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Noncommutative Integration of Hamiltonian Systems

by

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You have shown a solid mastery of this important work of M&F. Well done! A.
JM

This paper is concerned with the problem of reducing a Hamiltonian System on a Symplectic manifold given a k-dimensional Lie Algebra of functions on M, thinking of them as integrals of a given H.

To be a bit more clear, we have the situation (M^{2n}, ω) and k functions $f_1 \ldots f_k$ that are closed with respect to the Poisson Bracket induced by ω $\{f, g\} = \omega(X_f, X_g)$. The functions satisfy,

$$\{f_i, f_j\} = \sum_{j=1}^{k} C_{ij}^{\xi} f_{\xi} \quad \xi \in \{1, \ldots k\}. \quad C_{ij} \text{ are constants independent of } m \in M.$$

Thus provided $f_1 \ldots f_k$ are independent (to be defined) they form a k-dim Lie Algebra, $\mathfrak{g}$.

Let $\xi \in \mathfrak{g}^*$. Define $h_\xi = \{x \in \mathfrak{g} \mid \xi \circ \mathrm{ad}_x \equiv 0\}$. In other words $x \in h_\xi \iff \langle \xi, \{x, y\} \rangle = 0 \ \forall y \in \mathfrak{g}$. h_ξ is clearly a Linear Subspace of $\mathfrak{g}$.

Define: $\xi \in \mathfrak{g}^*$ is in general position provided h_ξ has smallest possible dimension.

Define: Rank $\mathfrak{g}$ = dim of h_ξ for ξ in general position.

Then, we will prove:

Theorem 1 (M^{2n}, ω) $f_1 \ldots f_k$ as above. Let T^r denote the common nonsingular level surface defined by $f_i(x) = \xi_i$ $\xi = (\xi_1 \ldots \xi_k) \in \mathbb{R}^k$. Assume the $f_i(x)$ are independent in the sense that $\forall m \in T^r$ $df_i(m)$ are k independent cotangent vectors in $T_m T^r$. Suppose the Lie Algebra $\mathfrak{g}$ satisfies:

$$\dim \mathfrak{g} + \mathrm{Rank}\, \mathfrak{g} = \dim M = 2n.$$

Then T^r is a smooth submanifold invariant with respect to every vector field X_h where $h \in h_\xi$, where $\xi = (\xi_1 \ldots \xi_k)$ is a covector of $\mathfrak{g}$.

Furthermore, if T^r is compact and connected, it is then diffeomorphic to an r-Torus on which X_h takes a simple form.

Before proceeding with the proof, I want to give some background on this subject. Given a Hamiltonian system X_H on (M^{2n}, ω) it is natural to attempt to find as many integrals as possible, in order to reduce the dimension of the submanifold on which a given trajectory lies. For example if f, g are 2 independent integrals, $\{H, f\} = \{H, g\} = 0$ Then considering $f(x) = \varepsilon_1 \quad g(x) = \varepsilon_2$ we get 2 Hypersurfaces of dim $2n-1$ which, in the case of general position, intersect transversally, in a submanifold $M^{2n-2}_{\varepsilon_1, \varepsilon_2}$. Since f and g are constant along the orbits of X_H we must have $X_H(x) \in T_x M^{2n-2}_{\varepsilon_1, \varepsilon_2} \quad \forall x \in M_{\varepsilon_1, \varepsilon_2}$. If we can find $2n-1$ such independent integrals, then in principle we can reduce the dynamics to a $2n - (2n-1) = 1$ dimensional submanifold.

Also, note that if f and g are integrals, then so is $\{f, g\}$ since we have $\{H, \{f, g\}\} = \{\{H, f\}, g\} + \{\{H, g\}, f\} = 0 + 0$ by Jacobi's identity. Of course, there is no guarantee that $\{f, g\}$ is independent of f and g.

<u>Liouville's</u> <u>Theorem</u>: Suppose there exists n independent functions $f_1 \ldots f_n$ such that $\{f_i, f_j\} = 0$. let M_ε be a nonsingular level surface defined by $f_i(x) = \varepsilon_i$. Then, this surface is invariant under the flow of any vector field X_f where $f \in \text{Span}\{f_1 \ldots f_n\}$. Furthermore, if M_ε is compact, connected, Then it is diffeomorphic to T^n The n-Torus. Also, There exist symplectic coords. in a nbhd. of T^n s.t.

