem 3.1.20 ***(Cartan's Magoc Formula)*** *Let $\alpha \in \Omega^k(M)$ and $\mathbf{v}$ be a vector field on M, then,*

$$\mathfrak{L}_{\mathbf{v}}\alpha = \mathbf{d}i_{\mathbf{v}}\alpha + i_{\mathbf{v}}\mathbf{d}\alpha \tag{3.1.22}$$

where $\mathbf{d}\alpha$ is the exterior derivative of α.

Definition 3.1.21 *A k-form on a manifold M is a skew-symmetric $\begin{pmatrix} 0 \\ k \end{pmatrix}$ tensor. The space of k-forms on M is denoted $\Omega^k(M)$.*

Theorem 3.1.22 ***(Change of Variables)*** *If $\varphi : M \longrightarrow N$ is a regular and orientation preserving C^1 map and $\alpha \in \Omega^k(\varphi(M))$, then,*

$$\int_M \varphi^*\alpha = \int_{\varphi(M)} \alpha \tag{3.1.23}$$

Theorem 3.1.23 ***(Stokes' Theorem)*** *Suppose $\alpha \in \Omega^{n-1}(M)$ and ∂M is positively oriented, then,*

$$\int_M \mathbf{d}\alpha = \int_{\partial M} \alpha \tag{3.1.24}$$

3.2 Geometric Continuum Mechanics

Definition 3.2.1 *A body is an open set $\mathcal{B} \in \mathbb{R}^3$ and a configuration of $\mathcal{B}$ is a mapping $\varphi : \mathcal{B} \longrightarrow \mathbb{R}^3$. The set of all configurations of $\mathcal{B}$ is denoted $\mathcal{C}$.*

Definition 3.2.2 *A motion is a curve in $\mathcal{C}$, i.e., a map,*

$$\begin{aligned} c : \mathbb{R} &\rightarrow \mathcal{C} \\ t &\mapsto \varphi_t \end{aligned}$$

For a fixed t, $\varphi_t(X) = \varphi(X,t)$ and for a fixed X, $\varphi_X(t) = \varphi(X,t)$, where X is position of material points in the undeformed configuration $\mathcal{B}$. Material velocity is the map $\mathbf{V}_t : \mathcal{B} \to \mathbb{R}^3$,

$$\mathbf{V}_t(X) = \mathbf{V}(X,t) = \frac{\partial \varphi(X,t)}{\partial t} = \frac{d}{dt}\varphi_X(t) \tag{3.2.1}$$

Similarly, material acceleration is defined by,

$$\mathbf{A}_t(X) = \mathbf{A}(X,t) = \frac{\partial \varphi(\mathbf{V},t)}{\partial t} = \frac{d}{dt}\mathbf{V}_X(t) \tag{3.2.2}$$

Here it is assumed that φ_t is invertible and regular. Spacial velocity of a regular motion φ_t is defined as,

$$\mathbf{v}_t : \varphi_t(\mathcal{B}) \to \mathbb{R}^3, \quad \mathbf{v}_t = \mathbf{V}_t \circ \varphi_t^{-1} \tag{3.2.3}$$

and spacial acceleration $\mathbf{a}_t$ is defined similarly.

Definition 3.2.3 *Let $\varphi : \mathcal{B} \to \mathcal{S}$ be a C^1 configuration of $\mathcal{B}$ in $\mathcal{S}$, where $\mathcal{B}$ and $\mathcal{S}$ are manifolds. Deformation gradient is defined to be $\mathbf{F} = T\varphi$, i.e., it is the tangent of φ. Thus,*

$$F(X) : T_X\mathcal{B} \to T_{\varphi(X)}\mathcal{S} \quad \forall X \in \mathcal{B} \tag{3.2.4}$$

