Interesting & clear paper!

Developments in the Theory of Implicit Hamiltonian Systems

Stephen Prajna
Control and Dynamical Systems
California Institute of Technology

CDS 205 Report

Abstract

This paper provides a survey of recent progress in implicit Hamiltonian systems theory. Relations between Dirac structures and implicit Hamiltonian systems are discussed, and conditions for closedness/integrability are presented. The discussion continues with symmetry and reduction of implicit Hamiltonian systems. An example of an inverted pendulum on a cart illustrates the use of the theory.

1 Introduction

The port-controlled Hamiltonian approach has been proposed in [MS92] as a way for modelling of physical systems. It originates from the network modelling of lumped parameter physical systems with independent storage elements. The notion *port-controlled* indicates that the system's energy is controlled through the ports of the system.

It has been previously shown that a power-conserving interconnection of port-controlled Hamiltonian systems will again yield a port-controlled Hamiltonian system when the energy variables are independent. However, in the case where the energy variables are *dependent*, an *implicit* Hamiltonian system will be obtained. This kind of systems is typically represented by a set of differential and algebraic equations. It includes mechanical systems with constraints and general interconnected electrical L–C circuits. This differential–algebraic representation is analogous to, or can be considered as a special case of, descriptor representations of dynamical systems.

The present paper attempts to provide a very concise survey of progress in modelling and analysis of implicit Hamiltonian systems. Our discussion begins with a geometric view to implicit Hamiltonian systems models. In this context, it is shown that implicit Hamiltonian systems can be modelled by Dirac structures on the space of energy variables. This relation and three representations of implicit Hamiltonian systems are the topic of Section 2. In Section 3, conditions for integrability (or equivalently, conditions for the Dirac structures to

BS99 appeared in J. Geom. & Phys.

be closed) are presented. Symmetry in implicit Hamiltonian systems, as well as some partial reduction results, is the subject of Section 4. Throughout the paper, the theory is illustrated by an example of a pendulum on a cart. This example, albeit simple, is rich enough to cover all the topics discussed in this paper.

Finally, interested readers are referred to [B00] for a detailed presentation of the same topics, as well as some other topics such as full reduction of implicit Hamiltonian systems, implicit port-controlled Hamiltonian systems, and relation between optimal control and implicit Hamiltonian systems.

2 Representations of Implicit Hamiltonian Systems

As mentioned in the introduction, a geometrical view to implicit Hamiltonian systems is provided by Dirac structures [C90], [D93] on the space of energy variables. This was explored in [SM95], [SDM96], [DS99]. Our notations here follow those references.

Let $\mathcal{X}$ be a manifold (which will be the space of energy variables) with tangent bundle $T\mathcal{X}$ and contangent bundle $T^*\mathcal{X}$. Furthermore, let $T\mathcal{X} \oplus T^*\mathcal{X}$ be the smooth vector bundle over $\mathcal{X}$ with $T_x\mathcal{X} \times T_x^*\mathcal{X}$ as the fiber at $x \in \mathcal{X}$, and let $\langle . | . \rangle$ denote the natural pairing between a one-form and a vector field. Then generalized Dirac structures are defined as follows.

Definition 1 *A generalized Dirac structure on $\mathcal{X}$ is a smooth vector subbundle $\mathcal{D} \subset T\mathcal{X} \oplus T^*\mathcal{X}$ such that $\mathcal{D}^\perp = \mathcal{D}$, where*

$$\mathcal{D}^\perp = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \langle \hat{\alpha} | X \rangle + \left\langle \alpha | \hat{X} \right\rangle = 0, \ \ \forall (\hat{X}, \hat{\alpha}) \in \mathcal{D} \}. \tag{1}$$

$\mathcal{D}$ is called a Dirac structure if it is closed (cf. Section 3). In the following discussion, we omit the word 'generalized' for brevity. If the closedness is crucial, it will be explicitly indicated.

An implicit Hamiltonian system that arises from a Dirac structure $\mathcal{D}$ on a manifold $\mathcal{X}$ is given below.