$\omega = \sum ds_i \wedge d\varphi_i \qquad \varphi_1 \ldots \varphi_n$ coords. on T^n and $s_1 \ldots s_n$ coords. in a transverse direction to T^n. In these coords. The dynamical system has The form $\quad \dot{s}_i = 0 \quad \dot{\varphi}_i = h_i(s_1 \ldots s_n)$.

<u>Remark 1</u>: The commutative n-dimal subalgebra of $C^\infty(M)$ is mapped by $\alpha : f_i \mapsto X_{f_i}$ to n linearly independent vector fields on M_ε. They are commutative since (depending on sign conventions),

is a (Lie) homomorphism. Thus $\alpha\{f_i, f_j\} = [\alpha(f_i), \alpha(f_j)]$.

Remark 2: The symplectic form ω, restricted to M_ξ, is zero since $\omega_x(X_{f_i}, X_{f_j}) = \{f_i, f_j\}(x) = 0$ and $X_{f_i}(x)$ form a basis for $T_x M_\xi$ by the independence condition.

Remark 3: There is a natural symplectic $\mathbb{R}^n$ action on T^n defined via the flows φ_i of each X_{f_i}: define $a(g): T^n \to T^n$ as follows, where $g = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$, $e_1, \dots e_n$ standard basis $\mathbb{R}^n$. $a(g)(t) = \varphi_1^{x_1} \circ \varphi_2^{x_2} \circ \cdots \varphi_n^{x_n}(t)$. Note that the φ_i's commute since the X_f's commute, ~~and so this is an algebra homomorphism.~~ Linear independence of X_{f_i} implies that the action is locally transitive, and hence $a: \mathbb{R}^n \to T^n$ is onto. This implies $\mathbb{R}^n / G_t \cong T^n$ where G_t is the isotropy subgroup of t which is a discrete subgroup of $\mathbb{R}^n$. The action is also symplectic since it acts by the flow~~ing along orbits~~ of Hamiltonian vector fields; each φ_i is symplectic and the composition of symplectic maps is symplectic. The action can be extended to a nbhd of T^n.

Remark 4: Theorem 1 on Noncommutative integrability contains Liouville as a special case.

Proof: Let $\mathfrak{g}$ denote the Lie Algebra spanned by the functions $f_1, \dots f_n$. Then $\mathfrak{g}$ is an abelian Lie Algebra. Let $\xi \in \mathfrak{g}^*$ be a covector in general position so that $\dim H_\xi = \dim\{x \in \mathfrak{g} \mid \xi \circ \mathrm{ad}_x = 0\}$ is minimum. Well since $\forall x$, $\mathrm{ad}_x \equiv 0$ for an abelian $\mathfrak{g}$ we have $H_\xi = \mathfrak{g}$. Thus $\mathrm{Ran}\, \mathfrak{g} = \dim \mathfrak{g} = k$ so the condition $\dim G + \mathrm{rank}\, G = \dim M \Rightarrow k + k = 2n \Rightarrow k = n$. Thus we conclude Liouville's result.

This Remark tells us that we can consider Theorem 1 as the noncommutative version of Liouville. A noncommutative algebra of integrals together w/

$$\dim \mathfrak{g} + \mathrm{Rank}\, \mathfrak{g} = \dim M$$

gives the same conclusion as Liouville.

We now turn to the proof of Theorem 1

Let us look at our general situation and compare/contrast to Liouville.
We have a k-dim. Lie Algebra $\mathfrak{g}$ of integrals and its simply connected Lie Group $\hat{C} = \exp(\mathfrak{g})$. Let M_ε denote the non-degenerate level surface.
How does $\hat{C}$ act on M_ε?

$$\begin{array}{c} \mathfrak{g} \\ \exp \downarrow \\ \hat{C} \end{array} \quad g \in \hat{C} \quad g = \exp(f) \; f \in \mathfrak{g}. \text{ Then } \Phi_g(m) = \varphi_f^1(m) \text{ where } \varphi_f^t \text{ is the flow of the vector field } X_f.$$

Then we have $\frac{d}{dt}\Big|_{t=0} \Phi_{\exp(tf)}(m) = \frac{d}{dt}\varphi_f^t(m) = X_f(m)$.

Thus, the infinitesimal generator $f_M(m) = X_f(m)$ from which we can calculate the Momentum map J for the action.
Lower case j is simply the identity $j(f) = f$. and $\langle J(m), f\rangle = j(f)(m)$
$= f(m)$.

