CDS 205 Project: Steering Laws for UAVs

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1 Introduction

This document presents a Lie Group formulation for some of the work done by Eric Justh and P.S. Krishnaprasad on steering laws for vehicle formations. This paper addresses the problem of developing steering laws to achieve relative equilibrium amongst a group of vehicles moving at unit speed. The problem has a natural Lie Group structure which is discussed below. With a suitably chosen Lyapunov function algorithm, one can control the vehicles' formation to form unique equilibria.

Consider n vehicles moving in the plane–as shown for 3 vehicles in Figure 1. The vehicles are assumed to be point particles moving at unit speed and each vehicle is given a Frenet-Serret frame (see O'Neill [1997] sec. 2.3) whose origin is located at the position of the vehicle. Each frame consists of two unit vectors: $\mathbf{x}_j$ being tangent, and $\mathbf{y}_j$ being normal, to the trajectory of the the j^{th} vehicle. Let the vector $\mathbf{r}_j$ describe the position of the j^{th} vehicle relative to some fixed coordinate frame.

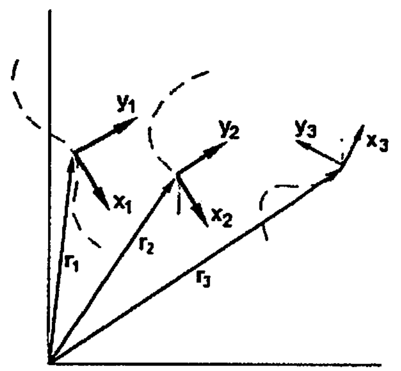

Figure 1.1: Frenet frames for three vehicles moving in the plane (trajectories show by dashed lines).

2 Lie Group Formulation

The configuration of each vehicle can be described as an element of $G = \mathrm{SE}(2)$, the Special Euclidean group Marsden and Ratiu [1994]. In particular, the configuration of the j^{th}

vehicle is given by $g_j \in G$ where in coordinates we have

$$g_j = \begin{bmatrix} \mathrm{x}_j & \mathrm{y}_j & \mathrm{r}_j \\ 0 & 0 & 1 \end{bmatrix} \tag{2.1}$$

Therefore the dynamics of the n vehicles evolve on the configuration submanifold $M_{cf} \subset G \times G \times \ldots \times G$. In fact, it can be shown that this manifold is collision free, i.e.

$$M_{cf} = \{(g_1, \ldots, g_n) \in G \times \ldots \times G \mid \mathrm{r}_{ij} = \mathrm{r}_i - \mathrm{r}_j \neq 0, \forall i \neq j\}. \tag{2.2}$$

Let $\mathfrak{g}$ denote the Lie Algebra of G. One can easily show that

$$\mathfrak{g} = \left\{ \begin{bmatrix} \Omega & \mathrm{r} \\ 0^T & 0 \end{bmatrix} \mid \Omega \in \mathfrak{so}(2) \ \& \ \mathrm{r} \in \mathbb{R}^2 \right\}. \tag{2.3}$$

Let

$$A_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.4}$$

be basis elements of the Lie Algebra $\mathfrak{g}$. Then we let the (first-order) dynamics of each vehicle be given by

$$\dot{g}_j = g_j \xi_j, \qquad j = 1, \ldots, n \tag{2.5}$$

where $\xi_j \in \mathfrak{g}$ has the form

$$\xi_j = A_o + A_1 u_j, \qquad j = 1, \ldots, n. \tag{2.6}$$

The quantity $u_j \in \mathbb{R}$ is the control input to the j^{th} vehicle. Notice these dynamics prescribe a "gyroscopic" control, that is, the vehicle is always pushed in a direction perpendicular to its direction of motion.