Definition 2 *Let $H : \mathcal{X} \to \mathbb{R}$ be a smooth function (the Hamiltonian), then the implicit Hamiltonian system corresponding to $(\mathcal{X}, \mathcal{D}, H)$ is described by*

$$(\dot{x}, dH) \in \mathcal{D}. \tag{2}$$

Different representations of implicit Hamiltonian systems have been derived in [DS99]. The authors also showed that the fibers of $\mathcal{D}$ has the same dimension as the manifold $\mathcal{X}$. From now on, let us assume that the dimension is n. The first representation is provided by the following theorem. Here we identify the tangent and cotangent space by $\mathbb{R}^n$.

Theorem 3 [Representation I] *Given a Dirac structure $\mathcal{D}$ on a manifold $\mathcal{X}$, at every $x \in \mathcal{X}$ we can find $n \times n$ matrices $E(x)$ and $F(x)$, which depends smoothly on x, such that locally*

$$\mathcal{D}(x) = \{(v, v^*) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \mid F(x)v = E(x)v^*\}, \tag{3}$$

where $E(x)$ and $F(x)$ satisfy the following conditions:

$$\text{rank } [F(x) \mid -E(x)] = n, \tag{4}$$

$$E(x)F^T(x) + F(x)E^T(x) = 0. \tag{5}$$

Conversely, given such matrices $E(x)$ and $F(x)$, Eq. (3) defines a Dirac structure.

Remark 4 *The implicit Hamiltonian system related to this representation is locally described by*

$$F(x)\dot{x} = E(x)\frac{\partial H}{\partial x}(x). \tag{6}$$

Before continuing with the other representations, we need to define the followings. For a Dirac structure $\mathcal{D}$, we have smooth distributions

$$G_0 = \{X \in T\mathcal{X} \mid (X, 0) \in \mathcal{D}\}, \tag{7}$$

$$G_1 = \{X \in T\mathcal{X} \mid \exists \alpha \in T^*\mathcal{X} \text{ such that } (X, \alpha) \in \mathcal{D}\}. \tag{8}$$

as well as smooth codistributions

$$P_0 = \{\alpha \in T^*\mathcal{X} \mid (0, \alpha) \in \mathcal{D}\}, \tag{9}$$

$$P_1 = \{\alpha \in T^*\mathcal{X} \mid \exists X \in T\mathcal{X} \text{ such that } (X, \alpha) \in \mathcal{D}\}. \tag{10}$$

It was proved in [DS99] that

$$G_0 = \ker P_1 \triangleq \{X \in T\mathcal{X} \mid \langle \alpha | X \rangle = 0, \forall \alpha \in P_1\} \tag{11}$$

$$P_0 = \text{ann } G_1 \triangleq \{\alpha \in T^*\mathcal{X} \mid \langle \alpha | X \rangle = 0, \forall X \in G_1\}. \tag{12}$$

Under some constant-dimensionality conditions, we have two other ways of representing implicit Hamiltonian systems.

Theorem 5 [Representation II] *Given a Dirac structure $\mathcal{D}$ on a manifold $\mathcal{X}$ with contant dimensional P_1, we can find a skew-symmetric vector bundle map $J(x) : P_1(x) \to (P_1(x))^*$ which locally can be extended to a skew-symmetric vector bundle map $J(x) : T_x^*\mathcal{X} \to T_x\mathcal{X}$ such that $\mathcal{D}$ is given by*

$$\mathcal{D} = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid X(x) - J(x)\alpha(x) \in G_0(x), \alpha \in \text{ann } G_0\}. \tag{13}$$

Conversely, given a constant dimensional smooth distribution G_0 and such a skew symmetric vector bundle map $J(x) : T_x^*\mathcal{X} \to T_x\mathcal{X}$, then Eq. (13) defines a Dirac structure.

Theorem 6 [Representation III] *Given a Dirac structure $\mathcal{D}$ on a manifold $\mathcal{X}$ with contant dimensional G_1, we can find a skew-symmetric vector bundle map $\omega(x) : G_1(x) \to (G_1(x))^*$ which locally can be extended to a skew-symmetric vector bundle map $\omega(x) : T_x\mathcal{X} \to T_x^*\mathcal{X}$ such that $\mathcal{D}$ is given by*

$$\mathcal{D} = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \alpha(x) - \omega(x)X(x) \in P_0(x),\ X \in \ker P_0\}. \tag{14}$$

Conversely, given a constant dimensional smooth distribution P_0 and such a skew symmetric vector bundle map $\omega(x) : T_x\mathcal{X} \to T_x^*\mathcal{X}$, then Eq. (14) defines a Dirac structure.

