1990
(?)

(Very interesting)

Professor Marsden:

This is a report on my project. I choose to write it in an informal letter format to avoid unnecessary boredom to you while reading this. I should actually say that this is a "progress" report. I have become very interested in my subject: the Berry-Hannay phase. When I was doing the research for my project I came across a great deal of quantum mechanical applications of the phase. However, I noted a lack of much work with the classical applications. Granted, my search was not exhaustive, but still, there seem to be many more quantum applications than classical ones. This seems odd to me because Jan Segert (1987) argues that Berry's phase should be "viewed as a classical rather than quantum effect". Even then, it seems only Hannay has done much work with the classical manifestations of the phase. There seems to be a wealth of topics to be investigated.

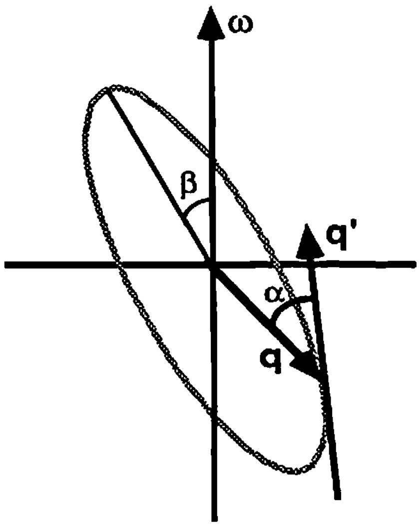
My original intent was to calculate a Berry's phase for particles circling a particle accelerator on the surface of the earth. I work at LBL with a group that does a great deal of work on particle accelerators. It seems no one has thought of this application. I believe that the phase would be observable after a day (one rotation of the earth). Storage rings could provide an unusual verification of the Berry-Hannay phase! Unfortunately, the calculation was too difficult for me to perform. I understand that the relativistic manifestation of the phase is do to "a lack of synchrony of clocks in the rotating frame." Apparently a similar calculation for ring gyroscopes has been done by Forder (1984). I was unable to find this paper. Another complication arose when I could not find simple derivations of any classical applications. In light of all this I had to set my goals to a much more modest level.

In the back of this letter is a hand written appendix of the calculation you gave in your notes (sec. 1.6). I have gone through and corrected the typographical errors in your draft. I noticed that there was one seemingly unnecessary assumption in the planer rigid hoop example. The calculation can be carried out without assuming the hoop is rotated in its plane. Below I go through the corrections necessary to prove this generalized case.

I begin with the equation marked with a (*) in the appendix.

$$\ddot{s} - (\vec{\omega} \times \vec{q}) \cdot (\vec{\omega} \times \vec{q'}) + \vec{q'} \cdot (\dot{\vec{\omega}} \times \vec{q}) = 0$$

Now I let the loop lie at an angle ß with respect to the axis of rotation. The diagram below illustrates the geometry.



[Diagram of the geometry]

The cross product terms may be simplified. First,

$$(\vec{\omega} \times \vec{q}) \bullet (\vec{\omega} \times \vec{q'}) = -\omega^2 (\vec{q} \bullet \vec{q'}) + (\vec{q} \bullet \vec{q'}) (\vec{\omega} \bullet \vec{q})$$

In the old example of rotation within the plane,

$$(\vec{\omega} \bullet \vec{q}) = 0$$

With the hoop at an angle to the plane of rotation,

$$(\vec{\omega} \bullet \vec{q}) = \omega\, q \cos \beta$$

Thus,

$$(\vec{\omega} \times \vec{q}) \bullet (\vec{\omega} \times \vec{q'}) = (\vec{q} \bullet \vec{q'}) \left[\omega\, q \cos \beta - \omega^2\right]$$