Note that unlike the setting of Liouville, it is not true that M_ε is invariant under the flow of X_f, $f \in \mathfrak{g}$. That is because $\{f_i, f_j\} \neq 0$ in general so that $X_{f_i}[f_j] \neq 0 \Rightarrow f_j$ is not constant along the orbit $\varphi_i^t(m)$, so the value $f_j(\varphi_i^t(m))$ changes. Thus φ_i^t takes us off M_ε.

However, we will consider the subalgebra $h_\varepsilon \subset \mathfrak{g}$ and its corresponding Lie Subgroup $\hat{H}_\varepsilon = \exp(h_\varepsilon)$. We will show that this Lie Subgroup of $\hat{C}$ gives a well defined free action on M_ε. That is $\Phi_h : M_\varepsilon \to M_\varepsilon$ and acts freely. Thus we will be able to consider the quotient $M_\varepsilon / \hat{H}$ which turns out to be a symplectic Manifold.

We now turn to the details: First we translate our situation into the framework of Marsden/Weinstein Symplectic reduction.
First, $J : M \to \mathfrak{g}^*$ our calculation above showed that

$m \xmapsto{J} \bar{m}$ where $\bar{m}(f) = f(m)$ - evaluation of the functions in the Lie Algebra on the point $m \in M$! What could be simpler.

Conversely, take $\xi \in \mathfrak{g}^*$. Consider $J^{-1}(\xi) = \{ m \in M \mid J(m) = \xi \}$
$= \{ m \in M \mid \bar{m}(f) = f(m) = \xi(f) \} = M_\xi$.

If ξ is a regular value of J then $J^{-1}(\xi)$ is a manifold and the $df_i(m)$ are k linearly independent covectors for each $m \in J^{-1}(\xi)$. Thus we have characterized M_ξ in terms of the momentum map.

Proposition 1.1: h_ξ is a Lie subalgebra of $\mathfrak{g}$.

Proof $h_\xi = \{ f \in \mathfrak{g} \mid \xi \circ ad_f = 0 \}$. Suppose $f, g \in h_\xi$. Then $\forall x \in \mathfrak{g}$ we have $\langle \xi, \{\{f,g\},x\} \rangle = \langle \xi, \{f,\{g,x\}\} + \{g,\{f,x\}\} \rangle$
$= \langle \xi, \{f,\{g,x\}\} \rangle + \langle \xi, \{g,\{f,x\}\} \rangle = \xi \circ ad_f \circ ad_g(x) + \xi \circ ad_g \circ ad_f(x)$
$= 0 + 0$.

Proposition 1.2: $f \in h_\xi$ Then $X_f(m) \in T_m M_\xi$

Proof: We unravel the definitions. $f \in h_\xi$ means $\langle \xi, \{f,g\} \rangle = 0 \ \forall g \in \mathfrak{g}$
$\Leftrightarrow \{f,g\}(m) = 0 \ \forall m \in M_\xi \ \forall g \in \mathfrak{g}$. This means all the functions in the algebra $\mathfrak{g}$ are integrals of X_f. Thus the flow of X_f preserves M_ξ
$\Rightarrow X_f(m) \in T_m M_\xi$.

Thus we have a subalgebra $h_\xi \subset \mathfrak{g}$ whose flows preserve M_ξ. $\hat{H}_\xi = \exp(h_\xi)$ is the corresponding Lie subgroup of $\hat{G}$ that acts on M_ξ.
Furthermore since $\hat{H}$ action is induced by X_f $f \in h_\xi$, the action is symplectic since the corresponding flow is a symplectic diffeo.

Note: $\omega|_{M_\xi}$, the restriction of ω to M_ξ. Then consider $(T_m M_\xi)^\perp$
$= \{ v \in T_m M_\xi \mid \omega_m(v,u) = 0 \ \forall u \in T_m M_\xi \}$. Then $(T_m M_\xi)^\perp = \{ X_h(m) \mid h \in h_\xi \}$.
it is clear that $X_h(m)$ is ω perp to $X_f(m)$ $f \in \mathfrak{g}$. since $\omega_m(X_h(m), X_f(m)) = \langle \xi, \{h,f\} \rangle$
$= 0 \ \forall f \in \mathfrak{g}$.

Assume that $\hat{H}$ acts locally freely on M_ξ. This means the isotropy subgroup of $m \in M$, is at most a discrete subgroup $\Gamma \subset \hat{H}$.

