A geometric analysis of Foucault's pendulum

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Abstract

We provide a geometric analysis of Foucault's pendulum. Such an analysis differs from conventional ones in that it explicitly demonstrates the reason for the pendulum's motion: the geometric constraint of moving the pendulum on the Earth. Such an analysis frees one from the potentially confusing notion of a "fictitious" force and is the foundation for solving similar problems with geometric constraints.

A geometric analysis of Foucault's pendulum

Foucault's pendulum is perhaps one of the most elegant dynamical systems in Hamiltonian mechanics. It combines the richness of the harmonic oscillator, an undamped simple pendulum, with that of a seemingly innocent constraint, adibiatic motion (with respect to the pendulum frequency) around the earth $\approx S^2$ at a fixed latitude. This setup is shown in figure 1, and it is something virtually anyone who has ever been to a natural science museum has seen. The interesting thing about this system is that as the pendulum swings, after one day (one Earth revolution) its plane of swing rotates by an amount $2\pi\sin(\theta)$, where θ is the latitude of swing.

The reason for the rotation, or as we shall henceforth call it, the phase shift of the pendulum, is explained in virtually any mechanics textbook[1]. Briefly, the usual procedure is to introduce a body frame **x**, **y**, and **z** (for example, with respect to an observer watching the pendulum in the museum). By assuming the pendulum is "long", so that we may ignore vertical **z** motion in the body frame and by ignoring centripetal force on the pendulum we can model the pendulum as a two-dimensional harmonic oscillator and obtain the equations of motion

$$\begin{aligned} \ddot{x} &= -\omega^2 x + 2\dot{y}\Omega_z \\ \ddot{y} &= -\omega^2 y - 2\dot{x}\Omega_z \end{aligned} \tag{1}$$

where $\Omega_z = |\Omega|\sin(\theta)$, and Ω is the angular velocity of the earth. We let $z = x + iy$ and reduce (1) to the equivalent complex ordinary differential equation

$$\ddot{z} + i2\Omega_z\dot{z} + \omega^2 z = 0. \tag{2}$$

We can solve this equation in the standard way, and upon using the adibiatic assumption $\Omega_z << \omega$ simplify the solution to obtain

$$z(t) = e^{-i\Omega_z t}(Ae^{i\omega t} + Be^{-i\omega t}),$$

or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(\Omega_z t) & \sin(\Omega_z t) \\ -\sin(\Omega_z t) & \cos(\Omega_z t) \end{pmatrix} \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix}, \tag{3}$$

[1] For example, see Arnold, pp. 132–33 or Landau and Lifshitz, pp. 129–130.

where $x_0(t)$ and $y_0(t)$ denote the body coordinates of our long pendulum if the Earth's rotation is ignored.

Upon analyzing (3), one discovers that indeed, after one day ($t = t_0 = \frac{2\pi}{|\Omega|}$), $x(t)$ and $y(t)$ have rotated from "where they should be" in the body frame by an amount $\Omega_z t_0 = 2\pi \sin(\theta)$, as claimed.

Many reputable mechanics texts usually insist on explaining this counterintuitive result by modeling the phase shift as a result of a "fictional" coriolis force due to the constant angular velocity of the Earth and hence of the rotating body frame.[2] The reason it is called a fictional force, as opposed to a "real" force, is because it is the force that must be applied to the pendulum in the body frame to turn the body frame into an inertial one. Viewed relative to an inertial frame (for example, the Sun, ignoring the Earth's orbit about the Sun), this force "does not exist." For an analogy, suppose you are at the North Pole and you throw a ball. Relative to you in the body frame, the ball behaves as if it is being acted on by a force equal to the coriolis force. This force is "fictional" in the sense that if we viewed this experiment from the sun, the ball would not appear to experience this force.

But there is a serious problem with this way of thinking in our pendulum example; for after one day, the earth returns to its original position in space and hence, relative to an *inertial frame* like the sun the pendulum has phase shifted as well. Hence a supposedly "fictional" force has accounted for a very real action in an inertial frame. Introductory mechanics is full of this confusion between rotating observers (throwing a ball off the North pole) and rotating systems (our Foucault pendulum example).

