The most fascinating thing about Geometric Mechanics is that it brings together seemingly disparate fields such as rigid body dynamics, fluid mechanics, and electromagnetism. Also, a complete understanding of a system cannot be gained by looking at it from one perspective – one needs to consider the system from both the Lagrangian and Hamiltonian points of view. This report attempts to study the rigid body abstractly. Specifically, in the report, the rigid body is explored in the context of Lie algebras and their duals. Equations on the Lie algebras are termed the Euler-Poincaré equations and those on the dual are known as the Lie-Poisson equations. These shall be considered in turn.

LIE-POISSON EQUATIONS

Sophus Lie first introduced the concept of a Poisson manifold and defined a Poisson structure (now called the Lie-Poisson bracket) on the dual of a general Lie algebra. This bracket plays an important role in the Hamiltonian description of the rigid body. Lie's work in this area appears to have been ignored by many important schools, such as Cartan's. However, others, notably Hamel, built upon the foundation that Lie had laid.

Setup

Let G be a Lie group and $\mathfrak{g} = T_e G$ its Lie algebra with the associated bracket. The dual space $\mathfrak{g}^*$ is a Poisson manifold with the brackets

$$\{f, k\}_{\pm}(\mu) = \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\right] \right\rangle ,$$

where the functional derivative $\frac{\delta f}{\delta \mu} \in \mathfrak{g}$ is defined by

$$\left\langle \nu, \frac{\delta f}{\delta \mu} \right\rangle = Df(\mu)\cdot \nu$$

where $\nu \in \mathfrak{g}^*$ and D is the Frechet derivative. The choice of sign is dictated by Lie-Poisson reduction:

$$\lambda : T^*G \to \mathfrak{g}^* \quad \text{defined by} \quad p_g \longmapsto (T_e L_g)^* p_g \in T_e^* G \cong \mathfrak{g}^*$$
$$\rho : T^*G \to \mathfrak{g}^* \quad \text{defined by} \quad p_g \longmapsto (T_e R_g)^* p_g \in T_e^* G \cong \mathfrak{g}^*$$

λ is a Poisson map if one takes the minus Lie-Poisson structure on $\mathfrak{g}^*$ and ρ is a Poisson map if one takes the plus Lie-Poisson structure on $\mathfrak{g}^*$

Rigid body

The Euler equations of motion for a rigid body are

$$\dot{\Pi} = \Pi \times \Omega \quad , \quad \text{where}$$

$\Pi = I\Omega$ is the body angular momentum, I is the moment of inertia tensor and Ω is the body angular velocity. Euler's equations are Hamiltonian relative to the minus Poisson structure. In this case

$G = SO(3)$, $\mathfrak{g} \cong (\mathbb{R}^3, \times)$, and using the standard Euclidean inner product $\mathfrak{g} \cong \mathfrak{g}^*$. The minus Lie-Poisson bracket on $\mathbb{R}^3$ is given by

$$\{f, k\}(\Pi) = -\Pi \cdot (\nabla f \times \nabla k) .$$

For the rigid body, the minus bracket is preferred because the Hamiltonian is left invariant and is mapped by λ to $\mathfrak{g}^*$.

Each position of the rigid body is specified by an Euclidean motion giving the location and orientation of the body. An element $R \in SO(3)$ takes a point X in the reference configuration to a point x in the current configuration, $x = R(X)$. When the rigid body is in motion, R is time dependent and the velocity of a point of the body is $\dot{x} = \dot{R}X = \dot{R}R^{-1}x$. The quantity $\dot{R}R^{-1}$ is skew and hence has associated with it an angular velocity vector, ω . ie.

$$\dot{x} = \dot{R}R^{-1}x = \omega \times x .$$

ω is the spahal angular velocity, the corresponding body angular velocity is defined by $\Omega = R^{-1}\omega$, i.e $R^{-1}\dot{R}v = \Omega \times v$. The kinetic energy is given by

$$K = \frac{1}{2}\int_{B} \rho(X) \, \|\dot{R}X\|^2 \, d^3X .$$

Making use of the fact that $\|\dot{R}X\| = \|\omega \times x\| = \|R^{-1}(\omega \times x)\| = \|\Omega \times x\|$, K can be written as a quadratic function of Ω :

$$K = \frac{1}{2}\Omega^T I \Omega$$

This function is taken to be the Lagrangian of the system on $TSO(3)$. The corresponding Hamiltonian description on $T^*SO(3)$ can be obtained by means of the Legendre transformation. The Hamiltonian in body representation is

$$H(\Pi) = \frac{1}{2}\Pi \cdot (I^{-1}\Pi) .$$

Euler's equations are in fact equivalent to the following equation

$$\dot{f} = \{f, H\} \qquad \forall f \in \mathcal{F}(\mathbb{R}^3) .$$

For this case any function $C: \mathbb{R}^3 \to \mathbb{R}$ of the form $C(\Pi) = \Phi\|\Pi\|^2$ is a Casimir function, i.e $\{C, f\} = 0 \quad \forall f \in \mathcal{F}(\mathbb{R}^3)$. The Casimir function represents a constant of motion for the system. In other words, $\|\pi\|$ is a constant or is conserved. In fact, the Hamiltonian $H = \frac{1}{2}\langle \pi, I^{-1}\pi \rangle$ is another constant of the motion. Due to the conservation of $\|\pi\|$

The Euler equations describe a 2-dimensional dynamical system on an invariant sphere. The curves H = constant are, in general, ellipsoids. The solution curves, the trajectories of the rigid body, lie at the intersection of these two surfaces.

