very nice report!

Equations of Motion in the Field of a Magnetic Monopole: An Example of Phase-Space Reduction on Cotangent Bundles

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Abstract In this report, I survey some results on the Hamiltonian formulation of equations of motion for a particle in the field of a magnetic monopole. The Hamilton equations are trivial for a divergence-free magnetic field, but it turns out that for the magnetic monopole, the intrinsic formulation can be obtained by regarding the magnetic potential as a connection on a nontrivial fiber bundle over $\mathcal{R}^3 \setminus \{0\}$. This provides a simple but interesting example of phase-space reduction over cotangent bundles as introduced by Marsden and Weinstein.

1. Introduction.

In [2], Dirac considered the concept of magnetic monopoles in an attempt to mathematically remove the existing asymmetry in Maxwell equations between the terms involving the magnetic and electric field. Since the magnetic field of a monopole can not be written globally as the curl of a magnetic vector potential, he chose a vector potential with a string of singularities. However, it was shown much later that a complete and intrinsic description of the vector potential can be found in a natural fashion, thus removing the string singularity considered in the Dirac paper. This achievement was due to the development of the physics of gauge theories.

The field of a magnetic monopole is a special example of what in physics is called the Yang-Mills field. To study the motion of a particle in a Yang-Mills field, the notion of gauge field and gauge potential have been introduced. As pointed out in [10], these physical terms are respectively equivalent to the notion of principal fiber bundle and connections on the fiber bundle used in mathematical terminology. For example the electromagnetism, or rather the electromagnetic potentials in Maxwell's equations, can be regarded as the connection on a bundle with a $U(1) \cong S^1$ group structure, which only in the absence of a magnetic

monopole is equivalent to a trivial $U(1)$ bundle. See [10] for more discussion, see also [3] for some historical and critical discussion of this subject.
In this report, I consider the simplest case of a Yang-Mills field corresponding to a nontrivial principal fiber bundle, which is the case of a single magnetic monopole. As mentioned, the structure group in this case is the Abelian group $U(1) \cong S^1$, the group of complex numbers with unit norm. This Abelian structure results in a considerably simpler analysis, however it should be mentioned that there exists a natural extension of these results to a general Yang-Mills field with non-Abelian group structure.

But first, we shall review the Hamiltonian equations of motion of a particle in a divergence-free magnetic field in $\mathcal{R}^3$:
Let $\mathbf{B}$ denote the magnetic field, then $B = *\mathbf{B}^\flat$ is a 2-form on $\mathcal{R}^3$. The Lorentz law maintains that for a particle with mass m and charge e, its velocity $\dot{x}$ satisfies

$$m\frac{d\dot{x}}{dt} = \frac{e}{c}\dot{x} \times \mathbf{B}.$$

Since B is divergence free, $\mathbf{B} = \nabla \times \mathbf{A}$, or equivalently $B = dA$ for the 1-form $A = \mathbf{A}^\flat$ on $\mathcal{R}^3$. The configuration space here is $T^*\mathcal{R}^3 \cong \mathcal{R}^3 \times \mathcal{R}^3$. With abuse of notation, let the same notation stand for B and its lift to $T^*\mathcal{R}^3$. It is easy to see that the particle motion can be regarded as the flow of the Hamiltonian vector field X_H, corresponding to the conserved quantity

$$H(x,r) := \frac{1}{2m}\|r\|^2 \qquad (= \frac{1}{2}m\|\dot{x}\|^2), \tag{1}$$

with respect to the non-canonical symplectic form

$$\Omega_B = \Omega - \frac{e}{c}B. \tag{2}$$

In other words, $dH = X_H \rfloor \Omega_B$. † Here, $\Omega = dx^i \wedge dr_i$ is the canonical symplectic form on $T^*\mathcal{R}^3$ and $r = m\dot{x}$ by Legendre transform.
Since $B = dA$, one can see that Ω_B is the pull-back of Ω under the fiber translation $t_A : \ r \mapsto p = r + \frac{e}{c}A$. Thus, one can also write $X_H = X_{\widetilde{H}}$, where $d\widetilde{H} = X_{\widetilde{H}} \rfloor \Omega$ and

$$\widetilde{H} := H \circ (t_A)^{-1} = \frac{1}{2m}\|p - \frac{e}{c}A\|^2. \tag{3}$$

