Yong - nice project. You made a good use of Routh's method. I have another exposition of this problem by O'Reill that I can show you.

A Particle in a Rotating Hoop

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Abstract

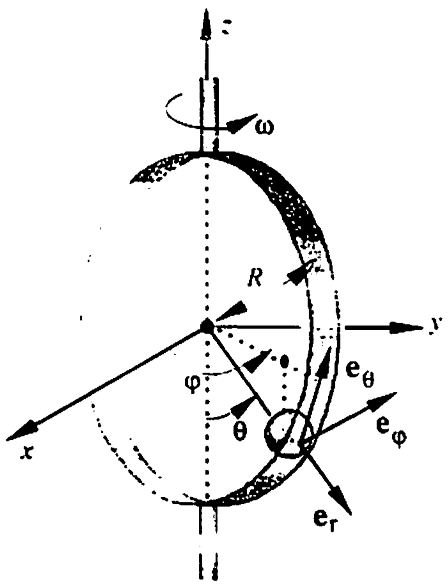
In this course project, I try to motivate the dynamics of the rotating string by studying an example in detail - the dynamics of a particle in a freely rotating hoop. It has been shown by reduction that there are super-critical bifurcations for the relative equilibrium $\theta = 0$ when the angular momentum reaches to certain values. The dynamics of this example is very similar to the example in Section 2.10 in the textbook.

1 Introduction

The study of the dynamics of the rotating string can go back to D. Bernoulli and L. Euler. They derived the solutions for the linearized equations of the whirling chain. As long as I know, the nonlinear effects on the rotating string has not been studied until 1955 by I. Kolodner. He studied the motion of a heavy string with one end free. He showed that the string can rotate at any frequency when $\omega > \omega_1$(ω_1 is frequency of the first mode) and there are exactly n distinct modes of rotation for each $\omega_n < \omega < \omega_{n+1}$. Caughey studied the same problem except that the boundary conditions for the upper end are circular periodic oscillations in the (x, y) plane. He showed that if $\omega > \omega_1$(ω is the frequency of the upper end), then for each mode, there are three solutions: tow of them are stable and the third is unstable. Caughey also consider the case when the upper end is fixed and the lower end the lower end is allowed to slide freely along the z-axis. He got the same result as those for the free rotating case by Kolodner. He further showed that the modes are orbitally stable. He also consider the same problem except that the string is elastic. He derived the exact solutions for the elastic string and shoed that the solutions are orbitally stable. Healey analyzed the stability of a family of axial motion solutions for homogeneous, nonlinealy elastic strings. He found that the total circulation,i.e., the integral of the tangent component of the velocity over the length of the string, is conserved for any sufficiently smooth motion.

2 An example for motivation - a particle on a rotating hoop

Consider the problem of a particle moves freely in a rotating hoop, while the hoop is not forced rotating as in Section 2.10 of the textbook. First we suppose that there are no frictions anywhere. We set the coordinate system as in the following figure which is taken from page 77 in the textbook.



Let M be the mass of the hoop, m be the mass of the particle. R is the radius of the hoop, $\omega = \dot{\phi}$ is the angular velocity of the hoop and $I = \frac{1}{2}MR^2$ is the inertia of the hoop rotating around z-axis. The kinetic energy of the hoop is

got this from elementary mechanics book?

$$K_h = \frac{1}{2}I\omega^2$$

The kinetic energy of the particle is

$$K_p = \frac{1}{2}m(R\dot{\theta})^2 + \frac{1}{2}m(R\sin\theta \cdot \omega)^2$$

The total kinetic energy is

$$K = K_h + K_p = \frac{1}{4}MR^2\dot{\phi}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\phi}^2\sin^2\theta$$

The total potential energy of the system is

$$V = -mgR\cos\theta$$

Hence, the Lagrangian of the system is

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = T - V = \frac{1}{2}mR^2\left(\frac{M}{2m} + \sin^2\theta\right)\dot{\phi}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta$$

Substitute the Lagrangian into the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

We then get the Euler-Lagrangian equations for the system

$$\ddot{\theta} + \frac{g}{R}\sin\theta - \sin\theta\cos\theta\dot{\phi}^2 = 0$$

$$\left(\frac{M}{2m} + \sin^2\theta\right)\ddot{\phi} + 2\dot{\phi}\dot{\theta}\sin\theta\cos\theta = 0$$

Now we derive the Hamilton's equations for the system. Let

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2(\frac{M}{2m} + \sin^2\theta)\dot{\phi}$$

Then

$$\begin{aligned}
\dot{\theta} &= \frac{p_\theta}{mR^2} \\
\dot{\phi} &= \frac{p_\phi}{mR^2(\frac{M}{2m} + \sin^2\theta)}
\end{aligned}$$

