THE KALUZA THEORY
of PROJECTED FIELDS

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12 May 1989

In 1921 Theodore Kaluza published a paper which unified Einstien's gravitation with electromagnetism. He used a five dimensional metric, four of which described physical space-time. I will discuss Kaluza's theory here in the context of 5-dimensional geometry and reduction to four dimensions.

Assume we have a Riemannian manifold, Z, which is five dimensional. Now define mappings from Z to a four dimensional manifold X, and a one dimensional ($\mathbb{R}$) manifold Y; both are Riemannian manifolds. So for $x \in X$ and $z \in Z$ define

$$x^a = \pi^a(z^\alpha).$$

Similarly for $y \in Y$: $\quad y = \varphi(z^\alpha)$

$$y = \varphi'(z^\alpha)$$

$$\vdots$$

where each φ is an element of a class Φ of mappings from Z to Y.

Sets of elements of Z mapped to a single element of X are called fibers. Sets of Z mapped to a single y are called sections of Z. And a set of sections corresponding to a φ is called a sectioning.

Assume there is a one parameter group of transformations on Z which preserves fibers and sectionings; i.e. $z^{\alpha'} = \gamma^\alpha(z^\beta, \gamma)$

so,
$$\pi^\alpha(\gamma^\alpha(z^\beta, \gamma)) = \pi^\alpha(z^\beta) \qquad (*)$$

and,

$$\varphi(\gamma^\alpha(z^\beta,\gamma)) = \varphi(z^\beta) + \gamma \qquad (\#)$$

where γ is a real parameter that satisfies

$$\gamma^\alpha(\gamma^\beta(z^\alpha,\gamma),\gamma') = \gamma^\alpha(z^\alpha;\gamma+\gamma').$$

We can now introduce a vector field, A^α, which is tangent to the fibers of Z:

$$A^\alpha = \left.\frac{\partial \gamma^\alpha(z^\beta,\gamma)}{\partial \gamma}\right|_{\gamma=0}.$$

A^α satisfies the following (differentiation of (*) and (#) with respect to γ):

$$A^\alpha \frac{\partial \pi^a}{\partial z^\alpha} = 0$$

$$A^\alpha \frac{\partial \varphi}{\partial z^\alpha} = 1$$

and indeed, these five conditions completely determine the A-field.

We are now in a position to introduce projection operators which allow us to "reduce" Z. The projection of a vector V^α parallel to (onto) the fiber direction is $V_{\parallel}{}^\alpha = A^\alpha A_\beta V^\beta$. The projection of V^α normal to the fiber direction is $V_\perp{}^\alpha = \epsilon^\alpha_\beta V^\beta$, where we have defined $\epsilon^\alpha_\beta = \delta^\alpha_\beta - A^\alpha A_\beta$. If we define A^α_a to be the "covariant inverse" of $\frac{\partial \pi^a}{\partial z^\alpha}$, that is

$$A_\alpha A^\alpha_a = 0 \quad \text{and} \quad \frac{\partial \pi^b}{\partial z^\alpha} A^\alpha_a = \delta^b_a, \qquad (\maltese)$$

then we can respecify our projections. $v = A^\alpha V_\alpha$

is the projection onto the fiber direction, and $v_a = A^\alpha_a V_\alpha$ is the projection normal to the fiber direction of V_α. Notice then that $V_\parallel^{\,\alpha} = v A^\alpha$ and $V_\perp^{\,\alpha} = v^a A^\alpha_a$ $\left[v^a = \frac{\partial \pi^a}{\partial z^\alpha} V_\parallel^{\,\alpha}\right]$. Although v^a is defined on Z it is not a vector on Z.

To prove that we have successfully "reduced" a vector on Z to a vector on X we must show that v^a is indeed a vector on X. Let's do a coordinate transformation on X:

$$x^{a'} = x^{a'}(x^b(z^\alpha)) \equiv \pi^{a'}(z^\alpha)$$

then we can say $v^{b'} = \frac{\partial \pi^{b'}}{\partial z^\alpha} V_\parallel^{\,\alpha}$,

and chain rule implies: $v^{b'} = \frac{\partial x^{b'}}{\partial x^a} \frac{\partial \pi^a}{\partial z^\alpha} V_\parallel^{\,\alpha} = \frac{\partial x^{b'}}{\partial x^a} v^a$

by definition of v^a. Thus v^a are defined on X and transform as the components of a vector on X.