If $\{x^i\}$ and $\{X^I\}$ are local coordinate charts on $\mathcal{S}$ and $\mathcal{B}$, respectively,

$$F^i{}_J(X) = \frac{\partial \varphi^i}{\partial X^J}(X) \tag{3.2.5}$$

Suppose $\mathcal{B}$ and $\mathcal{S}$ are Riemannian manifolds with inner products $\langle , \rangle_X$ and $\langle , \rangle_x$ based at $X \in \mathcal{B}$ and $x \in \mathcal{S}$, respectively. The transpose of deformation gradient is defined by,

$$\mathbf{F}^T : T_x\mathcal{S} \to T_X\mathcal{B}, \quad \langle \mathbf{F}\mathbf{V}, \mathbf{v} \rangle_x = \langle \mathbf{V}, \mathbf{F}^T\mathbf{v} \rangle_X \quad \forall \mathbf{V} \in T_X\mathcal{B},\ \mathbf{v} \in T_x\mathcal{S} \tag{3.2.6}$$

In components,

$$(F^T(X))^J{}_i = g_{ij}(x) F^j{}_K(X) G^{JK}(X) \tag{3.2.7}$$

where $\mathbf{g}$ and $\mathbf{G}$ are metric tensors on $\mathcal{S}$ and $\mathcal{B}$, respectively.

Definition 3.2.4 *The right Cauchy-Green deformation tensor is defined by,*

$$\mathbf{C}(X) : T_X\mathcal{B} \to T_X\mathcal{B}, \quad \mathbf{C}(X) = \mathbf{F}(X)^T\mathbf{F}(X) \tag{3.2.8}$$

In components,

$$C^I{}_J = (F^T)^I{}_k F^k{}_J \tag{3.2.9}$$

It can be shown that,

$$\mathbf{C}^\flat = \varphi^*(\mathbf{g}), \ i.e. \ \ C_{IJ} = (g_{ij} \circ \varphi) F^i{}_I F^j{}_J \tag{3.2.10}$$

Definition 3.2.5 ***(Conservation of mass)*** *Suppose $\mathcal{B} \subset \mathbb{R}^n$ is a body and $\varphi(X,t)$ is a motion of $\mathcal{B}$. Let $\rho(X,t)$ be the mass density per unit volume of $\varphi_t(\mathcal{B})$ at the point x. Then we say that ρ obeys conservation of mass if,*

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho(X,t)dv = 0 \quad \forall\, \mathcal{U} \subset \mathcal{B} \text{ with piecewise } C^1 \text{ boundary} \tag{3.2.11}$$

Localization of this gives us,

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{v}) = 0 \tag{3.2.12}$$

Definition 3.2.6 *The mass 3-form is defined by $\mathrm{m} = \rho dv$ in the deformed configuration and $\mathrm{m}_0 = \rho_0 dV$ in the undeformed configuration. Conservation of mass in terms of these 3-forms can be expressed as,*

$$\varphi^* \mathrm{m} = \mathrm{m}_0 \tag{3.2.13}$$

Theorem 3.2.7 ***(Transport Theorem)*** *Let $\varphi_t : \mathcal{B} \to \mathcal{S}$ be a regular motion of $\mathcal{B}$ in $\mathcal{S}$ and $\mathcal{P} \subset \mathcal{B}$ a k-dimensional submanifold. Then for any k-form α on $\mathcal{S}$,*

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{P})} \alpha = \int_{\varphi_t(\mathcal{P})} \mathbf{L}_{\mathbf{v}} \alpha \tag{3.2.14}$$

where $\mathbf{v}$ is the special velocity of the motion. In a special case when $\alpha = f dv$ and $\mathcal{P} = \mathcal{U}$ is an open set, then

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{P})} f dv = \int_{\varphi_t(\mathcal{P})} \left[\frac{\partial f}{\partial t} + div(f\mathbf{v})\right] dv \tag{3.2.15}$$

Definition 3.2.8 ***(Balance of Linear Momentum)*** *We say that a body $\mathcal{B}$ satisfies balance of linear momentum if for every nice open set $\mathcal{U} \subset \mathcal{B}$,*