Remark 7 *Representation II describes an implicit Hamiltonian system of the following form:*

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x) + g(x)\lambda, \tag{15}$$

$$0 = g^T(x)\frac{\partial H}{\partial x}(x), \tag{16}$$

where $g(x)$ is a full rank matrix such that $\mathrm{Im}\ g(x) = G_0(x)$. *On the other hand, Representation III, which is the dual of Representation II, has the following form:*

$$\frac{\partial H}{\partial x}(x) = \omega(x)\dot{x} + p^T(x)\lambda, \tag{17}$$

$$0 = p(x)\dot{x}, \tag{18}$$

where $p(x)$ is a full rank matrix such that $\mathrm{Im}\ p(x) = P_0(x)$.

We will describe the inverted pendulum example by these representations, and illustrate how to convert from some representations to the others.

Example 8 *Consider the inverted pendulum on a cart depicted on Figure 1. Let x_1, y_1 be the $x - y$ coordinates of the pendulum and x_2 be the x coordinates of the cart (the y coordinates of the cart is not important and is therefore neglected). Furthermore, let p_{1x}, p_{1y} and p_{2x} be the momenta corresponding to those coordinates. Modelling this system as a constrained mechanical system (cf. [MR99], Chapter 8) with the constraint on velocities*

$$(x_1 - x_2)\dot{x}_1 + y_1\dot{y}_1 - (x_1 - x_2)\dot{x}_2 = 0 \tag{19}$$

Holonomic constraint

will yield the following equations of motion:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{p}_{1x} \\ \dot{p}_{1y} \\ \dot{p}_{2x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial y_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial p_{1x}} \\ \frac{\partial H}{\partial p_{1y}} \\ \frac{\partial H}{\partial p_{2x}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ (x_1 - x_2) \\ y_1 \\ -(x_1 - x_2) \end{bmatrix} \lambda, \tag{20}$$

$$\begin{bmatrix} 0 & 0 & 0 & (x_1 - x_2) & y_1 & -(x_1 - x_2) \end{bmatrix} \frac{\partial H}{\partial x} = 0, \tag{21}$$

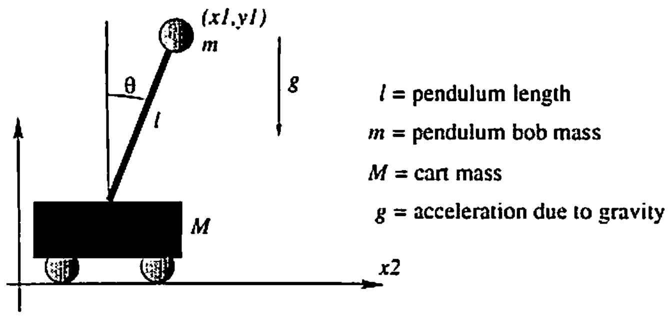

Figure 1: Inverted pendulum on a cart. Figure is taken from [MR99].

where the Hamiltonian is

$$H = \frac{1}{2m}(p_{1x}^2 + p_{1y}^2) + \frac{1}{2M}p_{2x}^2 + mgy_1. \tag{22}$$

Eqs. (20)-(21) are nothing but Representation II of the system. Next, premultiply Eq. (20) by $-J = \begin{bmatrix} 0 & -Id_3 \\ Id_3 & 0 \end{bmatrix}$ *and rearrange the equation to obtain*

$$\begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial y_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial p_{1x}} \\ \frac{\partial H}{\partial p_{1y}} \\ \frac{\partial H}{\partial p_{2x}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{p}_{1x} \\ \dot{p}_{1y} \\ \dot{p}_{2x} \end{bmatrix} + \begin{bmatrix} (x_1 - x_2) \\ y_1 \\ -(x_1 - x_2) \\ 0 \\ 0 \\ 0 \end{bmatrix} \lambda, \tag{23}$$

which, together with the constraint equation (19), constitutes Representation III of the system.

