Geometric Phases in the Motion

of Particles on Hoops

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Elegant & insightful!
Excellent Jerry M.

I. Introduction.

In class[1] we discussed the geometric phase resulting from slowly rotating a planar hoop about an axis normal to the plane of the hoop while a particle moved quickly around the hoop. In Section II of this paper, I consider the effects of transporting a nonplanar hoop along a path in SE(3). Although not necessary, it is possible to introduce the concepts of gauge freedom and a gauge-independent geometric phase, as described in Ref. 2. In Section III, I return to planar problems and consider changing the shape of the hoop, rather than having a rigid hoop rotate. I do this because the phase shift for the rotation of a rigid hoop one revolution about a constant axis results from the fact that SO(2) is not simply connected; it is not due to a nonvanishing 2-form (to be described below) as is often the case. The motion of particles on variable-shape hoops illustrates the concepts of geometric phases, gauge potentials, curvature forms, and gauge invariance as in Ref. 2. Section IV contains an explicit example of a two parameter family of curves. In Section V, it is shown that the rotation in 2D of a rigid hoop can be treated as a one-parameter family of deformable hoops.

II. Paths through SE(3).

In this section, we generalize the treatment in §8.7 of the text to the case of a nonplanar hoop transported along a path in SE(3). As before, we let $\mathbf{q}(s)$ be a closed curve parametrized by arc-length. This describes the shape of the rigid hoop in a frame fixed to the hoop. The position of the particle in the lab frame is

$$\mathbf{x}(t) = \mathbf{x}_0(t) + R(t)\ \mathbf{q}(s(t)), \tag{1}$$

where $\mathbf{x}_0(t)$ is a given slow function of time and R(t) is a given SO(3)-matrix-valued slow function of time. The velocity relative to the lab frame is

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_0(t) + \dot{R}(t)\ \mathbf{q}(s(t)) + R(t)\ \mathbf{q}'(s(t))\ \dot{s}(t)$$

$$= R\,[\,R^t\,\dot{\mathbf{x}}_0 + R^t\,\dot{R}\,\mathbf{q} + \mathbf{q}'\ \dot{s}].$$

If we define $\mathbf{y}(t) \equiv R^t(t)\ \dot{\mathbf{x}}_0(t)$ (another given function of time), and use

$R^t \dot{R} \mathbf{q} = \omega \times \mathbf{q}$, where ω is the instantaneous angular velocity vector relative to the frame fixed to the hoop, then we get for the Lagrangian

$$L(s,\dot{s},t) = \frac{1}{2} \| \mathbf{y}(t) + \omega(t) \times \mathbf{q}(s) + \mathbf{q}'(s)\, \dot{s} \|^2 \tag{2}$$

Next, we write down the Euler-Lagrange equations:

$$\frac{\partial L}{\partial s} = (\mathbf{y} + \omega \times \mathbf{q} + \mathbf{q}'\, \dot{s})\cdot(\omega \times \mathbf{q}' + \mathbf{q}''\dot{s})$$

$$= \mathbf{y}\cdot\omega \times \mathbf{q}' + (\omega \times \mathbf{q})\cdot(\omega \times \mathbf{q}') + \mathbf{y}\cdot\mathbf{q}''\dot{s} + \dot{s}\, \mathbf{q}''\cdot\omega \times \mathbf{q}$$

$$\frac{\partial L}{\partial \dot{s}} = (\mathbf{y} + \omega \times \mathbf{q} + \mathbf{q}'\, \dot{s})\cdot\mathbf{q}'$$

$$= \mathbf{y}\cdot\mathbf{q}' + \mathbf{q}'\cdot\omega \times \mathbf{q} + \dot{s}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \dot{\mathbf{y}}\cdot\mathbf{q}' + \mathbf{y}\cdot\mathbf{q}''\dot{s} + \dot{s}\, \mathbf{q}''\cdot\omega \times \mathbf{q} + \mathbf{q}'\cdot\dot{\omega} \times \mathbf{q} + \ddot{s}$$

$$\ddot{s} = \mathbf{q}'\cdot(\mathbf{y} \times \omega - \dot{\mathbf{y}}) + (\omega \times \mathbf{q})\cdot(\omega \times \mathbf{q}') - \dot{\omega}\cdot\mathbf{q} \times \mathbf{q}' \tag{3}$$