In this paper, we shall bypass this confusion by examining the phase shift of the pendulum in purely geometric terms. Briefly, we shall show the phase shift occurs due to the curvature of the earth, where the pendulum is constrained to lie. We shall first give a simple geometric explanation of this phase shift and then, after a brief explanation of parallel translation, we rigorously derive this intuitive geometric result.

We provide an intuitive explanation by first putting a cone "hat" on the earth, tangent to the earth at the latitude θ the pendulum swings at as in figure 2. We make this cone by first starting with a flat circular disk with

[2]See, for example, Arnolds book, pp. 130–31. The theorem there is not wrong of course, but it lends no insight as to the difference between rotating observers and rotating systems. The geometric approach explicitly underlines this difference.

center O and radius

$$R = E\cot(\theta),$$

where E denotes the radius of the Earth, and $\theta \neq 0$, and $\theta \neq \pi/2$, so that we are neither at the equator nor the North pole. Draw a bunch of parallel lines on the disk as in figure 3. Mark a point A on the edge of the disk, and measure around the disk a distance of

$$2\pi E\cos(\theta),$$

which is the distance around the Earth at its latitude θ, and call this point B. Since we are at neither the equator nor the North pole, $A \neq B$. We then remove the wedge from B to A from the disk, identify OA with OB, and we have made a conical hat for the Earth that fits at it's latitude θ, as desired.

The key to understanding the phase shift is to notice how the tangents to the parallel lines we have drawn on the cone at its base vary as we move around the base. Suppose we start the pendulum swinging at $B + \epsilon$, a point just to the right of OB as in figure 4, and suppose its plane of oscillation coincides with the direction of the line we have drawn there. We claim that after almost one period of rotation, the pendulum will of course return to a point $B - \epsilon$, but its plane of oscillation will still correspond to the direction of the line at $B - \epsilon$. A quick glance at figure 4 should convince the reader that the plane will not be where it began one day ago, but will in fact have phase shifted. In fact, by examining figure 3, we can calculate this phase shift as

$$\frac{2\pi E\cos(\theta)}{R} = 2\pi\sin(\theta), \tag{4}$$

which happily corresponds to the result we know to be true. Now if $\theta = 0$ (equator), this cone degenerates into a cylinder, so that the parallel lines experience no phase shift at all; similarly, if we are at the North pole, our cone degenerates into the flat disk of figure 3, where we see the phase shift is 2π, all as expected from (4).

This geometrical argument is intuitively appealing, but as yet we have proven nothing. We are suggesting that somehow the geometrical constraint of the pendulum swinging on a fixed latitude in S^2 is somehow responsible for the observed phase shift. Instead of calling the coriolis force that arises in the standard analysis a "fictitious force," we should instead think of it as a byproduct of this constraint, observable both in the inertial and body frames.

In order to prove that this simple geometric picture is the correct one and not some unfortunate fluke, we first need to understand (if only in an elementary way) the concept of parallel displacement of vectors on manifolds. We shall restrict our attention to the manifold $S^2 \approx$ Earth, but most of what we say can be extended to higher-dimensional Riemannian manifolds.

To begin our discussion of parallel displacement, embed S^2 in $\mathbb{R}^3$ and ask the following simple question: How can we move a vector $X \in T_xS^2$ along a curve $P_{xy} \subset S^2$ to a vector $Y \in T_yS^2$ so that, in some sense, X is parallel to Y? If we were doing this in on a flat manifold like $\mathbb{R}^2$ the answer is obvious: we just move X along P_{xy} parallel to itself with respect to a global coordinate system we introduce on $\mathbb{R}^2$, as in figure 5. For S^2, we simply exploit its locally flat nature and move X along P_{xy} parallel to itself with respect to a *local* coordinate system. This procedure is referred to as the *parallel displacement* of X along P_{xy}.

