GENERATING FUNCTIONS –

GOLDSTEIN VS MARSDEN

by

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PROJECT
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Abstract :

This project was suggested by Professor Marsden in class. A comparison of the definition and derivation of the generating functions of a contact transformation using the classical approach of Goldstein and the intrinsic approach of Marsden forms the theme of this project. First the Goldstein method is explained and then is interpreted in Marsden's framework.

GENERATING FUNCTIONS : GOLDSTENN AND MARSDEN

In this short note, I will present a comparison of generating functions as described by the classical approach in Goldstein, and the modern approach by Marsden.

Definition: (Goldstein)

A transformation when of phase space consisting of the $2n$ variables (q^i, p_i) to (Q^i, P_i), preserving the Hamiltonian, structure can be specified when half the new coordinates are the same as the old coordinates. The function bridging the two sets of variables is then called the generating function of the transformation.

Definition: (Marsden)

Consider a diffeomorphism

$$\varphi : P_1 \longrightarrow P_2$$

of one symplectic manifold (P_1, Ω_1) to another, (P_2, Ω_2). Let the graph of φ : $\Gamma(\varphi) \subset (P_1 \times P_2)$

Let $i_\varphi : \Gamma(\varphi) \longrightarrow (P_1 \times P_2)$.

Now, choose a form Θ such that

$\Omega = -d\theta$

Then $\quad i_\varphi^* \Omega = d i_\varphi^* \theta = 0$

So, locally, there exists

$$S : J(\varphi) \longrightarrow \mathbb{R}$$

such that

$$i_\varphi^* \theta = dS$$

Such an S is defined to be the generating function of the transformation φ.

Comparison

The elegant formulation in Marsden's approach says exactly the same conditions specified in Goldstein.

$J(\varphi)$ is an isotropic submanifold of $(P_1 \times P_2)$ in fact $J(\varphi)$ has half the dimension of $(P_1 \times P_2)$ and is maximally isotropic. This is a restatement of the requirement that the Generating function be specified when half the new coordinates are same as half the old coordinates.

Moreover, Marsden's formulation makes it very clear that the generating function is intimately related to the newly specified form more than anything else.

This is not clear in Goldstein and many times one wonders how the generating function was structured out of thin air.

Four Main Kinds Of Generating Functions
(Goldstein)

Goldsteins approach to the problem can be systematised in the following manner.

	Old Coordinates	New Independent Coord
1.	q, p	q, Q
2.	q, p	q, P
3.	q, p	Q, p
4	q, p	P, p

The corresponding generating functions are derived in the following manner.

1.
$$p_i \, dq^i - H = P_i \, dQ^i - K + \frac{dF}{dt} \qquad F(q, Q)$$

$$= P_i \, dQ^i - K + \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial Q^i} dQ^i + \frac{\partial F}{\partial t}$$

Noting that q^i, Q^i are independent coordinates.

we immedrately get

$$p_i = \frac{\partial F}{\partial q^i}$$

$$P_i = -\frac{\partial F}{\partial Q^i}$$

2.

$$p_i \, dq^i - H = P_i \, dQ^i - K + \frac{dF}{dt}$$

Goldstein says, the generating function has the form

$$F(q, P) - Q_i P_i$$

We wonder when the $-Q^i P_i$ all of a sudden came from.

$$p_i \, dq^i - H = \cancel{P_i \, dQ^i} - K + \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial P_i} dP_i - \cancel{dQ^i P_i}$$

$$Q^i dP_i + \frac{\partial F}{\partial F}$$

$$p_i \, dq^i - H = -K + \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial P_i} dP_i - Q^i dP_i + \frac{\partial F}{\partial t}$$

We now see

$$p_i = \frac{\partial F}{\partial q^i}$$

$$Q^i = -\frac{\partial F}{\partial p_i}$$

Case 3 (Q, p)

$$p_i dq^i - H = P_i dQ^i - K + \frac{dF}{dt}.$$

The generating function should have the form

$$F(Q, p) + p_i q_i$$

$$p_i \cancel{dq^i} - H = P_i dQ^i - K + \frac{\partial F}{\partial Q^i} dQ^i + \frac{\partial F}{\partial p_i} dp_i + p_i \cancel{dq^i} + dp_i q^i - + \frac{\partial F}{\partial t}$$

$$\therefore -H = P_i dQ^i - K + \frac{\partial F}{\partial Q^i} dQ^i + \frac{\partial F}{\partial p_i} dp_i + dp_i q^i + \frac{\partial F}{\partial t}.$$