As explained in the text, we need to average this around the hoop. The first term gives zero, since the hoop is closed. Thus, translations do not contribute to the effect. The second term is equal to $\frac{1}{2}\frac{\partial}{\partial s}\|\omega \times \mathbf{q}\|^2$, so its integral around the hoop is also zero. We are left with

$$\langle \ddot{s} \rangle = -\frac{2\mathcal{A}}{L}\cdot\dot{\omega}, \tag{4}$$

where the area vector $\mathcal{A}$ is defined as

$$\mathcal{A} = \frac{1}{2}\oint \mathbf{q} \times d\mathbf{q}.$$

$\mathcal{A}$ dotted into a unit vector gives the area of the plane curve obtained by projecting the curve onto the plane orthogonal to the unit vector.

Proceeding as in the text, we get

$$\Delta s = -\frac{2\mathcal{A}}{L} \cdot \int_0^T \omega \, dt, \qquad (5)$$

(We have assumed that $\omega(t=0)=0$. This assumption is discussed on p. 219 of the text.) We may write the result in terms of R(t) by using the relation $\omega = -\frac{1}{2}\mathrm{trace}(R^t \dot{R} \mathbf{J})$, where $\mathbf{J}$ is a 3-vector of 3×3 matrices given by $J_{(i)jk} = -\varepsilon_{ijk}$. This gives

$$\boxed{\Delta s = \frac{\mathcal{A}}{L} \cdot \int_0^T \mathrm{tr}(R^t \dot{R} \mathbf{J}) \, dt.} \qquad (6)$$

In this form it is clear that the time-parametrization of the path R(t) through SO(3) does not matter. The shift is a geometric quantity.

We have not assumed that R(T) equal R(0). It makes sense to talk about the shift Δs even if the path through SO(3) is not closed, because the hoop is rigid; the same reference point may be used for the hoop in each orientation. I would like to address the questions of gauge choice, gauge potentials, etc. as discussed in Ref. 2. This is why I consider changing the shape of the hoop in the 2D problem, in the next section. In the context of the present problem, we may consider gauge transformations of the following form:

$$s' = s + \Psi(R) \qquad (7)$$

A different reference point is used for each orientation of the hoop. If we introduce coordinates X^1, X^2, X^3 on SO(3), Eq. (6) may be written

$$\Delta s = \int_{path} A_i \, dX^i \tag{8}$$

with the vector potential

$$A_i = \frac{\mathcal{A}}{L} \cdot \mathrm{tr}(R^t \frac{\partial R}{\partial X^i} J)$$

Under the gauge transformation Eq. (7), the vector potential transforms as

$$A_i' = A_i + \frac{\partial \Psi}{\partial X^i}$$

just as in Ref. 2. For closed paths in SO(3), the net shift is independent of the choice of gauge.

III. Hoops of variable shape in 2D

As explained above, the geometric phase for the rotated rigid hoop in 2D is due to the fact that SO(2) is not simply connected, so that even though the one-form A is closed, its integral around the cycle in parameter space is non-zero. Also, there is a natural way to define the shift for a non-integer number of slow revolutions of the hoop, since the same reference point may be used for each orientation. In this section we look at a problem that does not have these properties.

Rather than rotating the hoop, we consider varying its shape. For each shape it is necessary to arbitrarily choose a reference point. Also, we take each shape to have the same perimeter, so that the quantity of interest is the difference in position Δs of a particle on a hoop that has gone through a cycle in shape space and one that has not, given that both processes take the same amount of time T. We consider an n-parameter family of shapes, parametrized by X^i, i=1, . . . , n. Given a choice of reference points for all of the shapes, the particle's position is $\mathbf{x}(X^i,s)$, where s is an arc-length parameter for each shape. (Sorry that $\mathbf{x}$ and X look similar!) The velocity is

$$\dot{\mathbf{x}}(t) = \frac{\partial \mathbf{x}}{\partial X^i} \dot{X}^i(t) + \frac{\partial \mathbf{x}}{\partial s} \dot{s}(t),$$

so the Lagrangian is (given $X^i(t)$)

$$L(s,\dot{s},t) = \frac{1}{2} \dot{s}^2 + \dot{s}\, F(s,t) + G(s,t),$$

where

$$F(s,t) = \frac{\partial \mathbf{x}}{\partial s} \cdot \frac{\partial \mathbf{x}}{\partial X^i} \dot{X}^i(t)$$

$$G(s,t) = \frac{1}{2} \left\| \frac{\partial \mathbf{x}}{\partial X^i} \dot{X}^i(t) \right\|^2$$