Then we have,

<u>Proposition 1.3</u>: The quotient $M_\xi/\hat{H}$ is a manifold w/ nondegenerate symplectic form Ω. Let $p: M_\xi \longrightarrow M_\xi/\hat{H}$ be the quotient map (projection). Then $p^*\Omega = \omega|_{M_\xi}$.

<u>Proof</u>: $\{M_\xi = J^{-1}(\xi)\}$ for ξ regular value. $\hat{H}$ acts locally freely on M_ξ. We can therefore Apply Theorem 1 from Marsden/Weinstein Symplectic reduction to conclude $M_\xi/\hat{H}$ Symplectic, $p^*\Omega = \omega|_{M_\xi}$.

<u>Proposition 1.4</u>. Let $f \in h_\xi$. Then the tangent vector field X_f restricted to M_ξ is left-invariant with respect to $\hat{H}$ action on M_ξ.

<u>Proof</u> We already know $X_f(m) \in T_m M_\xi$. f is invariant under the flow induced from any $f' \in h_\xi$ since $\{f, f'\}(m) = X_{f'}[f](m) = 0$. (in fact for all $f' \in \mathfrak{g}$). But $\hat{H}$ acts on M_ξ via flow of some f' s.t. $\exp(f') = g \in \hat{H}$.

so $\forall g \in \hat{H}$ $\Phi_g^* f(m) = f \circ \Phi_g(m) = f(m)$. Thus $\Phi_g{}^* X_f = X_f$. X_f left inv. $\square$

The above ideas inspire the following:

Suppose F is a fn. on M that is in involution with all of $\mathfrak{g}$. That is $\{F, f\} = 0$ $\forall f \in \mathfrak{g}$. Suppose also that $\hat{C}$ has only one type of Stationary subgroup when it acts in a nbhd of M_ξ. Then we can consider $M/\hat{C}$ and the projection $p: M \longrightarrow M/\hat{C}$. Since F commutes w/ all $f \in \mathfrak{g}$ it is clear that F is invariant under Φ_g $\forall g \in \hat{C}$. Thus F drops to the quotient as $\tilde{F}: M/\hat{C} \longrightarrow \mathbb{R}$. Consider $X_{\tilde{F}}$ the vector field on $M/\hat{C}$ induced by F. Note that any h, function on $M/\hat{C}$ lifts to $h \circ p$ on M automatically $\hat{C}$ invariant so $\{h \circ p, f\} = 0$ $\forall f \in \mathfrak{g}$. Suppose h is also an integral for $X_{\tilde{F}}$ on $M/\hat{C}$. Then $h \circ p$ will be a NEW integral of F on M. Thus, This is a method for constructing new integrals under these conditions.

We need to prove that J is an equivariant momentum map.

J is equivariant provided the diagram:

$$\begin{tikzcd} M \arrow[r, "\Phi_g"] \arrow[d, "J"'] & M \arrow[d, "J"] \\ \mathfrak{g}^* \arrow[r, "Ad^*_{g^{-1}}"'] & \mathfrak{g}^* \end{tikzcd}$$

commutes.

or $J(g \cdot m) = Ad^*_{g^{-1}}(J(m))$. infinitesimal equivariance says

$dJ(f_M(m)) = ad^*_f J(m)$ $f \in \mathfrak{g}$.

Claim: J is infinit. equivariant.

Since $f_M(m) = X_f(m)$ [$f_M(m)$ is infinit. generator of $f \in \mathfrak{g}$ at $m \in M$], we must show $X_f \xmapsto{dJ} ad^*_f(\xi)$ $\xi \in \mathfrak{g}^*$. To do this, let h be a fn. on $\mathfrak{g}^*$. We must show

$ad^*_f(h) \circ J = X_f[h \circ J]$

First, we have $ad^*_f(h)(\xi) = \langle ad^*_f(\xi), dh(\xi) \rangle = \langle \xi, ad_f(dh(\xi)) \rangle = \langle \xi, \{f, dh(\xi)\} \rangle$ where $dh(\xi) \in (\mathfrak{g}^*)^* \cong$. Thus $ad^*_f(h) \circ J(m) = \langle J(m), ad_f(dh(J(m))) \rangle$

$= \langle J(m), \{f, dh(J(m))\} \rangle = \{f, dh(J(m))\}(m)$.