To obtain Representation I, we need to eliminate the constraint force (or Lagrange multipliers) λ *from the equations of motion. For this purpose, we look for a full row rank* 5×6 *matrix that annihilates* $g(x)$ *in Representation II. Such a matrix is for example given by*

$$s(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & (x_1 - x_2) & y_1 \end{bmatrix}. \tag{24}$$

This matrix will always be full row rank because $(x_1 - x_2)$ and y_1 cannot be both equal to zero at the same time. Permultiplying Eq. (20) by $s(x)$ and combining it with Eq. (21), we obtain Representation I of the system.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & x_1 - x_2 & y_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{p}_{1x} \\ \dot{p}_{1y} \\ \dot{p}_{2x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -x_1 + x_2 & -y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 - x_2 & y_1 & -x_1 + x_2 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial y_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial p_{1x}} \\ \frac{\partial H}{\partial p_{1y}} \\ \frac{\partial H}{\partial p_{2x}} \end{bmatrix}. \tag{25}$$

3 Closedness Conditions

Closedness (or integrability) of a Dirac structure is analogous to the Jacobi identity of a Poisson structure or closedness of a symplectic form. This property is important in several aspects. For example, we will see below that a 'Darboux-like' theorem holds if and only the Dirac structure is closed. In this context, [D93] gives the following definition for closedness.

Definition 9 *A Dirac structure $\mathcal{D}$ on a manifold $\mathcal{X}$ is closed (or integrable) if for arbitrary (X_1, α_1), (X_2, α_2), $(X_3, \alpha_3) \in \mathcal{D}$,*

$$\langle \mathcal{L}_{X_1}\alpha_2 | X_3 \rangle + \langle \mathcal{L}_{X_2}\alpha_3 | X_1 \rangle + \langle \mathcal{L}_{X_3}\alpha_1 | X_2 \rangle = 0. \tag{26}$$

An equivalent definition is given by [C90] (this definition is presented as a theorem in [D93]), and this definition is useful to check whether a Dirac structure represented by Representation I is closed.

Definition 10 *A Dirac structure $\mathcal{D}$ on a manifold $\mathcal{X}$ is closed (or integrable) if for arbitrary (X_1, α_1), $(X_2, \alpha_2) \in \mathcal{D}$,*

$$([X_1, X_2], \mathrm{i}_{X_1}\mathrm{d}\alpha_2 - \mathrm{i}_{X_2}\mathrm{d}\alpha_1 + \mathrm{d}\,\langle \alpha_2 | X_1 \rangle) \in \mathcal{D}. \tag{27}$$

In particular, since from Eqs. (3)–(5) it follows that

$$\mathcal{D}(x) = \ker\,[F(x) \mid -E(x)] = \operatorname{im} \begin{bmatrix} E^T(x) \\ -F^T(x) \end{bmatrix}, \tag{28}$$

which implies that the distribution G_1 (the admissible vector fields, cf. Eq. (8)) and codistribution P_1 (the admissible one forms, cf. Eq. (10)) are locally spanned by the columns of $E^T(x)$ and $-F^T(x)$, we have the following test for closedness [DS99] of a Dirac structure represented by Representation I.

Theorem 11 *Consider a Dirac structure $\mathcal{D}$ given locally in Representation I. Define $(X_i, \alpha_i) \in \mathcal{D}$ in local coordinates by*

$$X_i(x) = E_i^T(x), \tag{29}$$

$$\alpha_i(x) = -F_i^T(x), \tag{30}$$

where $E_i^T(x)$ and $-F_i^T(x)$ are the i-th columns of $E^T(x)$ and $F^T(x)$. Then $\mathcal{D}$ is closed iff

$$([X_i, X_j], \mathrm{i}_{X_i}\mathrm{d}\alpha_j - \mathrm{i}_{X_j}\mathrm{d}\alpha_i + \mathrm{d}\langle\alpha_j|X_i\rangle) \in \mathcal{D} \text{ for all } i, j = 1, \ldots, n. \tag{31}$$

The paper [DS99] also gives two other tests for closedness of Representation I and Representation II.

Theorem 12 *Consider a Dirac structure $\mathcal{D}$ given in Representation II. Let $\{.,.\}$ be the (generalized) Poisson bracket asssociated with $J(x)$ and define*

$$\mathcal{A}_D = \{H \in C^\infty(\mathcal{X}) | \mathrm{d}H \in \text{ann } G_0\}. \tag{32}$$

Then $\mathcal{D}$ is closed iff the following three conditions are satisfied.