This naturally leads to

$$P_i = -\frac{\partial F}{\partial Q^i}$$

$$q^i = -\frac{\partial F}{\partial p_i}$$

Case 4 (P, p)

$$p_i dq^i - H = P_i dQ^i - K + \frac{dF}{dt}.$$

The generating function has the form

$$F = F(p, P) + p_i q^i - P_i Q^i$$

This gives

$$p_i \cancel{dq^i} - H = P_i \cancel{dQ^i} - K + \frac{\partial F}{\partial p_i} dp_i + \frac{\partial F}{\partial P_i} dP_i + p_i \cancel{dq^i} + q^i dp_i - P_i \cancel{dQ^i} - Q^i dP_i$$

Leaving us with

$$q^i = -\frac{\partial F}{\partial p_i}$$

$$Q^i = \frac{\partial F}{\partial P_i}$$

Comments

1. Goldstein also makes remarks that one can combine these generating functions to form new generating functions.

2. He also asserts that a generating function does not have to conform to any one of the four general types and can be a mixture of all four.

3. These remarks cause a great deal of confusion and one is left in the lurch not knowing where these generating functions arise from, and how does one go about looking for a good one.

It is now we pause and look at the same four cases in the light of the modern interpretation given by Marsden.

Let us identify the framework and the fundamental transformations in each case.

<u>Case 1</u> (q, Q)

M described (q^i)

N described (Q^i)

$$\varphi : T^*M \longrightarrow T^*N$$
$$(q^i, p_i) \qquad\qquad (Q^i, P_i)$$

The transformation is described by

$$J : M \times N \longrightarrow T^*M \times T^*N$$
$$(q^i, Q^i) \longrightarrow (p^i, P_i)$$

This means,

$$p_i = F_1(q^i, Q^i)$$
$$P_i = F_2(q^i, Q^i)$$

This implies that

$$J^*(\theta_M - \theta_N) = dF$$

Noting that F is a function of (q, Q) we get

$$p_i \, dq_i - P_i \, dQ_i = \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial Q^i} dQ^i$$

Comparing terms, it is evident that

$$p_i = \frac{\partial F}{\partial q^i}$$

$$P_i = \frac{\partial F}{\partial Q^i}$$

Let us now try to understand what exactly happened

Step 1

We have.

$$\pi_1 : M \times N \longrightarrow M.$$

θ_1 = form on M

Set $\theta_1 = p_i \, dq^i$

Now $\pi_1^* \theta_1$ pulls back θ_1 on M to a form on $M \times N$.

Step 2

We have

$$\Pi_2 : M \times N \longrightarrow N.$$

θ_2 = form on N.

Set $\theta_2 = P_i \, dQ^i$.

Now $\Pi_2^*(\theta_2)$ pulls back the form on N to $M \times N$.

Step 3

Now define

$$\theta = \Pi_1^*(\theta_1) - \Pi_2^*(\theta_2).$$

Note θ is a form on $M \times N$.

Step 4

Define $i_\varphi : J(\varphi) \longrightarrow M \times N$.

(q, p, Q, P).

Choose a set of coordinates, (one from M and one from N to form a basis for $J(\varphi)$.

Step 5

Say we have chosen q, Q.

Step 6

Certainly then F has a parametrization as

$$dF = \left(\frac{\partial F}{\partial q^i}\right) dq^i + \left(\frac{\partial F}{\partial Q^i}\right) dQ^i .$$

Set $dF = i_\varphi^{*}\theta$

Step 7

$$\frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial Q^i} dQ^i = p_i dq^i - P_i dQ .$$

This implies

$$\frac{\partial F}{\partial q^i} = p_i$$

$$\frac{\partial F}{\partial Q^i} = P_i$$

Which is exactly what we derived using Goldstein method .

Now repeating a similar analysis for the case when we choose to parametrize $J(\psi)$ using the coordinates q, P, we get.

$$\frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial P_i} dP_i = p_i dq^i - P_i dQ^i.$$

It is now clear, that for F to be an exact differential, it is necessary that F possess a component that cancels the $p_i dQ$, (the form on N)

$\therefore$ Choose

$$\tilde{F} = F(q, P) - PQ$$

$$\Rightarrow \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial p^i} dP^i - \cancel{P^i dQ^i} - Q^i dP^i = p_i dq^i - \cancel{P_i dQ^i}$$

$$\Rightarrow \quad \frac{\partial F}{\partial q^i} = p_i$$

$$-\frac{\partial F}{\partial p_i} = Q^i$$

It is interesting to see that we can now freely change the forms either on M or on N and still systematically derive the generating function.

This is in sharp contrast to the black box type approach in Goldstein.

But the class notes on generating functions are a bit inadequate, and some further illustrative examples will be helpful.

Conclusions:

For the first time, I actually began to understand where the generating functions came from, how were they related to the forms and how does one find them. There is an intrinsic geometric sense to them that is obscured in the classical formulation but becomes evident in Marsdens framework.

Acknowledgement:

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References

1. Goldstein : Classical Mechanics

2. Marsden : Class notes.