Phase Portrait Analysis for Physical Systems

CDS205
Amber Thweatt
6/2/99

Introduction:

When I began to read through the paper on Heteroclinic Connections between Periodic Orbits and Resonance Transitions in Celestial Mechanics by Koon, Lo, Marsden, and Ross, I realized that I didn't have a good understanding of heteroclinic orbits. Not having had CDS 140, I haven't covered this material or the material of phase planes and fixed points. I decided that a good starting place would be to work some simple examples to help increase my understanding. However, this ended up blossoming into an entire project.

Background:

To determine all possible trajectories of a physical system, one often uses a phase portrait. A phase portrait plots the rate an object is moving on the y-axis versus its position on the x-axis. By plotting the trajectories in this way, certain characteristics can be observed. First, one will determine the fixed points. Fixed points are points where the flow is fixed or stagnant. If the system equation is of the form $v = f(x)$, where $v = \partial x/\partial t$, then the fixed points are the points where $v = 0$. In other words, with respect to the original differential equation, fixed points are equilibrium points. There are two types of fixed points: stable and unstable. A stable fixed point is one in which the flow is moving towards the point, i.e. attractors or sinks. An unstable fixed point is one in which the flow is moving away from the point, i.e. repellors or sources. Trajectories flowing towards a saddle are considered stable trajectories, while trajectories moving away from saddle points are unstable. It is important to note that stability in this case only refers to local stability about that fixed point, not global stability.

Once the fixed points have been determined, the trajectories joining the fixed points can be analyzed. Heteroclinic trajectories are paths that join two saddle points. Heteroclinic trajectories are more common in reversible systems than other systems. Trajectories that start and end on the same fixed point are called homoclinic orbits. Closed orbits (not necessarily through fixed points) correspond to periodic solutions to the differential equation.

Simple Pendulum:

To determine the equations of motion for the simple pendulum, first calculate the Lagrangian. L = T - V

$$T = \frac{1}{2} m\dot{\Theta}^2 \qquad V = m\frac{g}{l}\cos\Theta$$

$$L = \frac{1}{2} m\dot{\Theta}^2 - m\frac{g}{l}\cos\Theta$$

$$\frac{\partial L}{\partial \dot{\Theta}} = m\dot{\Theta} \qquad \frac{\partial L}{\partial \Theta} = m\frac{g}{l}\sin\Theta$$

$$\frac{d}{dt}(m\dot{\Theta}) + m\frac{g}{l}\sin\Theta = 0$$

$$m\ddot{\Theta} + m\frac{g}{l}\sin\Theta = 0$$

$$\ddot{\Theta} = -\frac{g}{l}\sin\Theta$$

Now we make this system into a linear system by setting

$x = \dot{\Theta}$

$\dot{x} = \ddot{\Theta} = -\frac{g}{l}\sin\Theta$

Now, let's look at the fixed points for this system. The fixed points are the points such that $\dot{x} = -\sin\Theta = 0$. We can see that this is true when $\Theta = k\pi$. Therefore the fixed points are located at $(k\pi,0)$. To determine what the phase portrait looks like at each fixed point, we determine the Jacobian and evaluate it at that point. For the system above, the Jacobian looks like:

$$\begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \Theta} \\ \frac{\partial \dot{\Theta}}{\partial x} & \frac{\partial \dot{\Theta}}{\partial \Theta} \end{bmatrix} = \begin{bmatrix} 0 & -\cos\Theta \\ 1 & 0 \end{bmatrix}$$

Looking at specific fixed points, the phase portrait can be determined. At (0,0) the Jacobian takes on the form

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues of the Jacobian are $\lambda_1 = +i$ and $\lambda_2 = -i$ The Determinant of the matrix equals one. Because the determinant is greater than zero, the fixed point at this point is either a center or a spiral. Look at another variable τ, where $\tau = \lambda_1 + \lambda_2$, to determine whether the point is a spiral or a center. If τ is less than zero, this implies that both the eigenvalues have negative real parts, so the point is a stable spiral, i.e. spiraling into the fixed point. If τ is greater than zero, this implies that the eigenvalues have positive real parts, so the point is an unstable spiral, i.e. spiraling away from the fixed point. If $\tau = 0$, then the point is a center. In the case of (0,0), the two eigenvalues add up to zero, so this point must be a stable center.