It is clear from figure 5 that in a globally flat manifold like $\mathbb{R}^2$, parallel displacement does not depend on the path P_{xy} taken. It might be tempting to conclude that the path is irrelevant in any manifold. In fact, however, a simple example will show that P_{xy} is crucial. Consider for example, two points x and y on the Earth's equator, and a vector X in T_xS^2 pointing North as in figure 6. Consider parallel translation of X along two different paths: the first path P_1 goes from x to y around the equator; the second path P_2 goes up the geodesic through x to the North Pole, and then comes down the geodesic through y. Parallel translation along geodesics is easy, because for all $x \in S^2$, there is a local coordinate system at x that preserves geodesics; by this we mean that locally, images of geodesics through x on S^2 map to geodesics (straight lines) through the origin in the coordinate system. As a result, parallel translation of X along a geodesic can be locally performed by maintaining the angle between X and the geodesic. [3]

Thus, parallel translation of X along the equator (P_1) results in a vector Y_{P_1} pointing North at y. Along P_2, we first parallel translate X through the geodesic to the North pole, which results in a vector pointing West at the North pole, and then translate this vector down through the geodesic to y, which gives us a vector Y_{P_2} pointing West, not North, as shown in figure 6.

[3] In fact, Arnold uses this as a definition of parallel displacement on geodesics of S^2, and then defines parallel displacement along any curve in S^2 as a limiting procedure, where the curve is approximated by piecewise geodesic arcs (Arnold, pp. 301–302).

Hence parallel translation along two different paths has given two different results.

In the Foucault pendulum, we are particularly interested in parallel translating a vector X around the latitude θ. We prove that such a translation results in a phase shift of the vector by $2\pi\sin(\theta)$. An easy way to see this is recalling that about each point on the latitude in S^2, there is a local coordinate system that preserves geodesics. Such a coordinate system can be made by putting a cone hat tangent to the Earth as we did earlier and as is shown in figures 2–4. The cone has no curvature, hence geodesics in the plane (straight lines) in figure 3 are the geodesics on the cone in figure 4. As a result, we parallel translate a vector around a latitude on the Earth by simply parallel translating it with respect to the geodesic lines we have drawn on the cone, which in turn is equivalent to parallel translating it with respect to the straight lines of the unrolled cone in figure 3, which in turn gives us the proper phase shift $2\pi\sin(\theta)$.

Another way to show this is by using results from metric geometry[4]. This method is important because it introduces techniques and terminology we will use later. To begin, introduce curvilinear coordinates $x^i = (\phi_1, \phi_2)$ on S^2 as shown in figure 9 with a metric given by

$$ds^2 = g_{uv}dx^u dx^v,$$

where we shall frequently employ the summation convention, in which repeated indices (in this case, u and v) are summed over. Since $ds^2 = dx^2 + dy^2 + dz^2$, in the embedding space $\mathbb{R}^3$ we can find g_{uv} by expressing x, y, and z in terms of our curvilinear coordinate system obtaining

$$g_{uv} = \begin{pmatrix} \cos^2(\phi_2) & 0 \\ 0 & 1 \end{pmatrix}. \qquad (5)$$

The inverse of this matrix will be denoted

$$g^{vu} = \begin{pmatrix} 1/\cos^2(\phi_2) & 0 \\ 0 & 1 \end{pmatrix}. \qquad (6)$$

Note the coordinate system singularity at $\phi_2 = \pi/2$ (the North pole), but this presents no problem if $\theta \neq \pi/2$. Now it is a result from metric geometry[5] that

[4]See, for example, Dirac, pp. 9–14.
[5]Dirac, p. 14

parallel translation of a vector X^i along a (local) path dx^i of a Riemannian manifold M is given by

$$dX^i = -\Gamma^i_{uv} X^u dx^v, \tag{7}$$

(sum on u and v) where the Γ^i_{uv} are called the *Christoffel symbols* or the *connection* on M and are given by

$$\Gamma^i_{uv} = \frac{1}{2}\left(\frac{\partial g_{u\lambda}}{\partial x^v} + \frac{\partial g_{v\lambda}}{\partial x^u} - \frac{\partial g_{uv}}{\partial x^\lambda}\right) g^{\lambda i}. \tag{8}$$