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho \mathbf{v} dv = \int_{\varphi_t(\mathcal{U})} \rho \mathbf{b} dv + \int_{\partial\varphi_t(\mathcal{U})} \mathbf{t} da \tag{3.2.16}$$

where $\rho = \rho(x,t)$ is mass density, $\mathbf{b} = \mathbf{b}(x,t)$ is body force vector field and $\mathbf{t} = \mathbf{t}(x,\hat{\mathbf{n}},t)$ is the traction vector. Note that Cauchy's stress theorem tells us that there is a second-order tensor $\mathbf{T} = \mathbf{T}(x,t)$ (Cauchy stress tensor) such that $\mathbf{t} = \langle \mathbf{T}, \hat{\mathbf{n}} \rangle = \mathbf{T} \cdot \hat{\mathbf{n}}$. Equivalently, balance of linear momentum can be written in the undeformed configuration as,

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \mathbf{V} dV = \int_{\mathcal{U}} \rho_0 \mathbf{B} dV + \int_{\partial\mathcal{U}} \mathbf{P} \cdot \hat{\mathbf{N}} dA \tag{3.2.17}$$

where, $\mathbf{P} = J\varphi^* \mathbf{T}$ (the first Piola-Kirchhoff stress tensor) is the Piola transform of Cauchy stress tensor. Note that $\mathbf{P}$ is a two-point tensor. Note that this is the balance of linear momentum in the deformed (physical) space written in terms of some quantities that are defined with respect to the reference configuration.

Definition 3.2.9 *(Balance of Angular Momentum) A body $\mathcal{B}$ satisfies the balance of angular momentum if for every nice open set $\mathcal{U} \subset \mathcal{B}$,*

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho \mathbf{x} \times \mathbf{v} dv = \int_{\varphi_t(\mathcal{U})} \rho \mathbf{x} \times \mathbf{b} dv + \int_{\partial\varphi_t(\mathcal{U})} \mathbf{x} \times \langle \mathbf{T}, \hat{\mathbf{n}} \rangle da \tag{3.2.18}$$

Definition 3.2.10 *(Balance of Energy) For every nice open set $\mathcal{U} \subset \mathcal{B}$,*

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho\Big(e + \frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle\Big) dv = \int_{\varphi_t(\mathcal{U})} \rho\Big(\langle \mathbf{b}, \mathbf{v}\rangle + r\Big) dv + \int_{\partial\varphi_t(\mathcal{U})} \Big(\langle \mathbf{t}, \mathbf{v}\rangle + h\Big) da \tag{3.2.19}$$

where $e = e(x,t), r = r(x,t)$ and $h = h(x, \hat{\mathbf{n}}, t)$ are internal energy per unit mass, heat supply per unit mass and heat flux, respectively.

4 Variational and Hamiltonian structures of continuum mechanics

5 Configurational Forces from a Geometric Point of View

Suppose the reference configuration evolves in time. This evolution can be represented by a one-parameter family of mappings that map $\mathcal{B}$ (reference configuration at $t = 0$) to $\mathcal{B}_t$ (reference configuration at at time t),

$$\psi_t : \mathcal{B} \to \mathcal{B}_t \tag{5.0.20}$$

We call these maps the *configurational deformation maps.*[14] The configuration space for evolution of reference configuration is,

$$\tilde{\mathcal{C}} = \{\psi | \ \psi : \mathcal{B} \to \mathcal{B}_t\} \tag{5.0.21}$$

Evolution of the reference configuration is a curve $\tilde{c}$ in $\tilde{\mathcal{C}}$, i.e.,

$$\tilde{c} : I \to \tilde{\mathcal{C}} \tag{5.0.22}$$

Physical deformation is represented by a one-parameter family of mappings,

$$\chi_t : \mathcal{B}_t \to \mathcal{S}_t \tag{5.0.23}$$

The physical configuration space is defined by,

$$\mathcal{C} = \{\chi | \ \chi : \mathcal{B}_t \to \mathcal{S}_t\} \tag{5.0.24}$$

[14] I think the configurational deformation map need not be a map from $\mathcal{B}$ to $\mathcal{B}$, i.e., the evolved reference configuration could be a completely different manifold. Is this right?