Then using the Legendre Transform, we get the Hamiltonian of the system

$$\begin{aligned}
H(\theta, \phi, p_\theta, p_\phi) &= (p_\theta\dot{\theta} + p_{phi}\dot{\phi} - L(\theta, \phi, \dot{\theta}, \dot{\phi}))|_{(\theta,\phi,\dot{\theta},\dot{\phi})\mapsto(\theta,\phi,p_\theta,p_\phi)} \\
&= \frac{{p_\phi}^2}{2mR^2}(\frac{M}{2m} + \sin^2\theta) + \frac{{p_\theta}^2}{2mR^2} - mgR\cos\theta
\end{aligned}$$

ence, the Hamilton's equations for the system are

$$\begin{aligned}
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \\
\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2(\frac{M}{2m} + \sin^2\theta)} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{\sin\theta\cos\theta}{mR^2(\frac{M}{2m} + \sin^2\theta)} - mgR\sin\theta \\
\dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = 0
\end{aligned}$$

Now we make reductions to the system. We have the following famous Routh's theorem:

Theorem 2.1 *Suppose the Lagrangian of a system can be written as*

$$L = L(q_1, \ldots, q_m, \dot{q}_1, \ldots, \dot{q}_m, \dot{q}_{m+1}, \ldots, \dot{q}_n)$$

Let

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = const = \beta_\alpha, \alpha = m + 1, \ldots, n.$$

Define the Routhian

$$R = -(\sum_{r=m+1}^{n} p_r\dot{q}_r - L)|_{(\dot{q}_{m+1},\ldots,\dot{q}_n)\mapsto(p_{m+1},\ldots,p_n)}$$
$$= R(q_1,\ldots,q_m,\dot{q}_1,\ldots,\dot{q}_m,\beta_{m+1},\ldots,\beta_n)$$

Then we have the reduced-order Euler-Lagrange equations, called the Routh's equations

$$\frac{d}{dt}(\frac{\partial R}{\partial \dot{q}_i}) - \frac{\partial R}{\partial q_i} = 0, i = 1,\ldots,m$$

Now we make reduction on the Euler-Lagrange equations of the example using the Routh's theorem. In the example, the Lagrangian looks like

$$L = L(\theta,\phi,\dot{\theta},\dot{\phi}) = L(\theta,\dot{\theta},\dot{\phi})$$

so, from the Hamilton's equations

$$p_\phi = const := G$$

i.e.

$$mR^2(\frac{M}{2m} + \sin^2\theta)\dot{\phi} = G$$

Then, by Routh's theorem

$$R(\theta,\dot{\theta};G) = (L(\theta,\dot{\theta},\dot{\phi}) - p_\phi\dot{\phi})|_{p_\phi=G,\dot{\phi}=\frac{p_\phi}{mR^2(\frac{M}{2m}+\sin^2\theta)}}$$
$$= -\frac{\frac{G^2}{2mR^2}}{\frac{M}{2m} + \sin^2\theta} + \frac{1}{2}mR^2\dot{\theta}^2 + mR\cos\theta$$

Substitute the Routhian into the Routh's equation

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = 0$$

we get the reduced-order orderEulerEuler-Lagrange equatios:

$$\ddot{\theta} - \frac{(\frac{G}{mR^2})^2}{(\frac{M}{2m} + \sin^2\theta)^2}\sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0$$

Then we get

1. If $G \leq \frac{MR^2}{2}\sqrt{\frac{g}{R}}$, then the equilibria are

$$\theta = 0, \pm\pi;$$

2. If $G > \frac{MR^2}{2}\sqrt{\frac{g}{R}}$, then the equilibria are

$$\theta = 0, \pm\pi, \pm\theta_e (0 < |\theta_e| < \frac{\pi}{2}).$$

Now we begin to study the stability of the equilibria. We first introduce the Lagrange-Dirichlet Theorem for Hamiltonian systems.

Theorem 2.2 *If the $2n \times 2n$ matrix $\delta^2 H$ of second partial derivatives evaluated at (q_e, p_e) is positive- or negative-difinite, then (q_e, p_e) is stable.*

A powerful,yet direct conclusion of the Lagrange-Dirichlet Theorem is the following theorem.

Theorem 2.3 *For a classical mechanical system, $(q_e, 0)$ is a stable equilibrium, provided the matrix $\delta^2 V(q_e)$ of second order partial derivatives of the potential V at q_e is positive-definite(or, more generally,q_e is a strictly local minimum for V).If $\delta^2 V$ at q_e has a negative-definite direction, then q_e is an unstable equilibrium.*