The next cross product term can also be rewritten,

$$\vec{q'} \bullet (\dot{\vec{\omega}} \times \vec{q}) = \dot{\vec{\omega}} \bullet (\vec{q} \times \vec{q'}) = \dot{\omega} q (\sin \alpha) (\cos\ \gamma)$$

where

$$\cos \gamma = \dot{\hat{\omega}} \bullet (\hat{q} \times \widehat{q'})$$

I am now ready to re-insert d^2s/dt^2 back into the Taylor expansion (marked with a (#) in the appendix). Again, applying the averaging law, I find,

$$s(T) = s_0 + \dot{s}_0 T + \int_0^T dt\ (T-t) \left[\omega^2 \left(\frac{1}{L} \int_0^L ds\ (\vec{q} \cdot \vec{q}') \right) - \omega\ (\cos\gamma) \left(\frac{1}{L} \int_0^L ds\ \ q(\vec{q} \cdot \vec{q}') \right) + \dot{\omega}\ (\cos \beta) \left(\frac{1}{L} \int_0^L ds\ (q \sin\alpha) \right) \right]$$

This complicated mess reduces down quite a bit. As before,

$$\int_0^L ds\ (\vec{q} \cdot \vec{q}') = 0$$

Also,

$$\int_0^L ds\ \ q(\vec{q} \cdot \vec{q}') = 0$$

where integration from 0 to L is once around to hoop. This leaves a single term,

$$\int_0^L ds\ (q \sin\alpha) = 2A$$

where A is the area enclosed by the loop just as in the planer example. Putting this all together yields,

$$s(T) = s_0 + \dot{s}_0 T + \int_0^T dt\ \left[(T-t)\ \ \dot{\omega}\ (\cos \beta)\ 2A \right]$$

Integration by part allows me to write this in a form similar to the planer example. In fact,

$$s(T) = s_0 + \dot{s}_0 T + \frac{2A \cos \beta}{L}\ \omega_0 T - \frac{4\pi A \cos \beta}{L}$$

In other words, the formula is modified from that of the planer example by taking the "projected area" of the loop in the plane of rotation. This can be rewritten in terms of the difference in angle, $\Delta\theta$, rather than arc length. For $\omega_0 = 0$, or after averaging over the velocity terms,

$$\Delta\theta = - \frac{8\pi A \cos \beta}{L^2}$$

The special case of the circle rotated at an angle ß gives $\Delta\theta = -2\pi$ (cos ß). For the tilted rotation, even the circular hoop yields a phase difference. Of course, this is just the bead "slipping" by the hoop. But, since in general $\Delta\theta \neq -2\pi$ then the phase difference would be detectable without having to count the number of loops the bead made around the hoop.

Apparently, if the hoop is rotated at 90° then there is no phase change. This is what would be expected since there would be not Euler force contribution.

This calculation could be generalized to rotations which were not in a single plane. But, there is a great deal to be explored even with the above calculation. Hannay (1984) gives a derivation which is more compact. Yet, he does not apply it to rotation outside the plane. His calculation is coordinate independent and is given by,

$$\mathbf{d}\,\langle d\theta\rangle = \frac{1}{2}\frac{d^2}{dI^2}\int_0^{2\pi} d\theta' \int_0^{2\pi} d\chi\,\chi\,[\mathbf{d}I(\theta' - \chi) \wedge \mathbf{d}I(\theta')]$$

where $\mathbf{d}$ is the exterior derivative (in parameter space) and I is the adiabatically invariant action of the particle.

This analysis can be extended to a simple well known example, the spinning top. A spinning top on the surface of the Earth is much like the superposition of two Focoult pendulums. The solution is that the extra angle turned by the top is just the angle caused by the parallel transport around the Earth's surface. On a sphere the curvature (two) form is just $\Omega=1/r^2 dS$. So, the angle change is just equal to the solid angle subtended. But, this problem is identical to the non-relativistic version of the particle in the particle accelerator! In other words, the phase difference will be dependent on the longitude of the accelerator. Relativistic correction need to be considered, but the effect should be similar.

I plan to continue with a relativistic calculation and then I will try to make some predictions about the "phases" for various machines. With the advent of particle accelerators in space (BEAR is a linear accelerator proposed for space basing) this effect may be important.

