# Chaos in Buckling Beams and Duffing's equation

*Nils Tarnow*

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## 1. What is Chaos ?

Chaos in dynamical systems denotes a behavior that is extremely sensitive to changes in initial conditions. In a mechanical system it means a motion for which trajectories starting from slightly different initial conditions diverge exponentially.

## 2. Buckling Beams and Duffing's equation

The differential equation of a buckled beam undergoing forced lateral vibrations can be written in non dimensional form [1] as:

$$u^{iv} + \Gamma u'' - K\left[\int_0^1 (u'(\xi))^2 d\xi\right] u'' + \delta \dot{u} + \ddot{u} = P(x,t) \qquad (1)$$

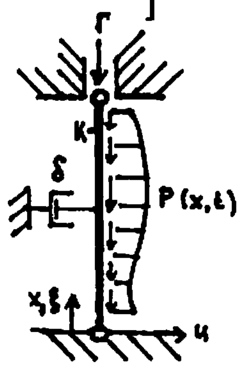


Figure 1

Where u(x,t) is the lateral deflection, δ is the viscous damping coefficient, K is the membrane stiffness, Γ is a constant axial compressive load and P is a distributed, time dependent load.

For a simple one mode model of this system where P(x,t) has sinusoidal spatial distribution coinciding with this mode and a sinusoidal time dependence, we get the o.d.e.:

$$\ddot{u} + \delta\dot{u} - \pi^2(\Gamma - \pi^2)u + \frac{1}{2}K\,\pi^4 u^3 = f\cos(\omega t) \qquad (2)$$

For a snap through oscillator one of the well known principles of mechanics leads to the exactly valid equation of motion [2]:

$$m\,\ddot{u} + \delta\,\dot{u} + \frac{2EA}{l}u\,(1 - \frac{l}{\sqrt{b^2+u^2}} = f\,cos(\omega t) \qquad (3)$$

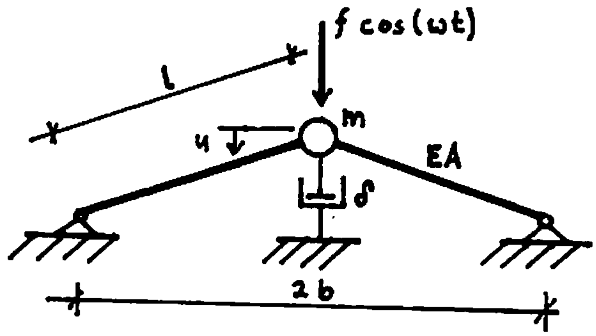


Figure 2

Where EA is the axial stiffness and m is a lumped mass.
After suitable normalization we obtain the following simplified equation, where higher than cubic nonlinearities are neglected.

$$\ddot{u} + 2\delta\,\dot{u} + \frac{1}{5}(-u + \frac{8}{3}u^3) = f cos(\omega t) \qquad (4)$$

Both examples can be understood as special forms of Duffing's equation:

$$\ddot{u} - \beta\,u + \alpha\,u^3 = \epsilon\,(\gamma cos(\omega t) - \delta\,\dot{u}\,) \qquad (5)$$

In the following parts of this report it will be shown how the chaotic properties of this equation can be discovered, both, in numerical experiments and in using analytical tools.

## 3. Numerical experiments with Duffing's equation

### 3.1. Period doubling

One way to discover chaos in Duffing's equation is through the period doubling phenomenon. A system parameter, in our case the forcing term $\gamma$ ( EQ. [5] ), is varied. Eventually we will reach a critical value of this parameter, the solution becomes unstable, bifurcates and starts oscillating between two values. Further changes of the parameter lead to successive bifurcations and finally to unperiodic, chaotic behavior. The period doubling criterion is applicable to dynamical systems whose behavior can be described exactly or approximately by a first order difference equation of the form $x_{n+1} = A\,x_n$. For one dimensional maps the critical values where the solutions bifurcate have the following universal property:

$$\lim_{m\to\infty}\frac{\Lambda_{m+1} - \Lambda_m}{\Lambda_m - \Lambda_{m-1}} = \frac{1}{\delta}, \qquad \delta = 4.6692 \qquad (6)$$

However, as Holmes [4] points out, in our case of a two dimensional map into the phase plane to which Duffing's equation can easily be transformed, period doubling is not the only source of bifurcation sequences and the relation does not hold. A numerical study with a 4th order Runge-Kutta scheme shows initially for a small load a one period motion around one of its two stable equilibrium points (Figure 3a). By that is meant as the response goes through one period the forcing function goes through one period as well. For increasing values of the force two (Figure 3b), four and eight period motion can be observed before finally chaos occurs (Figure 4).