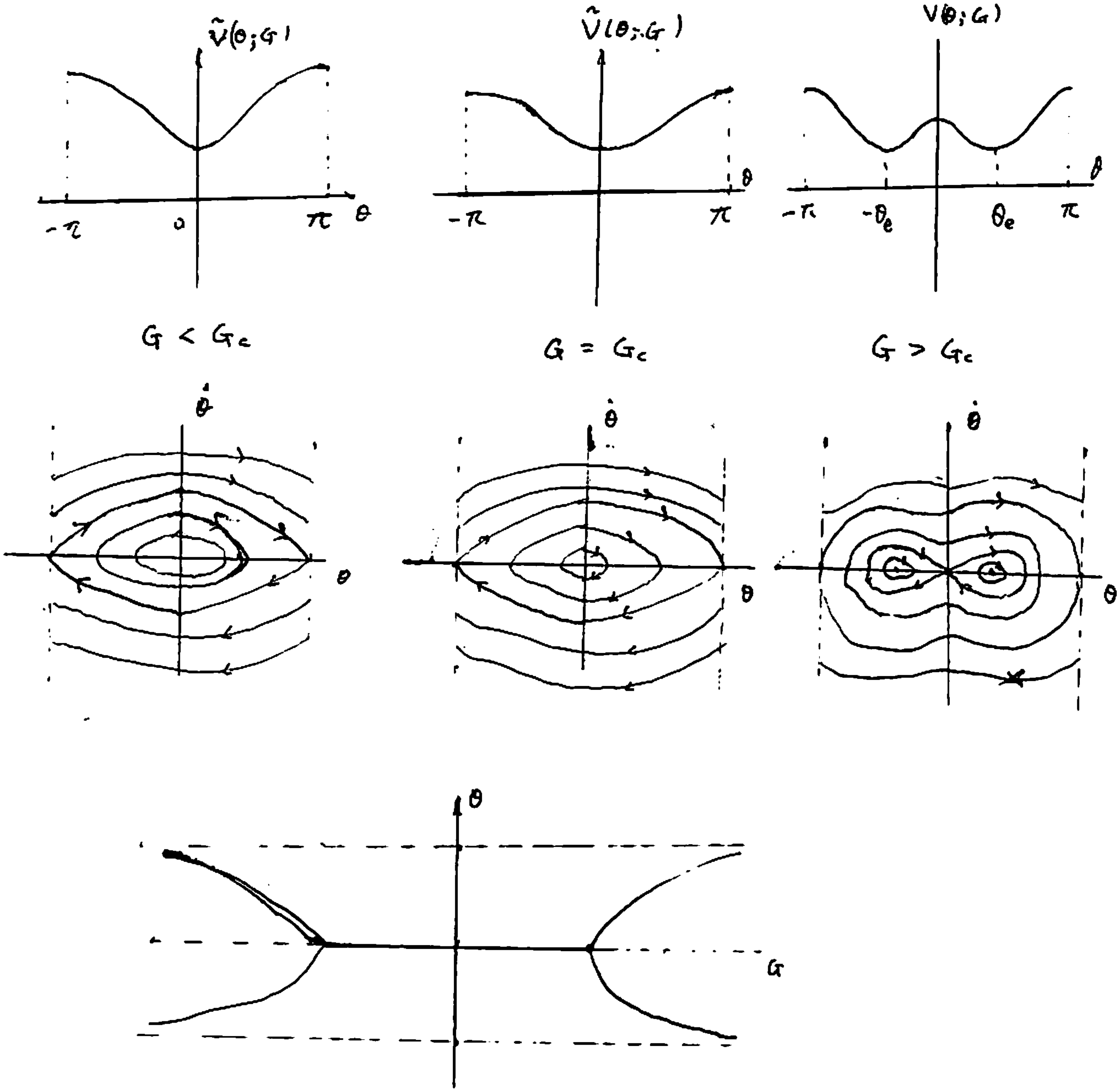
Now we apply the Lagrange-Dirichlet Theorem to the example be computing the second derivatives of $\tilde{V}$ at the equilibria. We have

$$\begin{aligned} \frac{\partial^2 \tilde{V}}{\partial \theta^2}(0, G) &= \frac{g}{R} - (\frac{2G}{MR^2})^2 \\ \frac{\partial^2 \tilde{V}}{\partial \theta^2}(\pm\pi, G) &= -(\frac{g}{R} + (\frac{2G}{MR^2})^2) \end{aligned}$$

If $G > \frac{Mr62}{2}\sqrt{\frac{g}{R}} := G_c$, then

$$\frac{\partial^2 \tilde{V}}{\partial \theta^2}(\pm\theta_e) > 0.$$

Therefore, we conclude that if $G \leq G_c$, then there are only two equilibria: $\theta = 0$ is a center, $\theta = \pi$ is a saddle; If $G > G_c$, there are four equilibria: $\theta = 0$ is a saddle, $\theta = \pi$ is a saddle, $\theta = \pm\theta_c$ are centers. So supercritical pichfork bifurcations for the equilibrium $\theta = 0$ occur when $G = G_c$.



Bifurcation Diagrams (supercritical pitchfork bifurcatio

Hence, If we compare this problem with the example in Section 2.10 of the textbook, in which the hoop is forced to rotate around z-axis with constant speed ω, we find that these two cases are very similar. Both of them have supercritical pichfork bifurcations when G or ω passes certain values.

Dr. Marsden,

I also got some preliminary results on the rotating string:

1. I came up with the Lagrangian and Hamiltonian for the example in Healey's paper. From these I derived the Euler-Lagrange equations and also the Hamilton's equations for the example.

2. I found that there are symmetries between the x-direction and y-direction in the whirling of a heavy chain and in the rotating string in which we do not consider the greavity. But I did not find either the Lagrangian or Hamiltonian for a general rotating string.

Because I'll go back to China today for about a month I did not have enough time to write these things down into the project. Next term I'll still select this course and I will continue to do this project.

Best wishes.

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