The Christoffel symbols may be thought of as telling us how to perform parallel translation on M. In our case, $M = S^2$, so that there are eight symbols, and by using (5) and (6), they are

$$\begin{aligned} \Gamma^2_{11} &= \cos(\phi_2)\sin(\phi_2) \\ \Gamma^1_{12} = \Gamma^1_{21} &= -\tan(\phi_2), \end{aligned}$$

with all others being 0. We expand the terms in (7) and get

$$\begin{aligned} dX^1 &= -\Gamma^1_{12} X^1 dx^2 - \Gamma^1_{21} X^2 dx^1 \\ &= \tan(\phi_2)(X^1 d\phi^2 + X^2 d\phi^1) \\ dX^2 &= -\Gamma^2_{11} X^1 dx^1 \\ &= -\cos(\phi_2)\sin(\phi_2) X^1 d\phi^1. \end{aligned}$$

Along a latitude, $\phi_2 = \theta$ and $d\phi^2 = 0$, so the motion of X^i as we parallel translate it around a latitude θ satisfies the differential equation

$$\begin{aligned} \begin{pmatrix} \dot{X}^1 \\ \dot{X}^2 \end{pmatrix} &= \begin{pmatrix} 0 & \tan(\theta) \\ -\cos(\theta)\sin(\theta) & 0 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \\ &= \mathbf{A} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \end{aligned}$$

where $\cdot$ denotes differentiation with respect to ϕ_1. Now (9) has the solution

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = e^{\mathbf{A}\phi_1} \begin{pmatrix} X^1_0 \\ X^2_0 \end{pmatrix},$$

which yields

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} \cos(\sin(\theta)\phi_1) & \frac{1}{\cos(\theta)}\sin(\sin(\theta)\phi_1) \\ -\cos(\theta)\sin(\sin(\theta)\phi_1) & \cos(\sin(\theta)\phi_1) \end{pmatrix} \begin{pmatrix} X^1_0 \\ X^2_0 \end{pmatrix},$$

where $(X_0^1, X_0^2)^T$ denotes the initial condition. Finally, by the change of variables $Y^1 = \cos(\theta)X^1$ and $Y^2 = X^2$, we obtain

$$\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} \cos(\sin(\theta)\phi_1) & \sin(\sin(\theta)\phi_1) \\ -\sin(\sin(\theta)\phi_1) & \cos(\sin(\theta)\phi_1) \end{pmatrix} \begin{pmatrix} Y_0^1 \\ Y_0^2 \end{pmatrix}. \tag{10}$$

By examining (10), it is obvious that after parallel translating a vector $Y = (Y^1, Y^2)^T$ about a latitude θ by an angle ϕ_1, the vector has rotated an amount $\phi_1 \sin(\theta)$. In particular, after one complete rotation of the Earth ($\phi_1 = 2\pi$) the vector has rotated an amount $2\pi \sin(\theta)$, as claimed. Moreover, if $d\phi_1/dt$ is constant (as is true with the Earth's rotation), the angular velocity of the plane's procession is constant as well.

Our final job is to show that the plane of oscillation of Foucault's pendulum parallel translates around S^2. This has been done for a general path on an arbitrary two-dimensional surface by Hart, Miller and Mills[6]. To begin, by assuming the pendulum is long, we may approximate it by a two-dimensional harmonic oscillator

$$\ddot{x}_0^i = -\omega^2(x_0^i - y_0^i(t)), \tag{11}$$

where x_0^i are local Cartesian coordinates on the surface, and $y_0^i(t)$ denotes the trajectory of the base point in these local coordinates. In terms of curvilinear coordinates $x = (x^1, x^2) = (\phi_1, \phi_2)$, (11) becomes