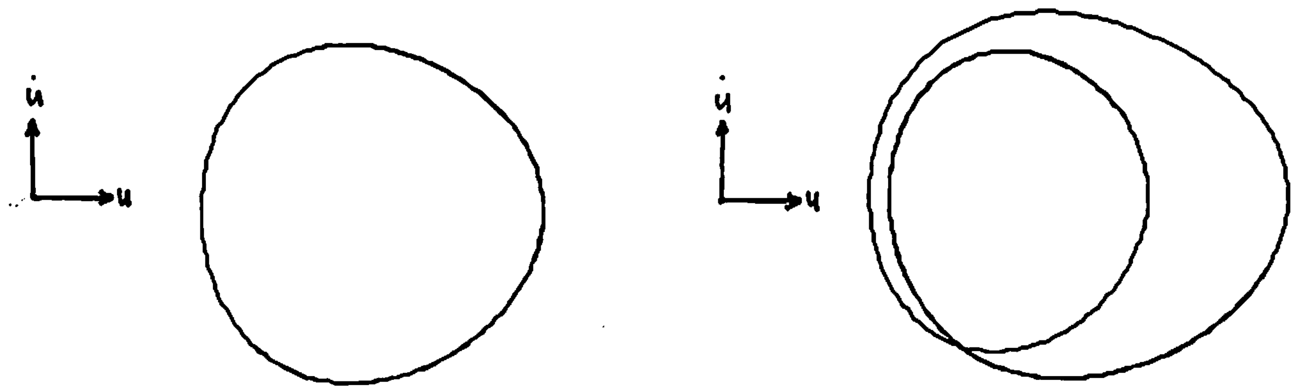


Figure 3a,b ( done on Macintosh computer ).

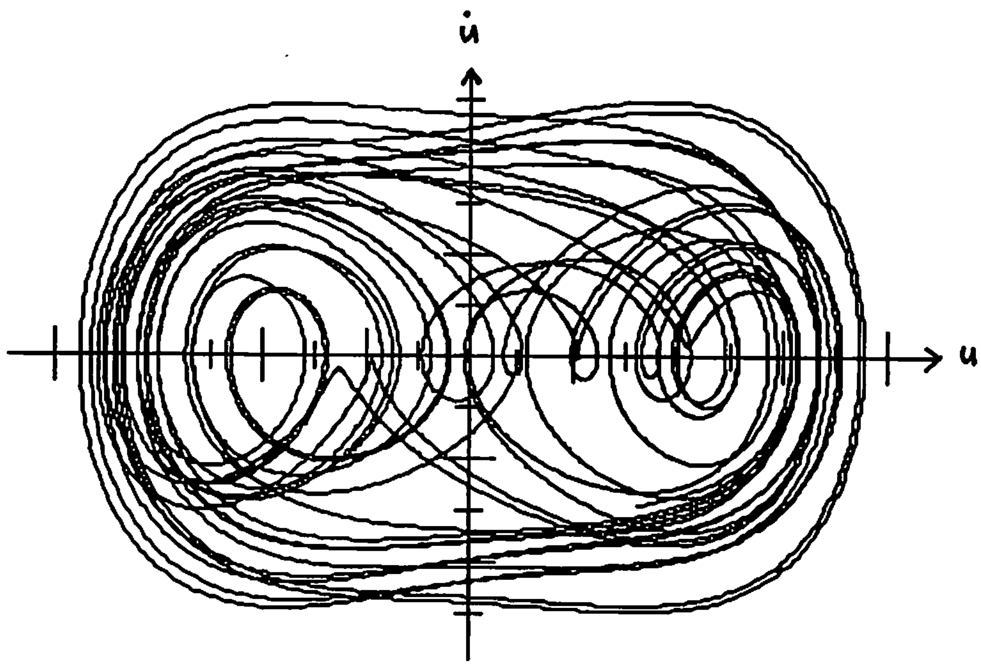


Figure 4 ( done on Macintosh computer ).

### 3.2. Poincare' Map

That Figure 4 really shows a nonperiodic chaotic motion becomes even more obvious in a Poincare' Map. That is in our case a map of a motion in the phase plane-time space after every period of the loading function back into the phase plane at time equal zero. In other words, instead of drawing a continuous picture of the motion in the phase plane as shown above, we plot only one point after every load cycle (Figure 5). While periodic motion results in a finite number of points and quasi-periodic behavior will give a closed orbit, chaotic motions appear as a cloud of infinite points occupying a certain part of the phase plane. These clouds often show highly organized structures with the geometrical property of **self-similarity** at different length scales. The appearance of these Cantor set-like patterns is a strong indicator for chaotic motions. While the finite set of points in the periodic case or the orbit in the quasiperiodic case respectively are considered to be attractors because the solutions are attracted by them as the transients die out, the pattern that is yield by chaotic motions is called a **strange attractor**.