Consider LHS: $X_f[h \circ J]$. First, since $J(m)(f) = f(m)$ it follows that $(dJ(X)(f))(m) = X(f)(m) = Df \cdot X(m)$. Therefore:

$X_f(h \circ J)(m) = \langle X_f(m), dh(J)(m)\, dJ(m) \rangle = \langle dJ(m)(X_f(m)), dh(J(m)) \rangle$

$= X_f(m)[dh(J(m))] = \{f, dh(J(m))\}(m)$ by def. of the bracket.

We Assume $\dim \mathfrak{g} + \mathrm{Rank}\, \mathfrak{g} = 2m$. Thus $\dim h_\xi = 2n - k = \dim M_\xi$.

Thus $\{X_f(m) \mid f \in h_\xi\}$ is a basis for $T_m M_\xi$. Since $\dim h_\xi = \dim M_\xi$

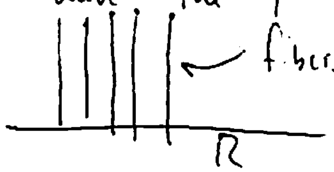
Then we expect, or atleast hope That a neighborhood U of M_ξ Can be chosen so that $U = M_\xi \times R$ where $\dim R = \dim \mathfrak{g} = k$.

We use Ad^* equivariance of J to do this:

Let W be a set of regular values of J containing $\xi \in \mathfrak{g}^*$. J equivariant $\Rightarrow$ W is invariant under Ad^* action of $\hat{G}$. Thus we get a locally trivial fibering $U = J^{-1}(W)$ So That $U \cong M_\xi \times R$.

$$\begin{tikzcd} U = J^{-1}(W) \arrow[d] \\ W \end{tikzcd}$$

We have the picture



Thus in a neighborhood of M_ξ, the momentum map decomposes into a projection $\pi: M_\xi \times R \to R$ and an embedding $\varphi: R \to V^*$.

Note the similarity to Liouville's Theorem - we have a local decomposition of M into $\hat{H}$ invariant small dimensional $(2n-k)$ spaces.

In fact it is possible to show that U can be further decomposed as $X_0 \times Y_0 \times M_\xi$ where X_0 is the orbit space of the $\hat{G}$ action on M

With this decomposition, we have

<u>Proposition 15</u> The symplectic form ω on M restricted to Y_0 coincides w/ the Coadjoint orbit $O(\xi)$ symplectic form.

<u>Proof</u>: $O(\xi) = \{ Ad_g^*(\xi) \mid g \in \hat{G} \} \subset \mathfrak{g}^*$. Let $U_\xi = J^{-1}(O(\xi))$.
Let $X_1 = ad_{f_1}^* \xi \quad X_2 = ad_{f_2}^* \xi \quad f_1, f_2 \in \mathfrak{g}$ so $X_1, X_2 \in T_\xi(O(\xi))$.
Then $\exists!\ Y_1, Y_2 \in T_m U_\xi$ s.t. $dJ(Y_1) = X_1, \quad dJ(Y_2) = X_2$. However, by the infinitesimal equivariance of J, we know $X_f \xmapsto{dJ} ad_f^* \xi$. Therefore,
$Y_1 = X_{f_1}, \ Y_2 = X_{f_2} \Rightarrow \omega_m(Y_1, Y_2) = \omega_m(X_{f_1}, X_{f_2}) = \{f_1, f_2\}(m)$
$= \langle \xi, \{f_1, f_2\} \rangle = \omega_\xi^{coad}(X_1, X_2)$.

We need one key result to complete the proof of Theorem 1.

<u>Lemma 1</u>: Suppose $\xi \in \mathfrak{g}^*$ is in general position meaning that h_ξ has min. dimension. Then the Lie Subalgebra h_ξ has trivial bracket. Thus the Lie Subgroup $\hat{H}_\xi$ is abelian.

<u>proof</u>: The proof is not easy and is found in an article by Duflo and Vergne (1969)

<u>Theorem 1</u>: if M_ξ is compact and connected then $M_\xi \cong T^{2n-k}$ the $2n-k$ Torus

<u>Proof</u>: Since Rank $\mathfrak{g}$ + dim $\mathfrak{g}$ = $2n$ we know $T_m M_\xi = \{ X_h \mid h \in h_\xi \}$
Furthermore, by Lemma 1 H is abelian and acts freely on M_ξ. $\dim M_\xi = \dim H$
... Let Γ be the isotropy subgroup of $m \in M_\xi$. Then Γ is discrete
Thus $M_\xi \cong H/\Gamma$ if M_ξ is compact this must be T^{2n-k}.

the splitting of U, the neighborhood containing T^{2n-k} into $R \times T^{2n-k}$ $\dim R = k$, allows us to write down the differential equation $\dot{x} = X_f(x)$ $f \in h_\xi$ as follows.