Using the Legendre transform, we also get the Lagrangian for this system:

$$L(x,\dot{x}) = \langle p, \dot{x} \rangle - \widetilde{H}(x,p) = \frac{1}{2}m\|\dot{x}\|^2 + \frac{e}{c}\langle A, \dot{x} \rangle. \tag{4}$$

As explained before the same formulation can be obtained if one considers the equations of motion in the principal fiber bundle $(\mathcal{R}^3 \times U(1),\ \mathcal{R}^3,\ \pi,\ U(1))$, where the fiber $U(1) \times \{x\}$

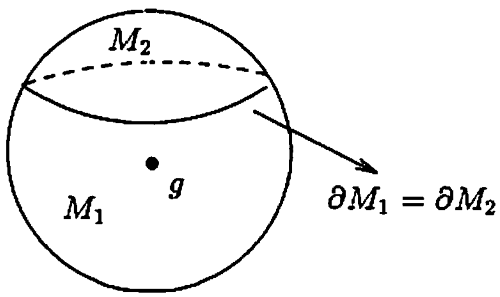
† $X \rfloor \omega := i_X \omega$ where X is any vector field and ω any differential form.

is the set of values which the particle wave function can take at point x. Obviously, the wave function does not affect the equations of motion, which is equivalent to saying that $U(1)$ represents a symmetry group. Then, assuming appropriate connection on the fiber bundle and reducing the phase-space with respect to this symmetry group, one arrives at either of the above Hamiltonian formulations. I defer an explanation of the relationship between these formulations from the point of view of the theory of connections to the next sections.

2. The magnetic Monopole.

Consider a magnetic monopole of strength g at origin in $\mathcal{R}^3$. The generated magnetic field for this monopole is given by

$$\mathbf{B} = g\frac{\mathbf{r}}{\|\mathbf{r}\|^3} \quad \text{where} \quad \mathbf{r} := (x^1, x^2, x^3) \qquad \text{or} \qquad B = *\mathbf{B}^\flat = \sin\theta\, d\theta \wedge d\phi. \tag{5}$$



One may easily show that the above vector field can not be written "globally" as curl of another vector field. To prove this consider the surface of the $S^2 \subset \mathcal{R}^3$ sphere divided in 2 parts M_1, M_2, as in the following figure. Now suppose that one can write $\mathbf{B} = \nabla \times \mathbf{A}$, or equivalently $B = dA$ globally on $\mathcal{R}^3$. Then by Stokes' theorem, we have

$$\int_{M_1} B = \int_{M_1} dA = \int_{\partial M_1} A = \int_{\partial M_2} A = \int_{M_2} B.$$

But as M_1 is enlarged, the left hand side converges to $4\pi g$, while the right hand side converges to 0 which is a contradiction.

However, note that $\nabla \cdot \mathbf{B} \cong dB = 0$ on $\mathcal{R}^3 \setminus \{0\}$. This means that in every simply connected open set U_i in $\mathcal{R}^3 \setminus \{0\}$ one may write $B = dA_i$ for some 1-form A_i. In particular, as shown in [12], one may consider the following open covering of $\mathcal{R}^3 \setminus \{0\}$: Let (r, θ, ϕ) be the spherical coordinates in $\mathcal{R}^3$ where ϕ is the azimuth angle, then

$$\mathcal{R}^3 \setminus \{0\} = U_1 \bigcup U_2 \qquad \text{where} \qquad \begin{cases} U_1 = \{r > 0,\ 0 \leq \theta < \pi - \alpha\}, \\ U_2 = \{r > 0,\ \alpha < \theta \leq \pi\}, \end{cases} \tag{6a}$$

and $0 < \alpha < \frac{\pi}{2}$. Clearly, U_i's are simply connected and there are 1-forms A_i on U_i such that $B = dA_i$. Needless to say, the choice is not unique. For example we may have

$$\begin{cases} A_1 = g(1 - \cos\theta)\, d\phi, & (\mathbf{A}_1 = \frac{g}{r\sin\theta}(1 - \cos\theta)\frac{\partial}{\partial\phi}) \\ A_2 = -g(1 + \cos\theta)\, d\phi, & (\mathbf{A}_2 = \frac{-g}{r\sin\theta}(1 + \cos\theta)\frac{\partial}{\partial\phi}) \end{cases} \tag{6b}$$