Next, we calculate the Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial s} = \dot{s} \frac{\partial F}{\partial s} + \frac{\partial G}{\partial s}$$

$$\frac{\partial L}{\partial \dot{s}} = \dot{s} + F$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}} = \ddot{s} + \dot{s} \frac{\partial F}{\partial s} + \frac{\partial F}{\partial t}$$

$$\ddot{s} = - \frac{\partial F}{\partial t} + \frac{\partial G}{\partial s}$$

As before, we need to average this around the hoop. This gives

$$\langle \ddot{s} \rangle = - \frac{d}{dt} \langle F \rangle$$

where $\langle F\rangle = \frac{1}{L}\oint F\,ds$ is a function only of t. Proceeding as before (again we take $\dot{X}^i(t=0) = 0$), we get

$$\boxed{\Delta s = \oint A_i\, dX^i}$$

where

$$A_i = -\frac{1}{L}\oint \frac{\partial \mathbf{x}}{\partial s}\cdot\frac{\partial \mathbf{x}}{\partial X^i}\, ds \tag{9}$$

An explicit example is worked out in the next section.

A gauge transformation is a redefinition of the reference point for each shape:

$$s' = s + \Psi(X^i)$$

As before, the corresponding transformation for the vector potential is

$$A_i' = A_i + \frac{\partial \Psi}{\partial X^i}$$

For a closed cycle in (a simply connected) shape space, Δs is gauge-independent.

IV. An Example

In this section we (Mathematica) work(s) out the gauge potential, Eq. (9), for a two-parameter family of C^1 curves of constant perimeter 2π. The parameters a and b are defined in Fig.1. The function $\mathbf{x}(a,b,s)$ is given for the choice of gauge indicated in Fig. 1 by piecing together the four C^∞ curves that are defined at the beginning of both App. 1 and App. 2. The allowed region in the a-b plane is a>0, b>0, a+b<1. App. 1 contains a fairly readable set of instructions to Mathematica 2.0 to calculate the gauge potential. App. 2 contains a streamlined version. The result is

$$A = \frac{\pi^2-4}{4\pi}\,(1+b,1-a).$$

We may define a curvature as in Ref. 2:

$$B = \frac{\partial A_b}{\partial a} - \frac{\partial A_a}{\partial b}$$

$$\boxed{B = \frac{4-\pi^2}{2\pi}}$$

By Stoke's Theorem, the geometric phase Δs for a cycle in shape space is given by this B times the area enclosed by the path in the a-b plane.

V. Rotations as a Special Case of Deformable Hoops

The geometric phase for the rotated rigid 2D hoop may be obtained from the treatment of variable shape 2D hoops in Section III as follows. We take the rotation angle θ as the parameter X^1 of a one-parameter family of curves. Using the "same" reference point for each value of θ, we have

$$\mathbf{x}(\theta,s) = R(\theta)\,\mathbf{q}(s).$$

The gauge potential, Eq. (9), is

$$A_\theta = -\frac{1}{L}\oint \frac{\partial \mathbf{x}}{\partial s}\cdot\frac{\partial \mathbf{x}}{\partial \theta}\,ds$$

$$= -\frac{1}{L}\oint (\mathbf{q}'\,R^t)(\frac{dR}{d\theta}\,\mathbf{q})\,ds$$

$$= -\frac{1}{L}\oint \mathbf{q}'\cdot\hat{\mathbf{z}}\times\mathbf{q}\, ds$$

$$= -\frac{2\mathcal{A}}{L}$$

So for one revolution, the geometric phase is

$$\boxed{\Delta s = -\frac{4\pi\mathcal{A}}{L}}$$

which is the same as the result we derived in class.

References

1. Jerrold E. Marsden and Tudor S. Ratiu, *An Introduction to Mechanics and Symmetry. Volume I.* September 1993 (Math 189).

2. Robert G. Littlejohn, "Phase anholonomy in the classical adiabatic motion of charged particles," Phys. Rev. A. **38**, 6034-6045 (1988).