Thank you,

Gil Tr

Gil Travish

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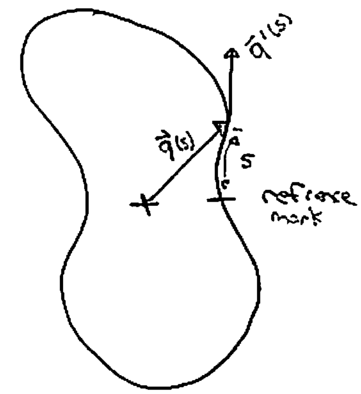
Appendix

Berry – Hannay Phase

Appendix

A loop being rotated in its plane

Notation



1) s is the arc length along a loop
   $0 \le s \le L$: L is the loop perimeter
2) $\vec{q}(s)$ parameterizes the loop. $\vec{q}'(s)$ is the unit tangent.
3) $\Theta(t)$ represents the angle the hoop has been rotated, in its plane, at time t.
4) $R_{\Theta(t)}$ represents the rotation through $\Theta(t)$
   So, $R_{\Theta(t)}\,\vec{q}(s(t))$ is the position of the refrence mark relative to the rotated, but inertial, co-ordinates.

- A bead is sliding without friction along a hoop.
- The shape of the hoop is described by $\vec{q}(s)$ where s is the arc length along the hoop measured w.r.t. some refrence mark.
- The hoop is rotated in its plane through an angle $\Theta(t)$ with angular velocity $\omega(t) = \dot{\Theta}(t)$; t is time.
  That is, $\omega(0) \neq 0$, $\dot{\omega}(t) \neq 0$
- No assumptions are made about $\omega(t)$ in general.

We want to calculate the Lagrangian of the bead:
The velocity of the particle (bead) is given by,

$$\vec{V} = \frac{d}{dt} R_{\theta(t)} \bar{q}(s(t))$$
$$= R_{\theta(t)} \frac{d\bar{q}}{ds}\frac{ds}{dt} + \left(\frac{d}{dt} R_{\theta(t)}\right) \bar{q}(t)$$
$$= R_{\theta(t)} \vec{q}\,'(s(t))\, \dot{s}(t) + R_{\theta(t)} (\bar{\omega} \times \bar{q})$$

where $\bar{\omega} = \dot{\theta}\hat{k}$ and $\hat{k}$ is the normal unit vector to the loop plane.

The Lagrangian is simply the kinetic energy of the particle,

$$L(s,\dot{s},t) = \frac{1}{2} m \|V\|^2$$
$$= \frac{1}{2} m \| \bar{q}' \dot{s} + \bar{\omega} \times \bar{q} \|^2$$

where I have suppressed the dependencies and I was able to cancle $R_{\theta(t)}$ because it is orthonormal.

The Euler-Lagrange formula states,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \frac{\partial L}{\partial s}$$

Calculating,

$$\frac{\partial L}{\partial \dot{s}} = m \| \vec{q}\,'\dot{s} + \bar{\omega} \times \bar{q} \| \, \bar{q}' \xrightarrow{\text{since } \|\bar{q}'\|^2 = 1} = m(\dot{s} + \bar{q}' \cdot (\bar{\omega} \times \bar{q}))$$

$$\frac{\partial L}{\partial s} = m \| \vec{q}\,'\dot{s} + \bar{\omega} \times \bar{q} \| (\vec{q}\,''\dot{s} + \bar{\omega} \times \bar{q}')$$

Thus,

$$\frac{d}{dt}\, m(\dot{s} + \vec{q}\,'\cdot(\vec{\omega}\times\vec{q})) = m\,\|\vec{q}\,'\dot{s} + \vec{\omega}\times\vec{q}\,\|\,(\vec{q}\,''\dot{s} + \vec{\omega}\times\vec{q}\,')$$

Continuing,

$$\ddot{s} + \vec{q}\,''\cdot(\vec{\omega}\times\vec{q})\dot{s} + \vec{q}\,'\cdot(\dot{\vec{\omega}}\times\vec{q}) + \overbrace{\vec{q}\,'\cdot(\vec{\omega}\times\vec{q}\,')\dot{s}}^{=0}$$

$$= q''\cdot(\vec{\omega}\times\vec{q})\dot{s} + \underbrace{\vec{q}\,'\cdot(\vec{\omega}\times\vec{q}\,')\dot{s}}_{=0} + (\vec{\omega}\times\vec{q}\,')\cdot(\vec{\omega}\times\vec{q})$$

Collecting terms:

$$\ddot{s} + \cancel{\vec{q}\,''\cdot(\vec{\omega}\times\vec{q})\dot{s}} + \vec{q}\,'\cdot(\dot{\vec{\omega}}\times\vec{q}) = \cancel{\vec{q}\,''\cdot(\vec{\omega}\times\vec{q})\dot{s}} + (\vec{\omega}\times\vec{q})\cdot(\vec{\omega}\times\vec{q}\,')$$