A similar analysis can be performed on the point $(0,\pi)$. This point has a Jacobian of the following form:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In this case, the determinant is less than zero. This immediately implies that the fixed point is a saddle. To determine which branches of the flow are moving in what directions, simply look at the eigenvalues and eigenvectors for the system:

$$\lambda_1 = -1,\ v_1 = (1,-1) \qquad \lambda_2 = 1,\ v_2 = (1,1)$$

By using the program xphased, it is easy to develop a phase portrait for this system. This phase portrait is shown below in Figure 1.

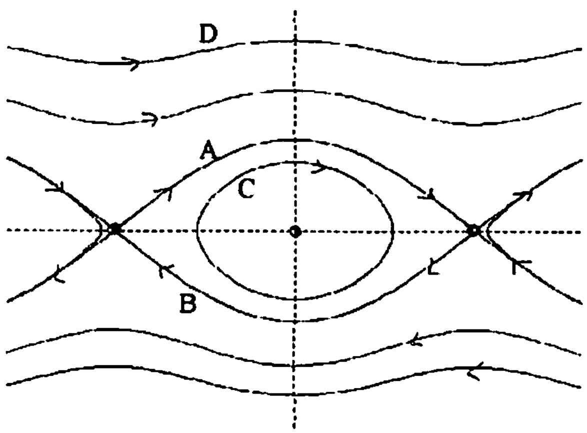

Figure 1: Phase Portrait for the Simple Pendulum

Looking at Figure 1, the two curves labeled A & B are seen to be heteroclinic orbits because they connect two saddle points. The point (0,0) is the fixed point representing the pendulum hanging straight down and at rest. This case also represents the case with the lowest energy state. The small closed curves around (0,0), like C, correspond to the trajectories of the simple pendulum oscillating back and forth through small angles. The saddle points represent the equilibrium points of the inverted pendulum at rest. The heteroclinic orbits are the orbits where the pendulum has the critical energy to cause it to come to rest in the vertical position. The remaining trajectories, an example one is labeled

D, are the trajectories where the pendulum is moving so rapidly that it doesn't slow to a rest in the inverted position; it simply rotates around and around. These trajectories have a higher energy state than the heteroclinic orbits.

As suggested in Strogatz [1994], if we were to project the phase portrait of the simple pendulum onto a 3-dimensional cylinder, the heteroclinic orbits shown above that are symmetric about the y-axis would wrap around the surface of the cylinder so that the two fixed points would meet and become the same point. This makes the heteroclinic orbits above become homoclinic orbits.

The Damped Simple Pendulum

The damped pendulum equation of motion is similar to the simple pendulum with one extra term: $\ddot{\Theta} + \dot{\Theta} + \sin\Theta = 0$

To linearize this system let,

$$x = \dot{\Theta}$$
$$\dot{x} = -x - \sin\Theta$$

The Jacobian for the system is then:

$$\begin{bmatrix} -1 & -\cos\Theta \\ 1 & 0 \end{bmatrix}$$

The fixed points for this system are the same as for the system above: (0,kπ). The Jacobian evaluated at (0,0) looks like:

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

The determinant of this matrix is 1 which is greater than zero and the eigenvalues are complex conjugate pairs, so the fixed point is either a spiral or a center. Evaluate $\tau = \lambda_1 + \lambda_2$, where

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

τ then takes on the value of -1 which is less than zero. This implies there is a stable spiral into the origin. A stable spiral makes sense because damping will cause the pendulum to loose energy and eventually come to rest at an equilibrium point. By doing a similar analysis for the points (0, $-\pi$) and (0, π) it can be seen that these points remain saddle points as seen before in the undamped case. The computer generated phase plane is shown below in Figure 2.

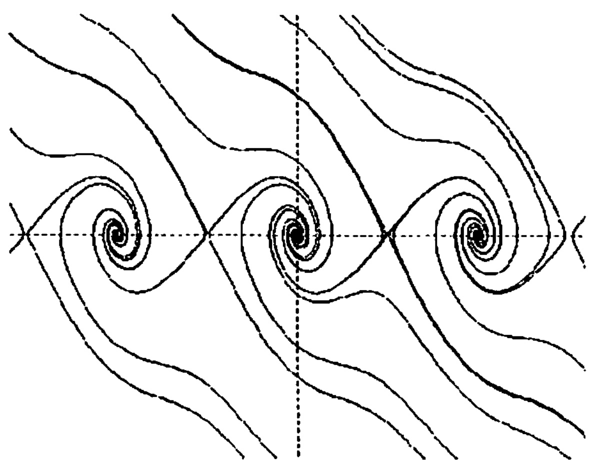

Figure 2: Phase Portrait for the Simple Damped Pendulum

Duffing Oscillator

It is interesting to compare the pendulum phase plane to that of the duffing oscillator. The duffing oscillator has the equation: $\ddot{x} + x + \rho x^3 = 0$ where ρ can be + or -. If we take the case where ρ is negative, then the phase plane looks very similar to that of the simple pendulum. Figure 3 shows the phase portrait for the duffing oscillator.