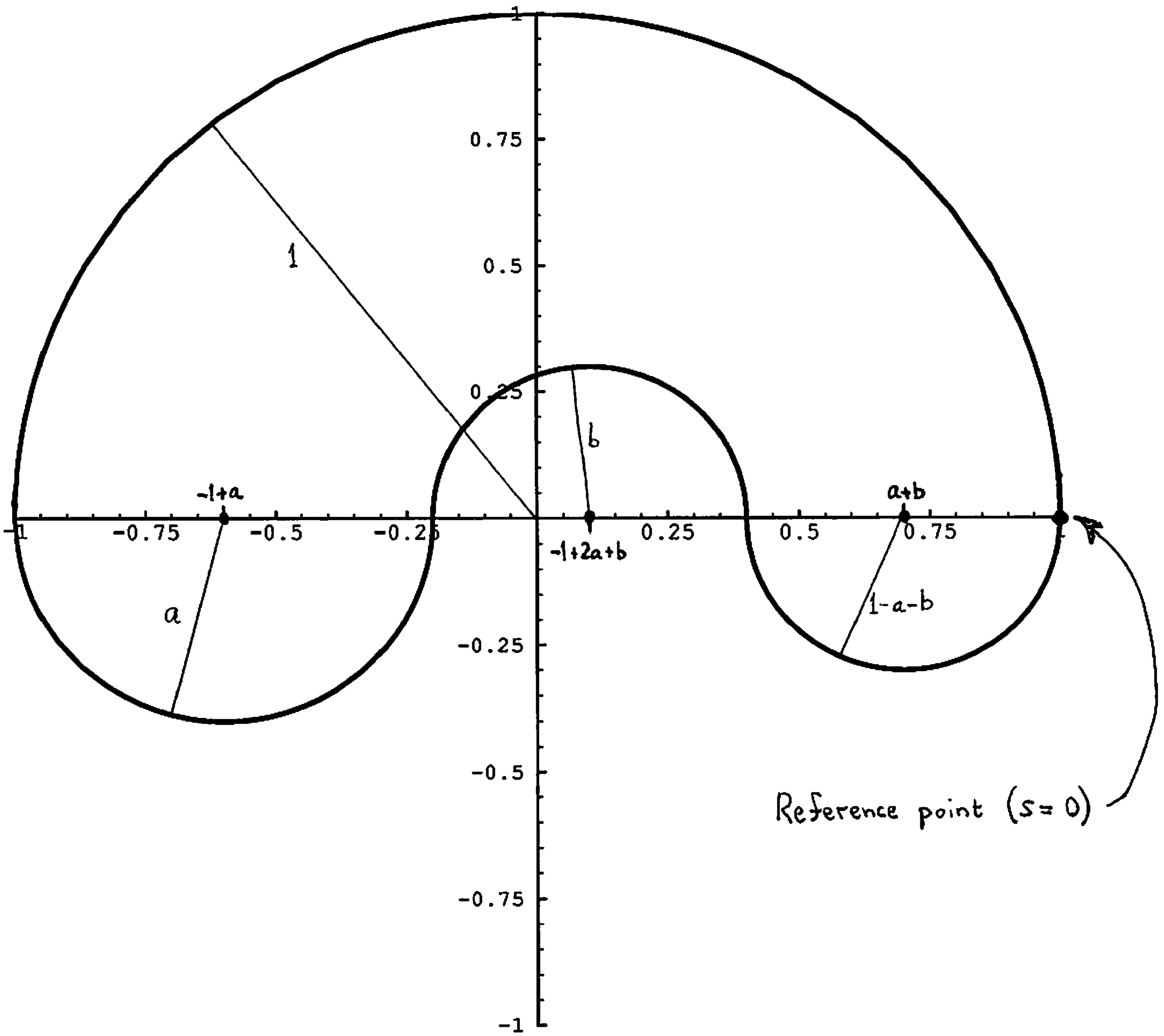

Fig. 1 : An element of the two-parameter family of curves of Section IV. ($a = 0.4$, $b = 0.3$)