The relative configuration of the n vehicles can be described by defining a *shape space*. There are several ways to go about this, but as an example consider the shape variables

$$\tilde{g}_j = g_1^{-1} g_j, \qquad j = 2, \ldots, n \tag{2.7}$$

Since G is a Lie Group, the shape variables evolve on the reduced space of n-1 products of G:

$$M_{shape} = \left\{ (\tilde{g}_2, \ldots, \tilde{g}_n) \in G \times \ldots \times G \mid (\tilde{g}_j^{-1}\tilde{g}_k)_{13}^2 + (\tilde{g}_j^{-1}\tilde{g}_k)_{23}^2 > 0, j \neq k \right\} \tag{2.8}$$

where we use the convention that g_{ij} denotes the (i,j)-component of the matrix g (for future reference, let g^{ij} denotes the (i,j)-component of the matrix g^{-1})

3 Two Vehicle Scenario

For ease of explanation, consider the case for $n = 2$; we will later return to the case of arbitrary n. Define the shape variable

$$g = g_1^{-1} g_2 \tag{3.1}$$

or in coordinates we have

$$g = \begin{bmatrix} \mathrm{x}_1 \cdot \mathrm{x}_2 & \mathrm{x}_1 \cdot \mathrm{y}_2 & (\mathrm{r}_2 - \mathrm{r}_1) \cdot \mathrm{x}_1 \\ \mathrm{y}_1 \cdot \mathrm{x}_2 & \mathrm{y}_1 \cdot \mathrm{y}_2 & (\mathrm{r}_2 - \mathrm{r}_1) \cdot \mathrm{y}_1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.2}$$

Similarly,

$$g^{-1} = g_2^{-1} g_1 = \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{x}_2 & \mathbf{x}_2 \cdot \mathbf{y}_1 & -(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{x}_2 \\ \mathbf{y}_2 \cdot \mathbf{x}_1 & \mathbf{y}_1 \cdot \mathbf{y}_2 & -(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{y}_2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.3}$$

Differentiating the shape variable w.r.t time gives

$$\begin{aligned} \dot{g} &= \frac{d}{dt}(g_1^{-1}) g_2 + g_1^{-1} \dot{g}_2 \\ &= -g_1^{-1} \dot{g}_1 g_1^{-1} g_2 + g_1^{-1} \dot{g}_2 \\ &= -\xi_1 g + g \xi_2 \\ &= g(\xi_2 - \mathrm{Ad}_{g^{-1}} \xi_1) \\ &= g\xi \end{aligned} \tag{3.4}$$

where $\xi \doteq \xi_2 - \mathrm{Ad}_{g^{-1}} \xi_1 \in \mathfrak{g}$. Hence if we allow the control inputs, u_1 and u_2, to be functions of the shape variable g only, then the dynamics given by (3.4) evolve on the (reduced) shape space M_{cf}/G.

The goal is to develop control inputs to drive the formation to some desired equilibrium. Notice that equilibria of the reduced dynamics given in (3.4) correspond to relative equilibria for the full dynamics give in (2.5). Let g_e denote an equilibrium of the dynamics given in (3.4), therefore,

$$\xi_2(g_e) - \mathrm{Ad}_{g^{-1}} \xi_1(g_e) = 0 \tag{3.5}$$

In coordinates, this is equivalent to

$$\begin{bmatrix} g_{12}u_2 & -g_{11}u_2 & g_{11} \\ g_{11}u_2 & g_{12}u_2 & -g_{12} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} g_{12}u_1 & -g_{11}u_1 & 1 - g_{23}u_1 \\ g_{11}u_1 & g_{12}u_1 & g_{13}u_1 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.6}$$

The equality given in (3.6) requires

$$\begin{aligned} u_2 &= u_1, \\ g_{11} &= 1 - g_{23}u_1, \\ g_{12} &= -g_{13}u_1. \end{aligned} \tag{3.7}$$

Since the Frenet-Serret are orthonormal,

$$g_{11}^2 + g_{12}^2 = 1. \tag{3.8}$$

Plugging in the conditions given by (3.7) into the equality (3.8) we obtain

$$\begin{aligned} 1 &= (1 - g_{23}u_1)^2 + (g_{13}u_1)^2 \\ &= 1 - 2g_{23}u_1 + g_{23}^2 u_1^2 + g_{13}^2 u_1^2 \end{aligned} \tag{3.9}$$

or

$$0 = u_1 \left[(g_{13}^2 + g_{23}^2) u_1 - 2g_{23} \right] \tag{3.10}$$

which is satisfied when

$$u_1 = \frac{2g_{23}}{g_{13}^2 + g_{23}^2}, \quad \text{or} \quad u_1 = 0. \tag{3.11}$$