1. *G_0 is involutive*
2. *$\{H_1, H_2\} \in \mathcal{A}_D$ for any $H_1, H_2 \in \mathcal{A}_D$.*
3. *$\{H_1, \{H_2, H_3\}\} + \{H_2, \{H_3, H_1\}\} + \{H_3, \{H_1, H_2\}\} = 0$ for any $H_1, H_2, H_3 \in \mathcal{A}_D$.*

Theorem 13 *Consider a Dirac structure $\mathcal{D}$ given in Representation II and let $\tilde{\omega}$ be the two-form associated with $\omega(x)$. Then $\mathcal{D}$ is closed iff the following conditions are satisfied.*

1. *ker P_0 is involutive*
2. *$d\tilde{\omega}(X_1, X_2, X_3) = 0$ for all $X_1, X_2, X_3 \in$ ker P_0.*

As mentioned before, when a Dirac structure is closed, a theorem similar to Darboux theorem holds. That is, locally we can find (canonical) coordinates (q, p, r, s), with $q, p \in \mathbb{R}^k$, $r \in \mathbb{R}^l$ and $s \in \mathbb{R}^m$, such that the $J(x)$ in Representation II and G_0 are of the following forms

$$J(x) = \begin{bmatrix} 0 & I_k & 0 & * \\ -I_k & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & * \end{bmatrix}, \tag{33}$$

$$G_0 = \text{span}\left\{\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_m}\right\}. \tag{34}$$

Furthermore, $m = n-$ dim P_1 and $l = n-$ dim G_1.

Remark 14 *The implicit Hamiltonian system corresponding to these canonical coordinates is described by*

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad i = 1, 2, ..., k, \tag{35}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad i = 1, 2, ..., k, \tag{36}$$

$$\dot{r}^i = 0, \qquad i = 1, 2, ..., l, \tag{37}$$

$$0 = \frac{\partial H}{\partial s_i}, \qquad i = 1, 2, ..., m. \tag{38}$$

Let us reconsider the inverted pendulum example. Since the constraint in the system is holonomic, we expect that the Dirac structure is closed.

Example 15 *Consider the inverted pendulum described previously in Example 8. We will show that the Dirac structure associated with this system is closed. Let us use Representation I for this purpose. Define the vector fields X_i and one forms α_i as in Theorem 11. Direct calculations will show that*

$$[X_5, X_6] = y_1 \frac{\partial}{\partial p_{1x}} + (-x_1 + x_2)\frac{\partial}{\partial p_{1y}} - y_1 \frac{\partial}{\partial p_{2x}}, \tag{39}$$

$$[X_6, X_5] = -[X_5, X_6], \tag{40}$$

$$[X_i, X_j] = 0, \quad \textit{otherwise}. \tag{41}$$

Similarly, we have

$$\mathrm{i}_{X_5}\mathrm{d}\alpha_6 - \mathrm{i}_{X_6}\mathrm{d}\alpha_5 + \mathrm{d}\langle\alpha_6|X_5\rangle = y_1(\mathrm{d}x_2 - \mathrm{d}x_1) + (x_1 - x_2)\mathrm{d}y_1, \tag{42}$$

$$\mathrm{i}_{X_6}\mathrm{d}\alpha_5 - \mathrm{i}_{X_5}\mathrm{d}\alpha_6 + \mathrm{d}\langle\alpha_5|X_6\rangle = -y_1(\mathrm{d}x_2 - \mathrm{d}x_1) - (x_1 - x_2)\mathrm{d}y_1, \tag{43}$$

$$\mathrm{i}_{X_i}\mathrm{d}\alpha_j - \mathrm{i}_{X_j}\mathrm{d}\alpha_i + \mathrm{d}\langle\alpha_j|X_i\rangle = 0, \quad \textit{otherwise}. \tag{44}$$