Appendix 1: Calculation of the vector potential for Section IV


```
(* Define the curves *)
ex[x_]={Cos[x],Sin[x]}
re={I,0}
x1=ex[s]                                  (* range: {s,0,Pi}                *)
x2=re (a-1)-a ex[(s-Pi)/a]                (* range: {s,Pi,Pi(1+a)}          *)
x3=re (2a+b-1)-b ex[(Pi(1+a)-s)/b]        (* range: {s,Pi(1+a),Pi(1+a+b)} *)
x4=re (a+b)+(1-a-b) ex[(s-2Pi)/(1-a-b)]   (* range: {s,Pi(1+a+b),2Pi}      *)

(* Plot an example *)
eg={a->.4,b->.3}
pl1=ParametricPlot[x1/.eg,{s,0,Pi}]
pl2=ParametricPlot[x2/.eg,{s,Pi,Pi(1+a)/.eg}]
pl3=ParametricPlot[x3/.eg,{s,Pi(1+a)/.eg,Pi(1+a+b)/.eg}]
pl4=ParametricPlot[x4/.eg,{s,Pi(1+a+b)/.eg,2Pi}]
Show[pl1,pl2,pl3,pl4,PlotRange->{{-1,1},{-1,1}},AspectRatio->1]

(* Check orientation: replot with intervals slightly clipped on right *)
pl1=ParametricPlot[x1/.eg,{s,0,Pi-.1}]
pl2=ParametricPlot[x2/.eg,{s,Pi,-.1+Pi(1+a)/.eg}]
pl3=ParametricPlot[x3/.eg,{s,Pi(1+a)/.eg,-.1+Pi(1+a+b)/.eg}]
pl4=ParametricPlot[x4/.eg,{s,Pi(1+a+b)/.eg,2Pi-.1}]
Show[pl1,pl2,pl3,pl4,PlotRange->{{-1,1},{-1,1}},AspectRatio->1]

(* Calculate the integral ds of D[x,s].D[x,a] *)
x2sx2a=Simplify[D[x2,s].D[x2,a]]
x2sx2aint=Simplify[Integrate[x2sx2a,{s,Pi,Pi(1+a)}]]
x3sx3a=Simplify[D[x3,s].D[x3,a]]
x3sx3aint=Simplify[Integrate[x3sx3a,{s,Pi(1+a),Pi(1+a+b)}]]
x4sx4a=Simplify[D[x4,s].D[x4,a]]
x4sx4aint=Simplify[Integrate[x4sx4a,{s,Pi(1+a+b),2Pi}]]
answer[a]=Simplify[x2sx2aint+x3sx3aint+x4sx4aint]

(* Calculate the integral ds of D[x,s].D[x,b] *)
x2sx2b=Simplify[D[x2,s].D[x2,b]]
x2sx2bint=Simplify[Integrate[x2sx2b,{s,Pi,Pi(1+a)}]]
x3sx3b=Simplify[D[x3,s].D[x3,b]]
x3sx3bint=Simplify[Integrate[x3sx3b,{s,Pi(1+a),Pi(1+a+b)}]]
x4sx4b=Simplify[D[x4,s].D[x4,b]]
x4sx4bint=Simplify[Integrate[x4sx4b,{s,Pi(1+a+b),2Pi}]]
answer[b]=Simplify[x2sx2bint+x3sx3bint+x4sx4bint]
```

Appendix 2: streamlined version of App. 1.


```
(* Define the curves *)
ex[x_]={Cos[x],Sin[x]}
x[1]=ex[s];                                     range[1]={s,0,Pi}
x[2]={a-1,0}-a ex[(s-Pi)/a];                    range[2]={s,Pi,Pi(1+a)}
x[3]={2a+b-1,0}-b ex[(Pi(1+a)-s)/b];            range[3]={s,Pi(1+a),Pi(1+a+b)}
x[4]={a+b,0}+(1-a-b) ex[(s-2Pi)/(1-a-b)];       range[4]={s,Pi(1+a+b),2Pi}

(* Plot an example *)
eg={a->.4,b->.3}
pl[i_,d_]:=ParametricPlot[Evaluate[x[i]/.eg],Evaluate[{0,0,-d}+range[i]/.eg]]
Show[Table[pl[i,0],{i,4}],PlotRange->{{-1,1},{-1,1}},AspectRatio->1]

(* Check orientation: replot with intervals slightly clipped on right *)
Show[Table[pl[i,.1],{i,4}],PlotRange->{{-1,1},{-1,1}},AspectRatio->1]

answer[c_]:=Simplify[Sum[Simplify[Integrate[Simplify[D[x[i],s].D[x[i],c]],
        range[i]]],{i,4}]]

Print[answer[a]]
Print[answer[b]]




(* Text output from run:

In[1]:= <<file2

                 2
(1 + b) (4 - Pi )
-----------------
        2
                   2
(-1 + a) (-4 + Pi )
-------------------
         2

*)
```