where A_1 and A_2 are well-defined 1-forms on U_1 and U_2 respectively. On $U_1 \bigcap U_2$, $dA_1 = dA_2 = B$, thus we must have $d(A_1 - A_2) = 0$. In fact, one can see that $A_1 - A_2 = 2g\, d\phi = d(2g\phi)$ is exact on $U_1 \bigcap U_2$. As we shall see, this condition guarantees that there exists a connection A related to A_1, A_2 on the corresponding principal fiber bundle. (Notice that one can not have $A_1 = A_2$, for example, $A = -\cos\theta\, d\phi$ is only well-defined over $\mathcal{R}^3 \setminus (\{0\} \times \{0\} \times \mathcal{R})$, where $dA = B$.)

Naturally (since Lorentz force law is still valid), the Hamiltonian (1) with respect to the symplectic form Ω_B (2) is still a Hamiltonian for the equations of motion in presence of the magnetic monopole (5). Also as I will show, one may still use the translated Hamiltonian (3) with the canonical symplectic form Ω to generate the vector field corresponding to the magnetic monopole. However, one should notice that in this case, A is not a 1-form on $\mathcal{R}^3 \setminus \{0\}$, but is obtained from a connection (1-form) on $\mathcal{R}^4 \setminus \{0\}$ by passing to the quotient $(\mathcal{R}^4 \setminus \{0\})/S^1$. It can be shown as a 1-form on U_1 or U_2 as given by (6). We shall see that although the configuration space for both Hamiltonian structures is $T^*\mathcal{R}^3 \setminus \{0\}$, but the symplectic manifolds $(T^*\mathcal{R}^3 \setminus \{0\}, \Omega_B)$ and $(T^*\mathcal{R}^3 \setminus \{0\}, \Omega)$ are in fact two different reduced structures which are reduced from $T^*\mathcal{R}^4 \setminus \{0\}$ under the group action $U(1)$!

For a more detailed discussion of the above computations see [11].

3. Reduction on Cotangent Bundles

In this section, I briefly review the main results of [6] and [5] chapters 2 and 3. Let us first introduce some notations:
Let Φ_g denote the left action of a Lie Group G on the manifold Q and $\Phi^*_{g^{-1}} := T^*\Phi_{g^{-1}}$ denote the left lift of this action to the cotangent bundle T^*Q. The coadjoint action on the dual Lie algebra $\mathcal{G}^*$ is denoted by $\mathrm{Ad}^*_{g^{-1}}$ and the symmetry subgroup of G over $\mathcal{G}^*$ at $\mu \in \mathcal{G}^*$ is shown by G_μ (*i.e.* for all $g \in G_\mu$, $\mathrm{Ad}^*_{g^{-1}}\mu = \mu$). The infinitesimal generator of the action $\Phi_{\exp t\xi}$ on Q for every $\xi \in \mathcal{G}$ is the vector field

$$\xi_Q(q) := \frac{d}{dt}\Big|_{t=0} \Phi_{\exp t\xi}(q) \qquad \forall q \in Q.$$

Then the reduction theorem of Marsden and Weinstein [6] on the cotangent bundles as symplectic manifolds simply states that if G is a symmetry group for a Hamiltonian vector field on the cotangent bundle (the flow of the Hamiltonian is invariant under actions of G), then one may reduce the phase space to a quotient space with respect to G. The statement of the theorem is as follows. Note that by the action of G on T^*Q, we mean $\overline{\Phi}_g : (q,p) \mapsto (\Phi_g q, \Phi^*_{g^{-1}} p)$ for all $(p,q) \in T^*Q$:

Theorem 1. (Marsden and Weinstein, 1974)
Let G act freely, properly and symplectically (canonically) on the symplectic manifold