Therefore, we only need to check the conditions in Theorem 11 for $i = 5$, $j = 6$, and $i = 6$, $j = 5$. Substituting $[X_i, X_j]$ into the left hand side of Eq. (31) and $\mathrm{i}_{X_i}\mathrm{d}\alpha_j - \mathrm{i}_{X_j}\mathrm{d}\alpha_i + \mathrm{d}\langle\alpha_j|X_i\rangle$ into the right hand side, we see that both sides match. Therefore

$$([X_i, X_j], \mathrm{i}_{X_i}\mathrm{d}\alpha_j - \mathrm{i}_{X_j}\mathrm{d}\alpha_i + \mathrm{d}\langle\alpha_j|X_i\rangle) \in \mathcal{D} \quad \textit{for } i = 5\ ,\ j = 6;\ \textit{and } i = 6\ ,\ j = 5. \tag{45}$$

For other i and j the conditions will trivially be satisfied, since $(0, 0) \in \mathcal{D}$.

Since the Dirac structure in this example is closed, we can find a set of canonical coordinates for the system, such that in these coordinates the system representation will be similar to the one in Remark 14. For example, we may choose $q^1 = \theta$, $q^2 = x_2$, $r = l$ (cf. Figure 1; we assume that r is the radial coordinate of the pendulum) and their generalized momenta as p_1, p_2, s.

4 Implicit Hamiltonian Systems with Symmetry

Studies of symmetries in Hamiltonian (and also Euler-Lagrange) systems are important, since the existence of such a symmetry implies that we can perform reduction on the dynamics of the system, and whenever needed, we can reconstruct the full-order dynamics from the reduced dynamics. Symmetries in implicit Hamiltonian systems and some (partial) results on reduction of implicit Hamiltonian systems with symmetry have been discussed in [S98]. They are summarized in this section.

Let us begin with infinitesimal symmetry of a Dirac structure. It is defined in [D93] as follows.

Definition 16 *Let $\mathcal{D}$ be a Dirac structure on $\mathcal{X}$. A vector field f on $\mathcal{X}$ is an infinitesimal symmetry of $\mathcal{D}$ if*

$$(L_f X, L_f \alpha) \in \mathcal{D}, \quad \text{for all } (X, \alpha) \in \mathcal{D}. \tag{46}$$

Whereas for a diffeomorphism φ to be a symmetry of a Dirac structure, we have the following definition.

Definition 17 *A diffeomorphism $\varphi : \mathcal{X} \to \mathcal{X}$ is a symmetry of $\mathcal{D}$ if*

$$(\varphi_*^{-1} X, \varphi^* \alpha) \in \mathcal{D}, \quad \text{for all } (X, \alpha) \in \mathcal{D}. \tag{47}$$

A sufficient condition for a class of infinitesimal symmetries is provided by the next proposition.

Proposition 18 *Let $\mathcal{D}$ be a closed Dirac structure, and let f be a vector field on $\mathcal{X}$ for which there exists a smooth function $F : \mathcal{X} \to \mathbb{R}$ such that $(f, dF) \in \mathcal{D}$. Then f is an infinitesimal symmetry of $\mathcal{D}$.*

In the case of explicit Hamiltonian systems, Noether theorem guarantees the existence of conserved quantities in systems with symmetry. Its generalization for implicit Hamiltonian systems exists. However, some regularity conditions are needed. We also need to define some notations.

Assumption 19 *Let $\mathcal{D}$ be a Dirac structure with constant dimensional P_1 and $G_0(x) = \text{span}\{g_1(x), g_2(x), ..., g_m(x)\}$, where $g_1(x), g_2(x), ..., g_m(x)$ are linearly independent. Furthermore, let the $m \times m$ matrix $\left[L_{g_i} L_{g_j} H(x)\right]_{i,j=1,...,m}$ be invertible for all x satisfying $L_{g_j} H(x) = 0$ for $j = 1, ..., m$.*

Remark 20 *This assumption is satisfied e.g. when the J matrix in Representation II is the canonical J matrix (equivalently, when ω in Representation III corresponds to the canonical two-form), and the Hamiltonian is $H(q,p) = \frac{1}{2}p^T G(q) p + V(q)$, where $G(q)$ is positive definite.*