Case 1. First consider the case $u_1 = 0$. From (3.7) we require

$$\begin{aligned} u_1 &= u_2 = 0 \\ g_{11} &= 1, \\ g_{12} &= 0. \end{aligned} \tag{3.12}$$

It's easy to verify that these equalities are satisfied only if $\mathbf{x}_1 = \mathbf{x}_2$, that is, both vehicles are moving in the same direction. We refer to this equilibrium state as *rectilinear motion.*

Case 2. Now consider the case where $u_1 = \frac{2g_{23}}{g_{13}^2+g_{23}^2} \neq 0$. The distance between the two vehicles is given by

$$r = \|\mathbf{r}_2 - \mathbf{r}_1\| = \sqrt{g_{13}^2 + g_{23}^2} = \sqrt{(g^{13})^2 + (g^{23})^2}. \tag{3.13}$$

It can be shown $\dot{u}_1 = 0$, which implies that each vehicle follows a circular orbit of radius $1/|u_1|$. Upon further computation (directly integrating the equations for $\mathbf{r}_1$ and $\mathbf{r}_2$ and using (3.7)) one can show that the centers of the two orbits coincide. Therefore, this second equilibrium configuration consists of two vehicles moving on the same circular orbit.

3.1 Two vehicle control law

Now that we have discussed the two types of possible equilibria for the dynamics given by (3.4), let us turn to the problem of ensuring that the vehicles reach of one of these equilibrium configurations. We focus our attention to the rectilinear motion case, as this case seems to have more practical importance. First we must choose a suitable control law that will force the vehicles to converge to the desired equilibrium. Then we require a Lyapunov function that one can use to prove that the vehicles converge to the desired equilibrium.

As previously mentioned, we would like the control input for each vehicle to be a function of the shape variable only (which describes the relative configuration of the two vehicles). Consider the following form of control inputs for the two vehicle system:

$$\begin{aligned} u_1(g) &= -\eta(r)\left(\frac{g_{13}g_{23}}{r^2}\right) + f(r)\left(\frac{g_{23}}{r}\right) + \mu(r)g_{21}, \\ u_2(g) &= -\eta(r)\left(\frac{g^{13}g^{23}}{r^2}\right) + f(r)\left(\frac{g^{23}}{r}\right) + \mu(r)g^{21} \end{aligned} \tag{3.14}$$

where r is the intervehicle distance given by (3.13). These controls are obviously only dependent on the shape variable g. Although the functional form of the controls given in (3.14) might seem arbitrary at first glance, each term plays a particular role, as we will explain shortly. First, let us make a few assumptions:

1. $\eta(r), \mu(r)$, and $f(r)$ are Lipschitz on $(0, \infty)$
2. $\exists h(r)$ such that $f(r) = dh/dr$
3. $\lim_{r\to 0} h(r) = \infty, \lim_{r\to\infty} h(r) = \infty$ and $\exists \hat{r}$ s.t. $h(\hat{r}) = 0$
4. $\eta(r) > 0, \mu(r) > 0$ and $2\mu(r) > \eta(r)$

Consider the (candidate) Lyapunov function

$$V_{pair} = \ln\left[1 - (g_{13}g^{13} + g_{23}g^{23})/r^2\right] + h(r). \tag{3.15}$$

We are now ready to present the following:

Proposition 1: Consider the system given by (3.4) with controls given in (3.14). Let $\Lambda \doteq \{g \mid [1-(g_{13}g^{13}+g_{23}g^{23})/r^2] \neq 0 \text{ for } 0<r<\infty\}$. Then any trajectory starting in Λ converges to the equilibrium given from (3.12).