Two terms cancel:

Hence

NOTE $\vec{q}\cdot\vec{q}\,' \neq 0$
$\vec{q}\,'\cdot\vec{q}\,'' = 0$

(*) — $\ddot{S} - (\vec{\omega}\times\vec{q})\cdot(\vec{\omega}\times\vec{q}\,') + \vec{q}\,'\cdot(\dot{\vec{\omega}}\times\vec{q}) = 0$

$\alpha = \alpha(S(t))$

Or,

$$\ddot{s} = \omega^2\,\vec{q}\cdot\vec{q}\,' - \dot{\omega}\, q \sin\alpha$$

NOTE:
$(\vec{\omega}\times\vec{q})\times(\vec{\omega}\times\vec{q}\,')$
$= -\omega^2\,\vec{q}\cdot\vec{q}\,' + (\vec{q}\cdot\vec{q}\,')(\vec{\omega}\cdot\vec{q})$
$= -\omega^2\,\vec{q}\cdot\vec{q}\,' + 0$
$\vec{\omega}\cdot\vec{q} = 0 \longrightarrow$ rotation in plane

Taylor expansion with remainder yields,

(#) — $s(t) = s_0 + \dot{s}_0 t + \int_0^t (t-t')\,\ddot{s}(t')\,dt'$

Now we want to know the arc length for one complete rotation. One might expect $s(T) = L$. This is not so.

In order to calculate $S(T)$, we need to average over the entire loop. We employ averaging theory:

Say $g(t)$ is a rapidly varying function } over $[a,b]$
$f(t)$ is a slowly varying function }

Then over one period of g, $[\alpha,\beta]$

$$\int_\alpha^\beta f(t)\, g(t)\,dt \approx \int_\alpha^\beta f(t)\,\bar{g}\,dt$$

NOTE: For continuous functions, the time average = space average.

where $$\bar{g} = \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\,dt$$

So, $$S(T) = S_0 + \dot{S}_0 T + \int_0^T (T-t')\left[\omega^2 \vec{q}\cdot\vec{q}' - \dot{\omega} q \sin\alpha\right] dt'$$

← t', $s(t')$ dependencies suppressed →

It is assumed that the hoop is rotated adiabatically.

$$S(T) = S_0 + \dot{S}_0 T + \int_0^T dt' (T-t')\left[\omega^2\left(\frac{1}{L}\int_0^L ds\, \vec{q}\cdot\vec{q}'\right) - \dot{\omega}\left(\frac{1}{L}\int_0^L ds\, q \sin\alpha\right)\right]$$

$$\int_0^L ds(\vec{q}\cdot\vec{q}') = 0 \quad ; \quad \int_0^L ds\, q(s)\sin\alpha(s) = 2A, \quad A = \text{area enclosed by loop}$$

Hence,

$$S(T) = S_0 + \dot{S}_0 T - \frac{2A}{L}\int_0^T dt' (T-t')\,\dot{\omega}$$

$$\int_0^T dt' (T-t') \dot{\omega} = [(T-t')\omega(t')]\Big|_0^T + \int_0^T \omega(t')dt'$$

$u = T - t'$ $\quad dv = \dot{\omega} dt'$

$du = -dt'$ $\quad v = \omega$

$$= -T\omega(0) + \theta(t)\Big|_0^T$$

$$= -T\omega(0) + 2\pi$$

Finally,

$$s(T) = s_0 + \dot{s}_0 T + \frac{2A}{L} T\omega(0) - \frac{4\pi A}{L}$$

Berry takes $\omega(0) = 0$: the hoop is assumed to begin at rest. Then,

$$\Delta s = s(T) - s_0 = -\frac{4\pi A}{L}$$

A similair result can be obtained for the average over the initial conditions (without assuming $\omega(0) = 0$).

Similarly,

$$\Delta\theta = \frac{2\pi}{L}\Delta s = -\frac{8\pi A}{L^2}$$

For a circle,

$$\Delta\theta = -2\pi$$

as one would expect, the hoop slips one turn past the bead.