When Assumption 19 holds, the constraint manifold of the system, which is determined by the set of algebraic constraints in the system and defined as

$$\mathcal{X}_c \triangleq \{x \in \mathcal{X} | dH(x) \in P_1(x)\}, \tag{48}$$

will be either empty or a submanifold of $\mathcal{X}$ with codimension m, and will be equal to

$$\mathcal{X}_c = \{x \in \mathcal{X} | g^T(x) \frac{\partial H}{\partial x}(x) = 0\} = \{x \in \mathcal{X} | L_{g_j} H(x) = 0,\ j = 1, \ldots, m\}, \tag{49}$$

where $g(x) = [g_1(x) \ \ldots \ g_m(x)]$. In this case, the implicit Hamiltonian system can be reduced to an explicit Hamiltonian system of the form

$$\dot{x}_c = X_{H_c}(x_c) = J_c(x_c) \frac{\partial H_c}{\partial x_c}(x_c), \tag{50}$$

where $H_c : \mathcal{X}_c \to \mathbb{R}$ is the restriction of H to $\mathcal{X}_c$.

Now we are ready to state the generalization of the Noether theorem. It is given in the following proposition.

Proposition 21 *Let $(\mathcal{X}, \mathcal{D}, H)$ be an implicit Hamiltonian system that satisfies Assumption 19. Let f be a vector field on $\mathcal{X}$ for which there exists a smooth function F such that $(f(x), dF(x)) \in \mathcal{D}(x)$ for all $x \in \mathcal{X}_c$. Furthermore, let f be a symmetry of H, i.e. $L_f H(x) = 0$ for $x \in \mathcal{X}_c$. Then $L_{X_{H_c}} = 0$ on $\mathcal{X}_c$. In other words, F is a conserved quantity for X_{H_c} on $\mathcal{X}_c$.*

The case where the symmetry is a Lie group is also treated in [S98]. In this case, we have a Lie group G acting on $\mathcal{X}$ by diffeomorphisms $\Phi_g : \mathcal{X} \to \mathcal{X}$ and Φ_g is a symmetry of $\mathcal{D}$ for every $g \in G$. Then we have the following result.

Proposition 22 *Let G be a symmetry Lie group of the Dirac structure $\mathcal{D}$ on $\mathcal{X}$, with quotient manifold $\bar{\mathcal{X}} = \mathcal{X}/G$ and smooth projection $\rho : \mathcal{X} \to \bar{\mathcal{X}}$. Then there exists a Dirac structure mathcalD on $\bar{\mathcal{X}}$, defined by $(\bar{X}, \bar{\alpha}) \in \bar{\mathcal{D}}$ if there exist X and α with $\rho_* X = \bar{X}$, $\alpha = \rho^* \alpha$ such that $(X, \alpha) \in \mathcal{D}$. Furthermore, $\bar{\mathcal{D}}$ is closed if $\mathcal{D}$ is.*

The corresponding result on reduction of an implicit Hamiltonian system $(\mathcal{X}, \mathcal{D}, H)$ is as follows.

Proposition 23 *Let $(\mathcal{X}, \mathcal{D}, H)$ be an implicit Hamiltonian system with a symmetry Lie group G, and let $\bar{\mathcal{X}}$, $\bar{\mathcal{D}}$ be as in the previous proposition. In addition, suppose that the action of G leaves H invariant, so that there exists a reduced Hamiltonian $\bar{H} : \bar{\mathcal{X}} \to \mathbb{R}$ with $H = H \circ \rho$. Then $(\mathcal{X}, \mathcal{D}, H)$ reduces to $(\bar{\mathcal{X}}, \bar{\mathcal{D}}, \bar{H})$.*

Example 24 *Consider again the inverted pendulum in Example 8. There exists a symmetry Lie group in this system, corresponding to translations in x_1 and x_2, i.e.,*

$$\Phi_g : (x_1, y_1, x_2, p_{1x}, p_{1y}, p_{2x}) \mapsto (x_1 + k, y_1, x_2 + k, p_{1x}, p_{1y}, p_{2x}). \tag{51}$$