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}(x(t))\dot{x}^\beta \dot{x}^\gamma = -\omega^2(x^\alpha - y^\alpha(t)), \tag{12}$$

where $\Gamma^\alpha_{\beta\gamma}(\cdot)$ is the familiar connection on our surface as in (8), evaluated at a point $x(t)$ on the surface. By using the adiabatic assumption, we may assume $x^\alpha = x_0^\alpha + \epsilon^\alpha$, where ϵ^α is a small (fast) oscillation about x_0^α, and approximate (12) by the linear equation

$$\ddot{\epsilon}^\alpha + 2\Gamma^\alpha_{\beta\gamma}(y(t))\dot{y}^\beta \dot{\epsilon}^\gamma + \omega^2 \epsilon^\alpha = 0. \tag{13}$$

At this point, it is instructive to compare (13) with (2). The former explicitly contains information on the geometry of the constraint via the $\Gamma^\alpha_{\beta\gamma}$'s, while the latter contains this information rather less explicitly via $\Omega_z = |\Omega| \sin(\theta)$. We consider the geometric information less explicit in (2) because it is a function of Ω, a non-geometric quantity.

[6]See Hart, p. 69

Finally, to solve (13), let $\epsilon^\alpha(t) = \zeta^\alpha(t)e^{-i\omega t}$, where (once again using the adibiatic assumption) $\zeta^\alpha(t)$ varies slowly with respect to $e^{-i\omega t}$. As a result, we may interpret $\zeta(t)$ as defining the plane of oscillation of the pendulum. Now (13) becomes

$$\ddot{\zeta}^\alpha - 2i\omega\dot{\zeta}^\alpha + 2\Gamma^\alpha_{\beta\gamma}(y(t))\dot{y}^\beta(\dot{\zeta}^\gamma - i\omega\zeta^\gamma) = 0.$$

Finally, because of the adibiatic assumption, $\ddot{\zeta}^\alpha$ and $\dot{\zeta}^\gamma$ are negligible compared to terms containing ω and we have

$$\dot{\zeta}^\alpha = -\Gamma^\alpha_{\beta\gamma}(y(t))\dot{y}^\beta\zeta^\gamma,$$

or, for a small time interval and dropping the explicit dependence of the $\Gamma^\alpha_{\beta\gamma}$ on $y(t)$,

$$d\zeta^\alpha = -\Gamma^\alpha_{\beta\gamma}dy^\beta\zeta^\gamma, \tag{14}$$

which is the result for parallel translation have from (7), since the Christoffel symbols are symmetric in their lower two indices. This means that the plane of oscillation of the pendulum, described by the vector ζ^α undergoes parallel translation as the Earth rotates, as hoped.

Performing the analysis geometrically in this fashion has shed some light onto exactly what is causing the phase shift in Foucault's pendulum. Notice that although the adibiatic assumption allows one to approximate the motion by (14), it is clear that the actual phase shift of the pendulum is due entirely to the (non adibiatic) procedure of parallel translating a vector on a curved geometric surface. It does not matter how fast we parallel translate a vector around the Earth; the result is the same. The only difference is without the adiabatic assumption, we can no longer claim the dynamics of the Foucault's pendulum is modeled by (14). The role of the geometric constraint in the phase shift of Foucault's pendulum is not obvious in the classical approach to solving this problem which results in equations (2) and (3). In fact, the Foucault's pendulum is only one of a large number of problems whose dynamics is significantly affected by similar geometric constraints.

References

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[2] Landau, L. E. and Lifshitz, E. M. *Mechanics,* third edition. Pergamon Press, New York, 1976.

[3] Hart, J. B., Miller, R. E., and Mills, R. L. "A simple geometric model for visualizing the motion of a Foucault pendulum." *Am. J. Physics*, 55 (1), pp. 67–70, January 1987.

[4] Dirac, P. A. M. *General Theory of Relativity*, John Wiley and Sons, Inc., New York, 1975.