(T^*Q,Ω), i.e. $\overline{\Phi}_g^*\Omega = \Omega$. Let $\mathbf{J} : T*Q \mapsto \mathcal{G}^*$ be an equivariant momentum map corresponding to the symplectic form Ω, that is

$$X_{\langle \mathbf{J},\xi\rangle}(p,q) = \left(\xi_Q(q),\ \frac{d}{dt}\Big|_{t=0} \Phi^*_{\exp -t\xi}(p)\right) \quad \text{and} \quad \mathrm{Ad}^*_{g^{-1}} \circ \mathbf{J} = \mathbf{J} \circ \overline{\Phi}_g,$$

and $\mu \in \mathcal{G}^*$ be any regular value of $\mathcal{G}^*$. Then there exists a unique symplectic form Ω_μ on the reduced phase space $P_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$ such that

$$\pi^*_\mu \Omega_\mu = i^*_\mu \Omega \qquad \text{where} \qquad \begin{cases} i_u : \mathbf{J}^{-1}(\mu) \mapsto T^*Q & \textit{is the inclusion and} \\ \pi_\mu : \mathbf{J}^{-1}(\mu) \longmapsto P_\mu & \textit{is the projection map.} \end{cases} \tag{7}$$

Moreover, if H is a Hamiltonian on T^*Q which is invariant with respect to the action of G, then the flow of X_H on T^*Q induces a flow on P_μ whose Hamiltonian is $\widetilde{H}$ given by $\widetilde{H} \circ \pi_\mu = H \circ i_\mu$.

If Ω is the canonical symplectic form, the standard momentum map corresponding to the canonical action $\overline{\Phi}_g$ is given by

$$\langle \mathbf{J}(q,p),\ \xi\rangle = \langle p,\ \xi_Q(q)\rangle \qquad \forall (p,q) \in T^*Q \quad \text{and} \quad \forall \xi \in \mathcal{G}^*. \tag{8}$$

(It is easy to verify that J is equivariant,:

$$\begin{aligned} \langle \mathbf{J} \circ \Phi_g,\ \xi\rangle &= \langle \Phi^*_{g^{-1}}p,\ \xi_Q \circ \Phi_g\rangle = \langle p,\ T\Phi_{g^{-1}}\xi_Q \circ \Phi_g\rangle \\ &= \langle p,\ \Phi^*_g \xi_Q\rangle = \langle p,\ \left(\mathrm{Ad}_{g^{-1}}\xi\right)_Q\rangle = \langle \mathbf{J},\ \mathrm{Ad}_{g^{-1}}\xi\rangle = \langle \mathrm{Ad}^*_{g^{-1}} \circ \mathbf{J},\ \xi\rangle. \qquad) \end{aligned}$$

Next, I proceed to explain the relationship between the Hamiltonian structure on $\mathbf{J}^{-1}(0)/G$ and the one on $P_\mu = \mathbf{J}^{-1}(\mu)/G$, both obtained by reduction from the Hamiltonian structure on (T^*Q,Ω), through the introduction of a connection $\overline{A}$ on the principal fiber bundle.
Let $X = Q/G$ and $\pi : Q \mapsto X$ be the projection on the quotient, then (Q,X,π,G) is a principal fiber bundle. The projection π, however, does not extend naturally to a map of T^*Q onto T^*X, such a map is given by a connection on the principal fiber bundle.
But if Ω is the canonical symplectic form on T^*Q and $\mathbf{J}$ the corresponding momentum map, one can identify $\mathbf{J}^{-1}(0)/G$ with X. To see this note that by definition, for all $(q,p) \in \mathbf{J}^{-1}(0)$ and $\xi \in \mathcal{G}$, $\langle p,\xi_Q(q)\rangle = 0$. Now, the infinitesimal generators of the G action are tangent to G_x fibers for $x \in X$ and span the tangent space to these fibers. Since every fiber bundle can locally be identified with a trivial bundle, this means that $p \in \mathbf{J}^{-1}(0)$ can be identified with an element of the dual of TX or T^*X.

A connection on a principal bundle leads to a correspondence between the fibers, it defines a horizontal lift of any curve on X to the corresponding fibers. In other words, it splits the tangent space TQ to a vertical space tangent to the fibers (isomorphic to $\mathcal{G}$) and a horizontal space, where the horizontal space is isomorphic to TX, the base tangent bundle. note that without the connection, there is no natural correspondence between TQ and TX.