The quotient manifold $\tilde{\mathcal{X}}$ here can be identified by $(y_1, p_{1x}, p_{1y}, p_{2x})$. Since the action of G also leaves H invariant, Proposition 23 holds. The reduced implicit Hamiltonian system is described by (in Representation II)

$$\begin{bmatrix} \dot{y}_1 \\ \dot{p}_{1x} \\ \dot{p}_{1y} \\ \dot{p}_{2x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial y_1} \\ \frac{\partial \tilde{H}}{\partial p_{1x}} \\ \frac{\partial \tilde{H}}{\partial p_{1y}} \\ \frac{\partial \tilde{H}}{\partial p_{2x}} \end{bmatrix} + \begin{bmatrix} 0 \\ (x_1 - x_2) \\ y_1 \\ -(x_1 - x_2) \end{bmatrix} \lambda, \tag{52}$$

$$\begin{bmatrix} 0 & (x_1 - x_2) & y_1 & -(x_1 - x_2) \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial y_1} \\ \frac{\partial \tilde{H}}{\partial p_{1x}} \\ \frac{\partial \tilde{H}}{\partial p_{1y}} \\ \frac{\partial \tilde{H}}{\partial p_{2x}} \end{bmatrix} = 0. \tag{53}$$

where

$$\tilde{H} = \frac{1}{2m}(p_{1x}^2 + p_{1y}^2) + \frac{1}{2M}p_{2x}^2 + mgy_1. \tag{54}$$

5 Concluding Remarks

In the previous sections, some notions related to implicit Hamiltonian systems have been presented. Different representations of implicit Hamiltonian systems have been given, conditions for closedness have been discussed, and finally symmetry and partial reduction of implicit Hamiltonian systems have been studied.

However, in this paper we have not covered the full reduction of implicit Hamiltonian systems with symmetry. This has actually been discussed in [BS99], and is a generalization of the classical reduction theorem of explicit Hamiltonian systems [MR99]. There are two approaches of performing the reduction; each of them consists of two steps. In the first approach, we begin by reducing the dynamics to a level set of the first integrals of the system, whose existence is guaranteed by Noether theorem. The reduced order system will again have some symmetry Lie groups, although they are just subgroups of the original Lie groups. The second step then accounts for reducing the dynamics of the (partially) reduced system to a fully reduced system on the quotient manifold. On the other hand, in the second approach we first reduce the dynamics of the system to the system on the quotient manifold. The reduced system will still have some first integrals, and therefore further reduction can be performed. It is shown in [BS99] that these two approaches are equivalent, in the sense that the reduced systems obtained by these approaches are isomorphic (see [BS99] for the definition).

References

[BS99] G. Blankenstein and A. J. van der Schaft. Symmetry and Reduction in Implicit Generalized Hamiltonian Systems. Memorandum 1489, Faculty of Mathematical Sciences, University of Twente, 1999.

[B00] G. Blankenstein. *Implicit Hamiltonian Systems: Symmetry and Interconnection.* Ph.D. Thesis. Faculty of Mathematical Sciences, University of Twente. 2000.

[C90] T. J. Courant. Dirac manifolds. *Trans. American Math. Soc.*, 319(2):631–661, 1990.

[DS99] M. Dalsmo and A. J. van der Schaft. On representations and integrability of mathematical structures in energy-conserving physical systems. *SIAM J. Control Optim.*, 37(1):54–91, 1999.

[D93] I. Dorfman. *Dirac Structures and Integrability of Nonlinear Evolution Equations.* John Wiley, Chichester, 1993.

[MR99] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry (2ed).* Springer–Verlag, New York, 1999.

[MS92] B. M. Maschke and A. J. van der Schaft. Port-controlled Hamiltonian systems: Modelling origins and system theoretic properties. In *Proc. 2nd IFAC Symp. on Nonlinear Control Systems Design. NOLCOS92*, pages 282–288, Bordeaux, France, 1992.

[SM95] A. J. van der Schaft and B. M. Maschke. The Hamiltonian formulation of energy-conserving physical systems with external ports. *Archive für Elektronik und Übertragungstechnik*, 49(5/6):362-371. 1995.

[SDM96] A. J. van der Schaft, M. Dalsmo, and B. M. Maschke. Mathematical structures in the network representation of energy-conserving physical systems. In *Proc. 35th Conf. Decision and Control*, pages 201–206, Kobe, Japan, 1996.

[S98] A. J. van der Schaft. Implicit Hamiltonian Systems with Symmetry. *Reports on Mathematical Physics*, 41(2):203-221, 1998.