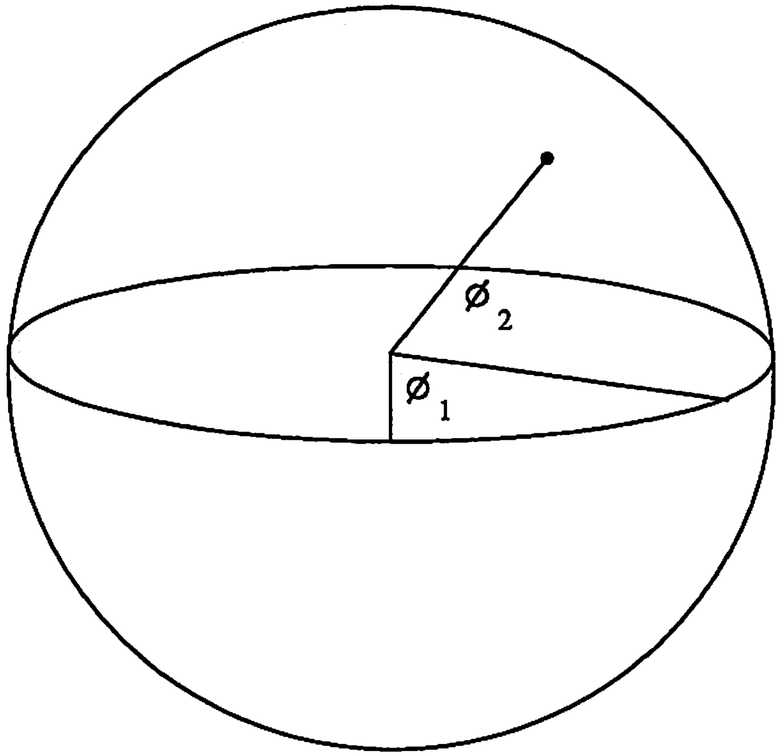

Figure 7. Coordinates on the Sphere

Figures

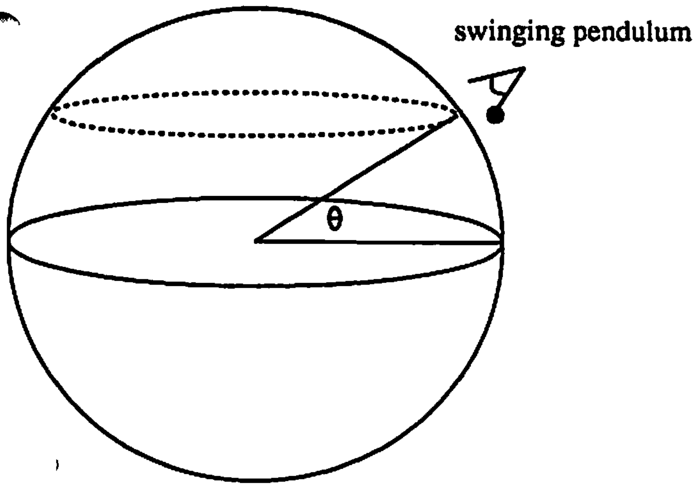

Figure 1. Foucault's pendulum

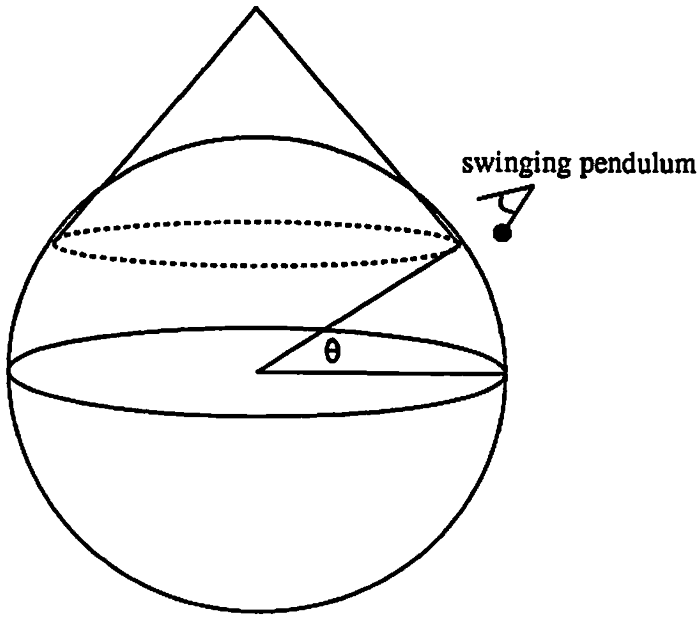

Figure 2. Putting a cone hat on the Earth

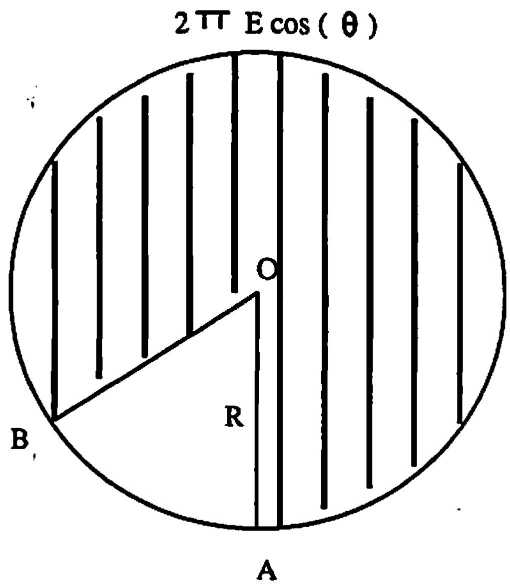