Let's see how we can arrive at the metric for space-time. Using the reduction technique just outlined it should be easily derivable from the metric on Z. Let $g_{\alpha\beta}$ be the metric on Z. Assume we have symmetry of the Riemannian structure along fibers; which is equivalent to

$$\underset{A}{\mathcal{L}}\, g_{\alpha\beta} = 0.$$

this condition can also be interpreted to mean that γ acts on Z as a one parameter group of motions. We also assume that

$$g_{\alpha\beta}(z^\gamma) = A_\alpha A_\beta + g_{ab}(\pi^c(z^\gamma)) \frac{\partial \pi^a}{\partial z^\alpha} \frac{\partial \pi^b}{\partial z^\beta} .$$

This makes it fairly obvious that the metric on Z decomposes orthogonally into:

$$g_{\alpha\beta} A^\alpha A^\beta = 1$$

$$g_{\alpha\beta} A^\alpha A^\beta_a = 0$$

$$\text{and} \quad g_{\alpha\beta} A^\alpha_a A^\beta_b = g_{ab}. \qquad (\text{c.f. } (5))$$

Thus we have obtained the metric on X.

Kaluza's theory associates X with physical space-time and g_{ab} is the space-time metric. Hence the electromagnetic field tensor, an object on X, should be expressible as a projection of a similar object on Z. From the vector field, A_α on Z, we form $F_{\alpha\beta}$:

$$F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha} .$$

And $f_{ab}(z^\alpha)$ on X immediately follows:

$$f_{ab} = A^\alpha_a A^\beta_b F_{\alpha\beta} .$$

Having arrived at an acceptable quantity on X we must check that it is indeed a tensor on X. It will suffice to show that $f_{ab}(z^\alpha) = f_{ab}(x^c)$.

By commutativity of Lie and partial differentiation we know

$$\underset{A}{\mathcal{L}} F_{\alpha\beta} = (\underset{A}{\mathcal{L}} A_\alpha)_{,\beta} - (\underset{A}{\mathcal{L}} A_\beta)_{,\alpha}$$

and $\underset{A}{\mathcal{L}} A_\alpha = 0$ for any A_α because the Lie derivative of a vector is zero. Hence $\underset{A}{\mathcal{L}} F_{\alpha\beta} = 0$. We also know that the Lie derivatives of a mixed tensor vanish so $\underset{A}{\mathcal{L}} A^\alpha_a = 0$. So, finally,

$$\underset{A}{\mathcal{L}} f_{ab} = \underset{A}{\mathcal{L}} A^\alpha_a A^\beta_b F_{\alpha\beta} = 0.$$

We can conclude this little proof by invoking a well known theorem (see Torrence + Tulczyjew; appendix therein). The theorem states that for the tensor T defined on Z and unit vector field, tangent to fibers on Z, $A^\alpha_{,}$ then $\underset{A}{\mathcal{L}} T^{a,\dots}_{b\dots} = T^{a,\dots}_{b,\dots;\gamma} A^\gamma$. So in our case this becomes:

$$\underset{A}{\mathcal{L}} F_{\alpha\beta} = \underset{A}{\mathcal{L}} f_{ab,} = f_{ab;\gamma} A^\gamma = 0.$$

Hence f_{ab} is a constant along fibers. It follows directly then that $f_{ab}(z^\alpha) = f_{ab}(\pi^c(z^\alpha)) = f_{ab}(x^c)$! f_{ab} is, in fact, a tensor on X.

This discussion illustrates the applicability of a five dimensional unified theory to four dimensional space-time. Kaluza's theory enjoyed widespread success during the early half of this century. In 1926 Oskar Klein rediscovered Kaluza's theory and his interest in applying it to quantum mechanics lead to the famous Klein-Gordon equation. Given the popularity of 4+n dimensional theories today it is apparent that Kaluza's 1921 paper (which was almost left unsponsered by Albert Einstien) started physicists and mathematicians into an abundant new field.

P.S. How did Kaluza learn to swim? I hear there is quite a story to the answer.

Bibliography

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