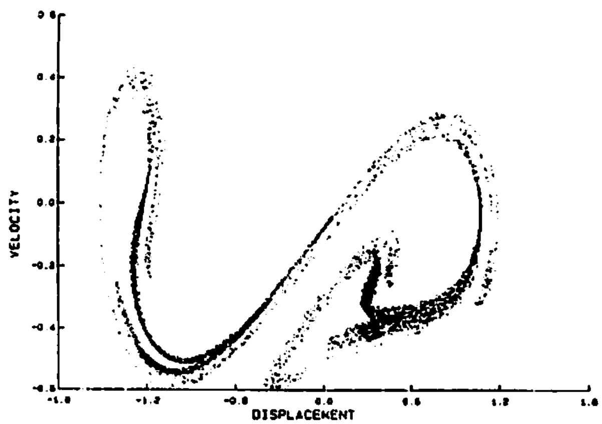


Figure 5 (from reference [6]).

### 3.3. Other numerical techniques to discover chaos

Other computational ways to detect chaos in Duffing's equation are simply the observation of time history of the system or a **Fourier analysis** of the response to the single valued harmonic input. More sophisticated methods are measurements of **Lyapunov exponents** or **fractal dimensions**. Positive Lyapunov exponents indicate extremely sensitivity to initial conditions and hence imply chaos. Fractal dimension of the orbit in the phase space implies the existence of a strange attractor, which is usually equivalent with existence of chaos.

## 4. Analytical methods to find chaos in Duffing's equation

Lyapunov exponents and the period doubling criterion can as well be used as theoretical criterions, but will not be discussed as such in this report. Instead we will use **homoclinic orbits** and **Melnikov's method** to proof the chaotic qualities of Duffing's equation.

### 4.1. Homoclinic orbits and Horseshoes

For zero external force and damping ($\gamma$=$\delta$=0) equation 5 reduces to an integrable Hamiltonian system, its orbits being the level curves of:

$$H = \frac{1}{2}\dot{u} - \frac{1}{2}\beta u^2 + \frac{1}{4}\alpha u^4 \qquad (7)$$

They are sketched in Figure 6. We especially note the homoclinic orbit $\Gamma$, points which tend to the saddle point p as $t \to \infty$.

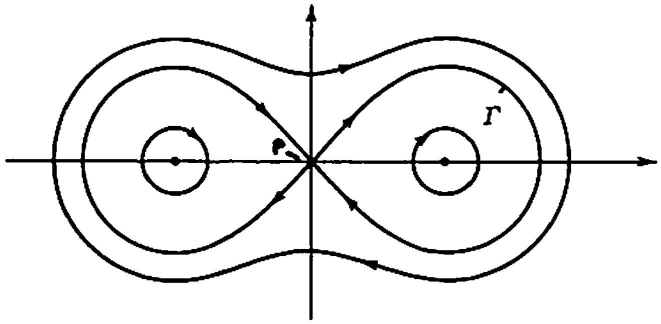


Figure 6 ( from reference [5] ).

Returning to the original system with damping and forcing the structure of this picture will change and plays an important role in determining the nature of solutions. For the damping coefficient being fixed and the forcing $\gamma$ increasing the following situation will occur. At first each loop of the homoclinic orbit will break down into a stable and a unstable manifold, that is the set of points which are asymptotic to p in a Poincare map P: $\lim_{n\to\infty} P^n(x) \to p$ in a forward or in a backward iteration, respectively. The stable manifold $M_\varepsilon^s$ passes either inside or outside the unstable one $M_\varepsilon^u$.


Figure 7 ( from reference [1]).

Finally for $\gamma$ being sufficiently big $M_\varepsilon^s$ and $M_\varepsilon^u$ meet. Since $M_\varepsilon^s$ and $M_\varepsilon^u$ are invariant under the map P the existence of one intersection implies the existence of infinitely many. The points of intersection of stable and unstable manifolds are called homoclinic points. The Melnikov method enables us to determine when these manifolds first intersect. When they do complicated invariant sets, **Smale horseshoes**, arise (Figure 8).