Again, a physical deformation is a curve in the physical configuration space. The total deformation map is composition of physical and configurational deformation maps,

$$\varphi_t = \chi_t \circ \psi_t : \mathcal{B} \to \mathcal{S}_t \tag{5.0.25}$$

This idea is clearly shown in the following diagram,

$$\begin{array}{ccc} \mathcal{B}_t & \xrightarrow{\chi_t} & \mathcal{S}_t \\ {\scriptstyle \psi_t}\uparrow & \nearrow_{\varphi_t = \chi_t \circ \psi_t} & \\ \mathcal{B} & & \end{array}$$

Fig.1 shows the same idea schematically. In term of mapping the material points, $x = \chi_t(X_t) = \chi_t \circ \psi_t(X)$ as is shown in the following diagram.

$$\begin{array}{ccc} X_t & \xrightarrow{\chi_t} & x \\ {\scriptstyle \psi_t}\uparrow & \nearrow_{\varphi_t = \chi_t \circ \psi_t} & \\ X & & \end{array}$$

The configuration space for the total deformation is defined as,

$$\mathcal{C}^{tot} = \{\varphi | \ \varphi = \chi \circ \psi, \ \chi \in \mathcal{C}, \psi \in \tilde{\mathcal{C}}\} = \tilde{\mathcal{C}} \circ \mathcal{C} \tag{5.0.26}$$

A deformation is a curve in the total configuration space, i.e.,

$$c : I \to \mathcal{C}^{tot} \tag{5.0.27}$$

Note that $\psi_t = id$ (identity map) in classical continuum mechanics. As it is seen, there are two independent deformation mappings φ_t and ψ_t (see Fig.1). An example is shown in Fig. 2, in which a bar is deformed and in the process of deformation it undergoes a phase transformation. It is seen that the phase boundary moves independent of the physical deformation.

Definition 5.0.11 *(Configurational Velocity) Lagrangian and Eulerian configurational velocities are defined by,*

$$\tilde{\mathbf{V}}(X,t) = \frac{\partial \psi_t}{\partial t}, \quad \tilde{\mathbf{v}}(X_t,t) = \tilde{\mathbf{V}} \circ \psi_t \tag{5.0.28}$$

Definition 5.0.12 *Total material velocity is defined by,*

$$\mathbf{V}^{tot}(X,t) = \frac{\partial \chi_t(X_t)}{\partial t}\Big|_{X \text{ fixed}} = \frac{\partial \chi_t}{\partial t} + \mathbf{F}\tilde{\mathbf{V}} \tag{5.0.29}$$

where $\mathbf{F} = \frac{\partial \chi_t}{\partial X_t}$ *is the physical deformation gradient.*