Let $r = 2n-k$. Let $\varphi_1, \dots \varphi_r$ be coordinates on T^r let $\xi_1, \xi_2, \dots \xi_k$ be coordinates on R. Then $X_f(x)$ takes the form, in these coords,

$$\dot{\varphi}_i = g_i(\xi_1, \dots \xi_k).$$

Again, this is in agreement w/ Liouville's Theorem.

To conclude, I want to describe a version of Theorem 1 with the added assumption that (M^{2n}, ω) is compact.

<u>Theorem 2</u> Let (M^{2n}, ω) be closed, compact symplectic. Let $\mathfrak{g}$ be a non-commutative algebra of functions satisfying the hypotheses of Thm. 1. That is $\dim \mathfrak{g} + \operatorname{Rank} \mathfrak{g} = 2n$. Then we can find a complete <u>commutative</u> set of functions $\mathfrak{g}_c$ s.t. $2 \dim \mathfrak{g}_c = 2n$

Thus in these circumstances we have reduced the noncommutative case to the Liouville case.

<u>Proof</u>: The idea of the proof is to show that the algebra $\mathfrak{g}$ behaves like a compact Lie Algebra. That is to show there is an invariant inner product $(\,,\,)$ on $\mathfrak{g}$. This means $\forall X, Y, Z \in \mathfrak{g}$ $([X,Y], Z) = (Y, [Z, X])$. From this it follows that $\mathfrak{g} = \mathfrak{C} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_\ell$ where $\mathfrak{C}$ = Center of $\mathfrak{g}$ and $\mathfrak{g}_i$ are distinct simple ideals. From here Theorem 2 is a consequence of [*] where it is proved that noncommutative $\longrightarrow$ commutative if $\mathfrak{g}$ can be split as above.

We define the inner product as follows: $f, g \in C_c^\infty(M)$ smooth fns. w/ compact support

define $(f, g) = \int_M f g \, \omega^n$ ω^n is the volume form on M^{2n}.

This inner product is invariant on the Lie Algebra $(C_0^\infty(M), \{,\})$

*] Mischenko + Fomenko, On the integration of the Euler eqns. on Semisimple Lie Algebras Soviet Math Dokl., 1976

We show if $f, g \in C_0^\infty(M)$, Then $\int_M \{f,g\}\omega^n = 0$

Then $(\,,\,)$ invariant will follow from $\{f, gh\} = g\{f,h\} + h\{f,g\}$ The derivation property of bracket.

to prove $\int_M \{f,g\}\omega^n = 0$ we show $\{f,g\}\omega^n = -d(nf\,dg \wedge \omega^{n-1})$ and Then use Stokes.

Choose symplectic coords $q_1 \ldots q_n, p_1 \ldots p_n$ $\omega = \sum dq_i \wedge dp_i$ Then in These coords,

$$\{f,g\}\omega^n = \left(-\sum_{i=1}^n \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} + \sum \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) n!\; dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 \wedge \cdots \wedge dp_n$$

on The other hand $d(nf\,dg \wedge \omega^{n-1}) = n\,df \wedge dg \wedge \omega^{n-1}$ since $d\omega = 0$

$$= \sum_{i,j} n \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial q_j} dq_i \wedge dq_j \wedge (n-1)! \sum_{j=1}^{n} \bigwedge_{i \neq j} dq_i \wedge dp_i$$

summing over all pairs of $q_1 \ldots q_n, p_1 \ldots p_n$ The only nonzero terms will be from pairs $q_i \leftrightarrow p_i$ and p_i, q_i

So we get $$n!\left(\sum_{i=1}^n \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \sum_{i=1}^n \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) dq_1 \wedge dp_1 \wedge dq_2 \wedge \cdots \wedge dp_n$$

$$= \{f,g\}\omega^n.$$

Since f and g vanish on ∂M by virtue of compact support we get $\int_M \{f,g\}\omega^n = 0$.

$\therefore\ \mathfrak{g}$ is a Lie Algebra with invariant inner product.

Note: The main class of examples of This Theory comes from considering The integrals of a left invariant Hamiltonian on T^*G where G is a Semisimple Lie Group. I regret that I have no time to get into These examples. Perhaps in The student Seminar over The Summer?

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