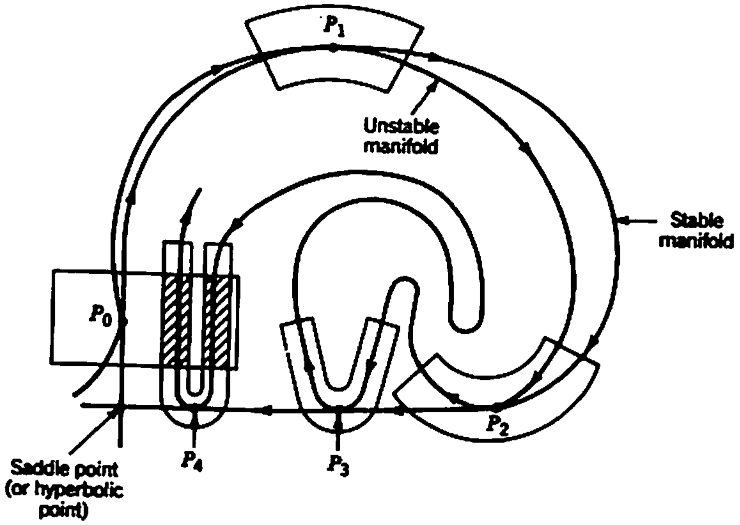


Figure 8 ( from reference [5] ).

As the picture above shows, in one circulation a originally rectangular area is stretched, folded and finally placed over the original area. If one follows a group of nearby points after many loops, the original neighboring cluster of points undergoes bifurcation after bifurcation and gets dispersed to all sectors of the rectangular area, leading to a structure with fractal properties.

### 4.2. Melnikov's method

Expressing eq. 1 as a first order system we obtain:

$$\frac{d}{dt}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \beta u - \alpha u^3 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ \Gamma \cos(\omega t) - \delta \dot{u} \end{pmatrix} \tag{8}$$

which can be rewritten as pseudo-Hamiltonian vector field in the form:

$$\dot{z} = \begin{pmatrix} \dfrac{\partial H_0}{\partial \dot{u}} \\ \dfrac{-\partial H_0}{\partial u} \end{pmatrix} + \epsilon \begin{pmatrix} \dfrac{\partial H_1}{\partial \dot{u}} \\ \dfrac{-\partial H_1}{\partial u} \end{pmatrix} \tag{9}$$

Melnikov derived a function that measures the distance between the stable and the unstable manifold. Thus simple roots of this function indicate intersections and are a sufficient condition for chaos. The Melnikov function is given by:

$$M(t_0) = \int_{-\infty}^{+\infty} \left\{ H_0, H_1 \right\} (z(t - t_0), t)\, dt \tag{10}$$

Where the Poisson brackets in our case are defined as:

$$\left\{ H_0, H_1 \right\} = \frac{\partial H_0}{\partial u}\frac{\partial H_1}{\partial \dot{u}} - \frac{\partial H_0}{\partial \dot{u}}\frac{\partial H_1}{\partial u} \tag{11}$$

Melnikovs function is applied on u and $\dot{u}$ being the explicit formulas for the homoclinic orbit, which can easily be obtained by integration of the Hamiltonian (eq. 7). The result turns out to be:

$$M = -\beta \sqrt{\frac{2}{\alpha}} \left( \delta \sqrt{2} \frac{\beta}{\alpha} \frac{2}{3} - \frac{\Gamma \omega \pi}{\beta} \frac{\sin(\omega t)}{\cosh\left(\dfrac{\omega \pi}{2\sqrt{\beta}}\right)} \right) \tag{12}$$

It follows that simple zeros occur under the the following condition,

$$\frac{\gamma}{\delta} > \frac{2\sqrt{2}\beta^{\frac{3}{2}}}{3\omega\sqrt{\alpha}} \cosh\left(\frac{\omega\pi}{2\sqrt{\beta}}\right) \tag{13}$$

At the same time this is the necessary and sufficient condition for the intersection of stable and unstable manifolds and with this for the appearance of chaos.

**5. Closure**

One may object that the simplification of the exactly valid equations leads to mathematical solutions having nothing to do with the real physical problem. However, as Holmes [1] outlines, higher order mathematical models do not affect the general behavior and it can be expected that more complex models display solutions at least as complicated. Moreover, Moon's experimental work [3] proofs the practical validity of the theoretical results.

**References**

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[2] Clemens,H. and Wauer,J. (1981),"Free and Forced Vibrations of a Snap-Through Oscillator", Report of the Institut fuer Technische Mechanik, Universitaet Karlsruhe.

[3] Moon,F.C. (1980),"Experiments on Chaotic Motions of a Forced Nonlinear Oscillator:Strange Attractors",ASME J. Appl. Mech. 47,638-644

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