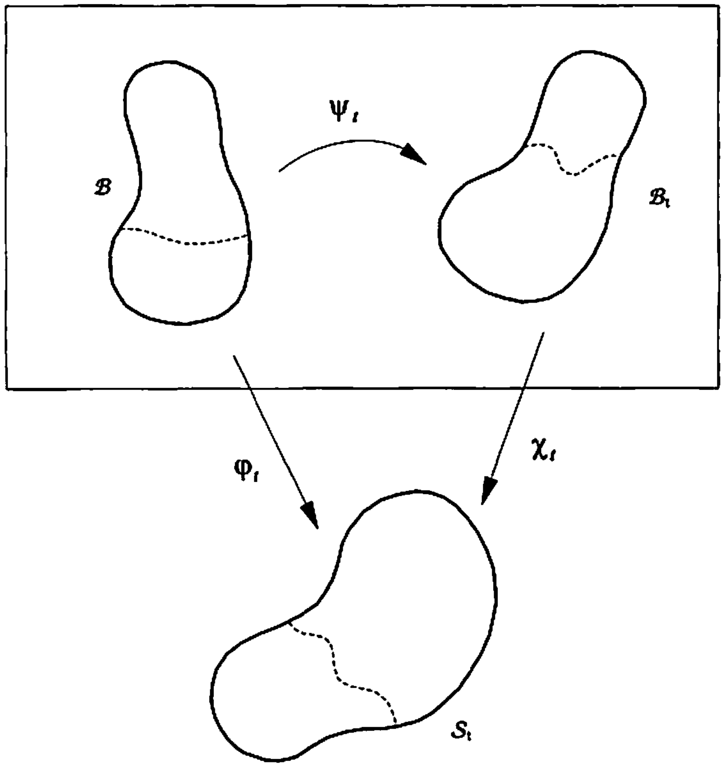

Figure 1: Configurational and physical deformation maps.

Note that,

$$\frac{\partial \varphi_t}{\partial X} = \frac{\partial \chi_t}{\partial X_t} \circ \frac{\partial \psi_t}{\partial X} \quad \Rightarrow \quad \mathbf{F}^{\mathrm{tot}} = \mathbf{F} \circ \tilde{\mathbf{F}} \tag{5.0.30}$$

Thus,

$$\tilde{\mathbf{F}} = \mathbf{F}^{-1} \circ \mathbf{F}^{\mathrm{tot}} \tag{5.0.31}$$

Now we postulate the conservation of configurational mass and balance of linear and angular configurational momenta.

- Question: Is this the right way of looking at evolution of reference configuration?
- Perhaps balance of configurational forces can be understood in terms of balance laws for configurational and physical deformation maps or a combination of them?
- Is the configurational deformation map ψ_t always invertible? Perhaps not always. An example would be the propagation of a crack.

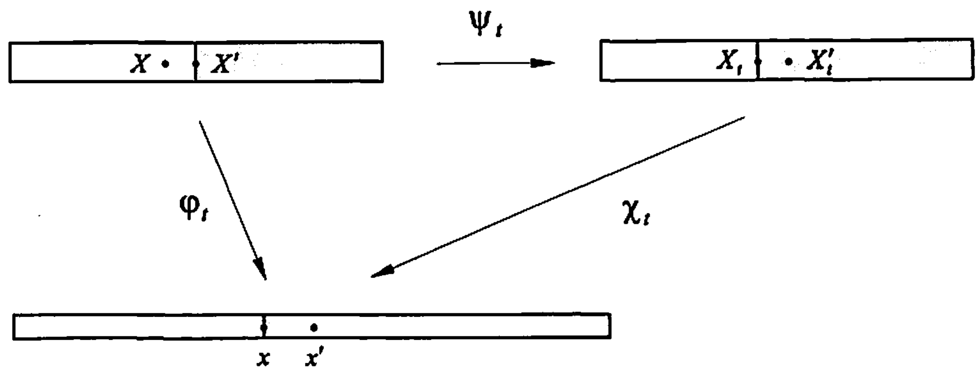

Figure 2: A bar that undergoes both physical and configurational (phase transformation) deformations.

6 Configurational Forces in Space-Time Continuum Mechanics

7 Configurational Forces in Multisymplectic Continuum Mechanics

8 Conclusions

A comprehensive review of the literature of configurational forces has been done in this report. We tried to see the similarities and differences between the different methods of studying configurational forces. It seems that a geometric study is lacking and this report is just the beginning of the project. It is hoped that having a geometric theory of configurational forces will make the continuum theory of driving forces clearer and some new things might be found in the process of developing the theory. The following is a tentative list of what should be done.

- Studying momentum maps corresponding to symmetries in the reference configuration.
- Studying configurational forces in spacetime continuum mechanics. Michael Ortiz has done some work on this to obtain the evolution equations of dislocations. His point of view is similar to that of Eshelby. After having a geometric theory of configurational forces generalizing it for spacetime should not be very hard.
- Studying configurational forces in multisymplectic continuum mechanics. I will read the relevant papers soon.

- Studying configurational force using the variational and Hamiltonian structure of continuum mechanics.

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