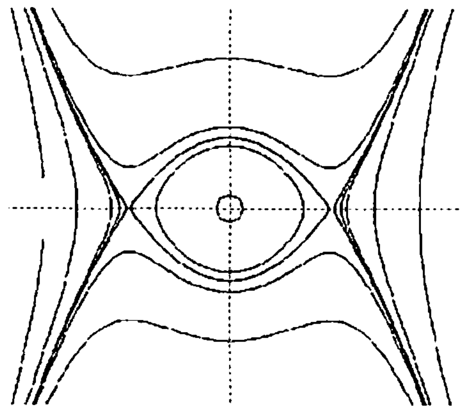

Figure 3: Phase Portrait for the Duffing Oscillator

As can be seen from the figure above, the section about the origin looks like the simple pendulum. This is not a surprise since the sine function can be approximated by $x - \frac{1}{6}x^3$. However, this approximation breaks down as x grows larger, so the duffing oscillator does not have the same periodic trajectories that the simple pendulum does. The trajectories beyond $(0,\pi)$ and $(0,-\pi)$ grow without bound, where the simple pendulum remains to have bounded, periodic trajectories.

Damped Duffing Oscillator

Due to the similarities seen between the duffing oscillator and the simple pendulum, one would expect to see similarities between the damped duffing oscillator and the damped simple pendulum. The damped duffing oscillator has the following equation of motion:

$$\ddot{x} + \dot{x} + x + \rho x^3 = 0$$

One would expect the damped duffing oscillator to have a phase portrait similar to the one in Figure 3, except instead of having closed trajectories about the origin, it would have stable spiral trajectories moving towards the origin caused by the damping. Again, this portion would mimic the damped simple pendulum; however, trajectories beyond (0,π) and (0,-π) would be unbounded. We would also expect to see the two saddle points at (0,π) and (0,-π) as in the previous examples. The phase portrait is shown in Figure 4.

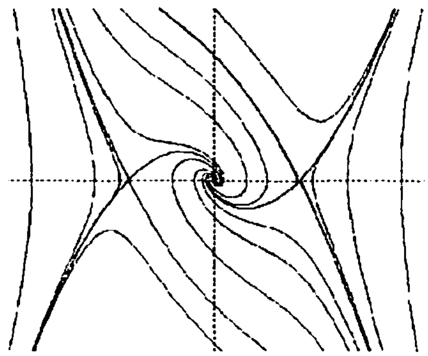

Figure 4: Phase Portrait for the Damped Duffing Oscillator

Ball In The Hoop

Finally, we will conclude our study with that of the Ball in the Hoop as done in Marsden [1999]. The ball in the hoop is a more complicated system with the following equations of motion:

$$\dot{x} = y$$

$$\dot{y} = \frac{g}{R}(\alpha \cos \Theta - 1)\sin x - \beta y$$

$$\alpha = \frac{R\omega}{g} \qquad \beta = \frac{v}{m}$$

First take the example where $\alpha = 0.5$ and $\beta = 0$. The phase portrait is shown in Figure 5.

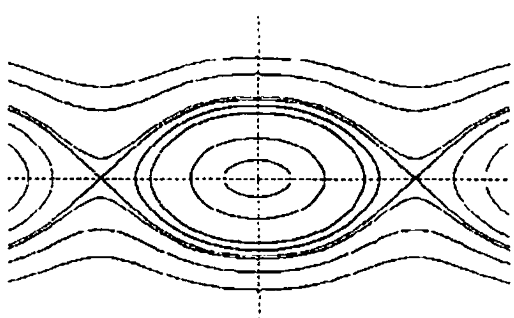

Figure 5: Ball in the Hoop with $\alpha = 0.5$ and $\beta = 0$

Comparing this to the simple pendulum, these phase portraits are identical. This can be seen analytically by looking at the equilibrium points as is done in Marsden [1999]. The equilibrium solutions, or fixed points are those satisfying $\dot{\Theta} = 0$. For this system, that corresponds to the following equation:

