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Klimontovitch function and Poisson structures

Let $P_n = \{(p_1 \dots p_n) \in P^n : p_i \neq p_j \ \forall i \neq j\}$, where P is some symplectic manifold. Let $\hat{P} = \mathcal{F}(P)^*$ be the Poisson manifold with Lie-Poisson brack. Let Φ_n be the map from P_n to $\hat{P}$ defined by Klimontovitch functions:

$$P_n \ni (p_1 \dots p_n) \xrightarrow{\Phi_n} \sum_{i=1}^{n} \delta_{p_i} \in \mathcal{F}(P)^* = \hat{P}$$

We are going to prove that Φ_n is an isomorphism of Poisson structures (P_n is consid. with the Poisson structure induced from the natural Poisson bracket on P^n.

$$\{F,G\}_{P^n} = \sum_{i=1}^{n} \{F(p_1 \dots \overset{\downarrow}{p_i} \dots p_n), G(p_1 \dots \overset{\downarrow}{p_i} \dots p_n)\}_P$$

where the arrow means that we regard a function as a function of p_i only.

Poisson structure on $\hat{P}$ is given by

$$\{F,G\}(\mu) = \mu\left(\left\{\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right\}_P\right), \quad \mu \in \hat{P}$$

because $\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \in \mathcal{F}(P)$ and $\mathcal{F}(P)$ is the Lie algebra of $Sym(P)$.

Case $n=1$: Let us put $\phi = \phi_1$. Let $F \in \mathcal{F}(P)$ and let $\hat{F} \in \mathcal{F}(\hat{P})$ be defined by

$$\hat{F}(\mu) = \mu(F) \quad \forall \mu \in \hat{P}$$

Then $\frac{\delta \hat{F}}{\delta \mu} = F$, since $\hat{F}$ is linear and

$$\{\hat{F}, \hat{G}\}_{\hat{P}}(\mu) = \mu(\{F, G\}_P)$$

In particular, if $\mu = \phi(p) = \delta_p$, we get

$$\{\hat{F}, \hat{G}\}(\phi(p)) = \{F, G\}_P(p)$$

We also have $\hat{F} \circ \phi = F$. This means on the image of ϕ (which is an orbit of the coadjoint representation if P is connected, because $Sym(P)$ acts on P transitively and commutes with ϕ) two Poisson structures coincide. On the other hand, we know that the Kirillov form determines the same Poisson structure as the Lie-Poisson structure.

Case $n > 1$. If P is connected, $\dim P \geq 2$ then P_n is connected, also, because the equation $p_i = p_j$ determines a submanifold of codim at least 2. So the action of $Sym(P_n)$ on P_n is transitive. That is why the image of ϕ_n is one coadjoint orbit.

For simplicity, we consider only functions on P^n with compact supports in P_n. We define, as in case of $n=1$, the extension of each $F \in \mathcal{F}(P)$ to a function on $\hat{P}$ by

$$\hat{F}(\mu) = \frac{1}{n!} (\underbrace{\mu \otimes \ldots \otimes \mu}_{n-\text{times}})(F)$$

Compact supports condition gives

$$\hat{F} \circ \phi_n = SF$$

where $SF(p_1 \ldots p_n) = \frac{1}{n!} \sum_{\sigma \in S_n} F(p_{\sigma(1)} \ldots p_{\sigma(n)})$,

is the symmetrization of F. In particular for symmetric functions:

$$\hat{F} \circ \phi_n = F$$

Indeed $\hat{F}(\phi_n(p_1 \ldots p_n)) = \frac{1}{n!} \sum_{(i_1 \ldots i_n)} F(z_{i_1}, \ldots, z_{i_n})$

Because $F = 0$ around $P^n \setminus P_n$ then the last sum reduces to SF. We have

$$D\hat{F}(\mu)\eta = \frac{1}{n!} \sum_i (\mu \otimes \ldots \otimes \overset{i}{\eta} \otimes \ldots \mu)(F)$$

by multilinearity of $\hat{F}$.

So, one can easily get

$$\frac{\delta \hat{F}}{\delta\mu}(p) = \frac{1}{n!}\sum_{i=1}^{n} (\mu\otimes\cdots\otimes \overset{i}{\delta_p}\otimes\cdots\otimes\mu)(F)$$

(δ_p means $\delta(p_i - p)$).

In particular, if $\mu = \sum \delta_{p_i}$ and F is symmetric, we get:

$$\frac{\delta \hat{F}}{\delta\mu}(p) = \frac{1}{n!}\sum_{(i_1 \dots i_n)} \sum_j F(p_{i_1}, \dots, \overset{j}{p}, \dots, p_{i_n}) =$$

$$= \sum_j F(p_1, \dots, \overset{j}{p}, \dots, p_n)$$

Hence

$$\left\{ \frac{\delta \hat{F}}{\delta\mu}, \frac{\delta \hat{G}}{\delta\mu} \right\}_p = \sum_{j,k} \{ F(p_1, \dots, \overset{\downarrow j}{p}, \dots p_n), G(p_1 \dots \overset{\downarrow k}{p}$$

and

$$\{\hat{F}, \hat{G}\}_{\hat{p}}(\phi(p_1 \dots p_n)) = \sum_{l,j,k} \{F(p_1 \dots \overset{\downarrow j}{p} \dots), F(p_1 \dots \overset{\downarrow k}{p} \dots p_n)\}(p$$

Sine if p is close to some of $p_1 \dots p_n$ then F, G are equal to 0, and so the brackets, then we get

$$\{\hat{F}, \hat{G}\}_{\hat{p}}(\phi(p_1 \dots p_n)) = \sum_l \{F(p_1 \dots \overset{\downarrow}{p_l} \dots p_n), G(p_1 \dots \overset{\downarrow}{p_l} \dots$$

$= \{F, G\}_{p_n}(p_1 \dots p_n)$. This gives us the desire result.