Figure 3. Making the hat

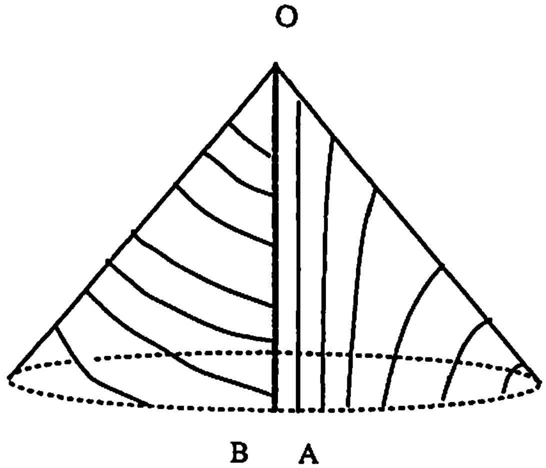

Figure 4. Identifying OA and OB.

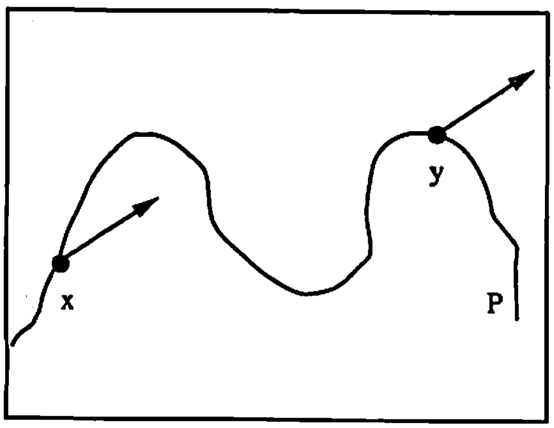

Figure 5. Parallel translation in the plane

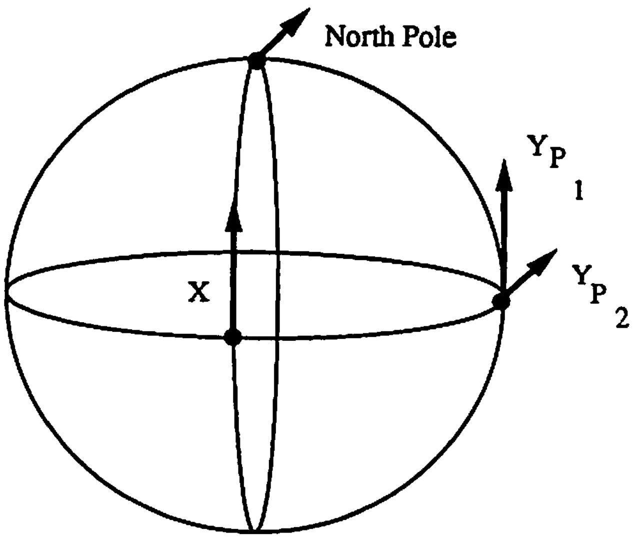

Figure 6. Parallel translation depends on path