Proof: First notice that the function given in (3.15) is a valid Lyapunov function, that is, it is continuously differentiable on Λ and it is radially unbounded. Next let θ_j denote the angle between the vectors $\mathbf{x}_j$ and $\mathbf{r}$ ($j=1,2$). Then we have

$$g_{13}g^{13} = -(\mathbf{r}\cdot\mathbf{x}_1)(\mathbf{r}\cdot\mathbf{x}_2) = -r^2\cos\theta_1\cos\theta_2 \tag{3.16}$$

Similarly,

$$g_{23}g^{23} = -(\mathbf{r}\cdot\mathbf{y}_1)(\mathbf{r}\cdot\mathbf{y}_2) = r^2\sin\theta_1\sin\theta_2 \tag{3.17}$$

Using the trig identity $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$, the Lyapunov function given in (3.15) can be written as $V_{pair} = \ln[1+\cos(\theta_1+\theta_2)] + h(r)$. If we define the quantities

$$\phi_1 = \pi/2 - \theta_1 \text{ and } \phi_2 = \pi/2 + \theta_2 \tag{3.18}$$

we can write

$$V_{pair} = \ln[1+\cos(\phi_2-\phi_1)] + h(r). \tag{3.19}$$

It is this Lyapunov function (3.19) that is considered by Justh and Krishnaprasad [2002 and 2003], where it is shown that $\dot{V}_{pair} \le 0$ and $\dot{V}_{pair} = 0 \Leftrightarrow \phi_1 = \phi_2$ (where they consider analogous control inputs (given in terms of r and $\phi_{1,2}$) to the control inputs we give in terms of shape variables in (3.14). Since $\dot{V}_{pair} \le 0$ on Λ, each trajectory starting in Λ remains in a compact sublevel set Ω of V_{pair} for all time. Hence the dynamics evolve on a collision free submanifold, as was asserted earlier without proof. By LaSalle's Invariance Principle Khalil [1992], the trajectory converges to the largest invariant set M of the set E of all points in Ω where $\dot{V}_{pair} = 0$. As shown in Justh and Krishnaprasad [2002 and 2003], the dynamics in E are given by

$$\begin{aligned} \dot{r} &= 0, \\ \dot{\phi}_1 &= -[\eta(r)\sin\phi_1 + f(r)]\cos\phi_1, \\ \dot{\phi}_2 &= -[\eta(r)\sin\phi_1 + f(r)]\cos\phi_1. \end{aligned} \tag{3.20}$$

Hence the *largest invariant set* contained in E is given by

$$M = \left(\left\{(r,\frac{\pi}{2},\frac{\pi}{2}),\ \forall r\right\} \cup \left\{(r,-\frac{\pi}{2},-\frac{\pi}{2}),\ \forall r\right\} \cup \{(\hat{r},0,0) \mid f(\hat{r})=0\}\right) \tag{3.21}$$

which describes rectilinear motion of either: vehicle 1 directly following vehicle 2; vehicle 2 following vehicle 1; or both vehicles moving in the same direction, perpendicular to the baseline between them, at a distance $\hat{r}$ apart. □

Now let us give physical intuition about the form of the controls given in (3.14). The main properties of several biological and swarming models (e.g. Reynolds [1987] or Tanner, et. al. [2003]) are (1) some method for heading alignment, (2) presence of a repulsion force that keeps the vehicles from colliding, (3) an attraction force that provides cohesion for the group and (4) decreased influence for neighbors at greater distances. The controls given in (3.14) fit somewhat into this convention. The term involving η tends to align each vehicle with the *baseline* between itself and it's neighbor. The term involving $f(r)$ forces the vehicle to steer toward, if $r<\hat{r}$, or steer away, if $r<\hat{r}$, from its neighbor. Finally, term involving $\mu(r)$ tends to align the heading of each vehicle with its neighbor.

4 Generalization to n vehicles

In the previous section we developed a control law for a formation of $n = 2$ vehicles. This control law and the results that were presented can be naively extended for arbitrary n– only making the computations and proof more arduous. In particular, one can choose the functional form for the control input for each vehicle to have the same from as in (3.14) except now there are identical contributions from each vehicle. Likewise, one can form and analogous Lyapunov function candidate as given in (3.15) only now summing over each vehicle. Then one can prove a convergence result analogous to Proposition 1, that is, one can show the vehicles achieve a relative equilibrium where each vehicle is moving in the same direction.

References

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