The preceding leads to the following theorem:

<u>Thm</u>: Let G be a Lie group and $H : T^*G \to \mathbb{R}$ be a left invariant Hamiltonian. $h : \mathfrak{g}^* \to \mathbb{R}$ is the restriction of H to the identity. For a curve $p(t) \in T^*_{g(t)}G$, let $\mu(t) = (T^*_{g(t)}G) \cdot p(t) = \lambda(p(t))$ be the induced curve in $\mathfrak{g}^*$. Then, the following are equivalent.

i) $p(t)$ is an integral curve of X_H; i.e. Hamilton's equations on T^*G hold.

ii) For any smooth function, $F \in \mathcal{F}(T^*G)$, $\dot{F} = \{F, H\}$, where $\{\,,\}$ is the canonical bracket on T^*G.

iii) $\mu(t)$ satisfies the Lie-Poisson equations

$$\frac{d\mu}{dt} = ad^*_{\delta h/\delta\mu}\,\mu \qquad \text{where}$$

$ad_\xi : \mathfrak{g} \to \mathfrak{g}$ is defined by $ad_\xi \eta = [\xi, \eta]$ and ad^*_ξ is its dual,

i.e.

$$\dot{\mu}_a = C^d_{ba} \frac{\delta h}{\delta \mu_b} \mu_d$$

iv) For any $f \in \mathcal{F}(\mathfrak{g}^*)$,
$\dot{f} = \{f, h\}$.

EULER - POINCARÉ EQUATIONS

Equations written directly on the Lie algebra are termed the Euler-Poincaré equations. These were first written down by Poincaré in 1901. Before writing the general Euler-Poincaré equations ~~a few~~ the special case of the rigid body ~~preliminaries~~ are presented in the following theorem.

Thm. The curve $R(t) \in SO(3)$ satisfies the Euler-Lagrange equations for $L(R,\dot{R}) = \frac{1}{2}\int_{B} \rho(X)\, \|\dot{R}X\|^2\, d^3X$ iff

$\Omega(t)$ defined by $R^{t}\dot{R}v = \Omega \times v$ for all $v \in \mathbb{R}^3$ satisfies Euler's equations $I\dot{\Omega} = I\Omega \times \Omega$.

Pf

By Hamilton's principle, $R(t)$ satisfies the E-L equations iff $\delta\int L\,dt = 0$. Let $\ell(\Omega) = \frac{1}{2}(I\Omega)\cdot\Omega$, $\ell(\Omega) = L(R,\dot{R})$ if R & Ω are related according to the above theorem. Hamilton's variational principle $\delta\int \ell\, dt = 0$ on $\mathbb{R}^3$ where the variations are of the form $\delta\Omega = \dot{\Sigma} + \Omega\times\Sigma$ with $\Sigma(a) = \Sigma(b) = 0$. ($\Sigma$ is arbitrary otherwise). With this,

$$\delta\int_a^b \ell\, dt = \int_a^b \langle I\Omega, \Omega\rangle\, dt = \int_a^b \langle I\Omega, \dot{\Sigma} + \Omega\times\Sigma\rangle\, dt$$

$$= \int_a^b \left[\left\langle -\frac{d}{dt} I\Omega, \Sigma\right\rangle + \langle I\Omega, \Omega\times\Sigma\rangle\right]$$

$$= \int_a^b \left\langle -\frac{d}{dt} I\Omega + I\Omega\times\Omega, \Sigma\right\rangle dt$$

Hence, $\delta\int \ell\, dt \Leftrightarrow I\dot{\Omega} = I\Omega\times\Omega$

The generalization of the above to an arbitrary Lie group is made in the following theorem :

Thm. Let G be a Lie group and $L : TG \to \mathbb{R}$ a left invariant Lagrangian. Let $l : \mathfrak{g} \to \mathbb{R}$ be its restriction to the identity. For a curve $g(t) \in G$, let $\xi(t) = g(t)^{-1} \cdot \dot{g}(t)$; i.e., ~~g(t)~~
$\xi(t) = T_{g(t)} L^{-1}_{g(t)} \dot{g}(t)$. Then, the following are equivalent

i) $g(t)$ satisfies the E-L eq'ns for L on G

ii) The variational principle $\delta \int L(g(t), \dot{g}(t))\, dt = 0$ holds for variations with fixed end points

iii) The E-P equations hold :
$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = ad^*_{\xi} \frac{\delta l}{\delta \xi}$$

iv) The variational principle $\delta \int l(\xi(t))\, dt = 0$ holds on $\mathfrak{g}$ using the variations of the form
$$\delta \xi = \dot{\eta} + [\xi, \eta] \text{, where } \eta \text{ vanishes at the endpoints}$$

v) conservation of spatial angular momentum holds :
$$\frac{d}{dt}\pi = 0 \quad \text{, where} \quad \pi = Ad^*_{g^{-1}} \frac{\delta l}{\delta \xi}$$

The ~~Lie~~- Poisson equations, $\frac{d\mu}{dt} = ad^*_{\delta h/\delta\mu}\, \mu$ are equivalent to the E-P equations $\frac{d}{dt}\frac{\delta l}{\delta \xi} = ad^*_{\xi} \frac{\delta l}{\delta \xi}$. This can be easily seen using the Legendre transformation, $\mu = \frac{\delta l}{\delta \xi}$, $h(\mu) = \langle \mu, \xi \rangle - l(\xi)$ and $\frac{\delta h}{\delta \mu} = \xi + \langle \mu, \frac{\delta \xi}{\delta \mu} \rangle - \langle \frac{\delta l}{\delta \xi}, \frac{\delta \xi}{\delta \mu} \rangle = \xi$.

<u>Reference</u>

MARSDEN & RATIU, <u>Mechanics & Symmetry</u> . chps 10, 13