$$R\omega^2 \sin\Theta \cos\Theta = g \sin\Theta$$

Again the fixed points are (0,kπ), where (0,0) represents the ball at rest at the base of the rotating hoop, and (0,π) is a saddle point representing the ball coming to rest at the top of the hoop. However, what happens if $\Theta \neq 0$ or π? Then the equation above looks like:

$$R\omega^2 \cos\Theta = g$$

Where the critical rotation is given by:

$$\omega_c = \sqrt{\frac{g}{R}}$$

However, this is the frequency of oscillations for the linearized simple pendulum where R = L. Therefore, we would expect the phase planes in this situation to be equivalent, which they are.

Now increase α to 1.5, but keep β at 0. Then the system has passed the critical rotation rate, so that it is no longer identical to the simple pendulum. Now the equations exhibit four fixed points as can be seen on the computer generated phase portrait shown in Figure 6.

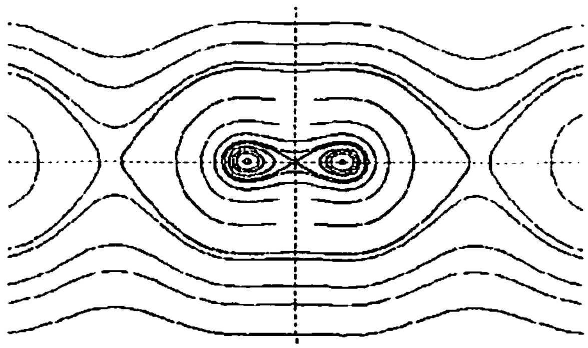

Figure 6: Ball in Hoop with $\alpha = 1.5$ and $\beta = 0$

When a system exhibits a change in dynamics like this, it is called a bifurcation. The point at which this bifurcation occurs, ω_c, is called the bifurcation point for the system. The above example exhibits the commonly seen Hamiltonian pitchfork bifurcation. Pitchfork bifurcations are common in systems that have symmetry.

Next we add damping to the system by setting $\beta = 0.1$ and α remaining at 1.5. The phase portrait is shown in Figure 7.

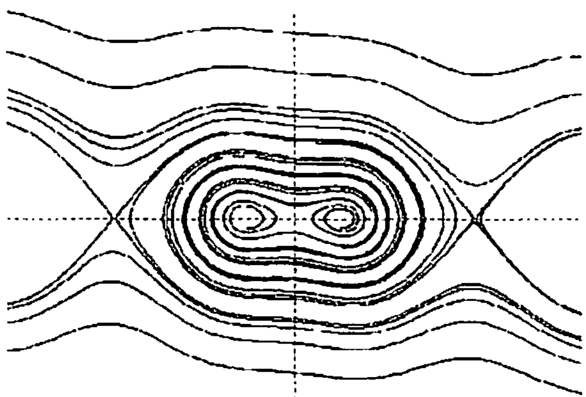

Figure 7: Ball in Hoop $\alpha = 1.5$ $\beta = 0.1$

As expected, this figure is similar to Figure 6 still containing the pitchfork bifurcation. The only change is that by adding damping, the system now decays to the fixed points.

Conclusions

Using the phase portrait, it is very easy to graphically visualize the possible trajectories and equilibrium points of a system. This method of classifing systems also makes its possible to quickly see similarities in systems, even though their equations of motion may not look alike at first glance. Phase planes also provide a good way of visualizing bifurcations, or changes in dynamics, that may occur in a system as you vary the parameters.

References

Koon, Lo, Marsden, Ross. Heteroclinic Connections between Periodic Orbits and Resonance Transitions in Celestial Mechanics., Not yet published.

Marsden J. E. & T. S. Ratiu [1999] Introduction to Mechanics and Symmetry 2nd Edition., Texts in Applied Mathematics, 17, Springer-Verlag.

Strogatz, S. H. [1994] Nonlinear Dynamics and Chaos., Addison-Wesley Publishing Company.

Wiggins, S. [1990] Introduction to Applied Nonlinear Dynamical Systems and Chaos., Texts in Applied Mathematics, 2, Springer-Verlag.