Following [1], a connection on the principal fiber bundle $\pi : Q \to X$ is defined formally as a linear G-invariant mapping $\sigma_q : T_x X \mapsto T_q Q$ for all $q \in Q$, such that $x = \pi(q) \in X$ and $T\pi \circ \sigma = \mathrm{id}$. Equivalently, a connection is a 1-form $\overline{A}$ on Q with values in $\mathcal{G}$, *i.e.* $\overline{A} : TQ \mapsto \mathcal{G}$, which satisfies

$$\begin{cases} \langle \overline{A},\, \xi_Q \rangle(q) = \xi & \text{for all } \xi \in \mathcal{G},\, q \in Q \quad \text{and} \\ \Phi_g^* \overline{A} = \mathrm{Ad}_{g^{-1}} \circ \overline{A}, & \text{for } \Phi_g \text{ a left action of } G \text{ on } Q. \end{cases} \tag{9a}$$

The definition of $\overline{A}$ naturally extends to an equivalent parametric 1-form A_μ on Q, with parameter $\mu \in \mathcal{G}^*$, according to the following:

$$\langle A_\mu(q),\, v \rangle = \big\langle \mu,\, \langle \overline{A}(q),\, v \rangle \big\rangle \qquad \forall (q, v) \in TQ. \tag{9b}$$

Note that by (9a), we have $\qquad \Phi_g^* A_\mu = A_{(\mathrm{Ad}^*_{g^{-1}} \mu)}$.

Now, consider the mapping $\mathrm{hor}_A : T^*Q \mapsto T^*Q$ such that

$$\mathrm{hor}_A(q, p) = \big(q, p - A_{\mathbf{J}(q,p)}(q)\big)\,. \tag{10}$$

It is easy to check that hor_A maps T^*Q into $\mathbf{J}^{-1}(0)$. We have

$$\langle p - A_{\mathbf{J}(q,p)}(q),\, \xi_Q(q) \rangle = \big\langle \mathbf{J}(q,p),\, \xi - \langle \overline{A},\, \xi_Q \rangle(q) \big\rangle = 0 \qquad \forall \xi \in \mathcal{G}$$

by (8) and (9a-b). Thus, $\mathbf{J}(\mathrm{hor}_A(q,p)) = 0$ for all $(q,p) \in T^*Q$. Now, if 0 and $\mu \in \mathcal{G}^*$ are regular values of $\mathbf{J}$, there exists Hamiltonian structures on the reduced manifolds $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ and $P_0 = \mathbf{J}^{-1}(0)/G \cong T^*X$. Let H_0 and Ω_0 be the corresponding Hamiltonian and symplectic form on P_0. This induces naturally a G-invariant Hamiltonian on $\mathbf{J}^{-1}(0)$, which by abuse of notation, I denote similarly. From (10), one gets the following mapping by restricting hor_A to $\mathbf{J}^{-1}(\mu)$:

$$t_{A_\mu} : \mathbf{J}^{-1}(\mu) \mapsto \mathbf{J}^{-1}(0), \qquad \text{such that} \qquad t_{A_\mu}(q,p) = (q, p - A_\mu(q)). \tag{11}$$

Therefore, the Hamiltonian on $\mathbf{J}^{-1}(0)$ induces the Hamiltonian

$$H_\mu := H_0 \circ t_{A_\mu} \tag{12a}$$

on $\mathbf{J}^{-1}(\mu)$, where $\pi_Q : T^*Q \mapsto Q$. The corresponding Hamiltonian structure on P_μ is obtained by passing to the quotient. Thus, we have the following symplectic form on P_μ where B_μ is the 2-form dA_μ on Q passed to the quotient: (This holds because of the properties of the connection (9a).)

$$\Omega_\mu := \Omega_0 + \pi_X^* B_\mu. \tag{12b}$$

It should be mentioned that $\mathbf{J}^{-1}(\mu)$ and $\mathbf{J}^{-1}(0)$ are not symplectic manifolds. See [5] for a more rigorous discussion.

A special case is when G is Abelian, in that case, $G_\mu = G$ for all $\mu \in \mathcal{G}^*$ and $P_\mu \cong P_0 \cong T^*X$. One example of this case is the Hamiltonian equations in a magnetic field. More generally if $\mu \in \mathcal{G}^*$ is G-invariant, *i.e.* $G_\mu = G$, then according to [4], we have the following result for the above system, where $\mathbf{J}$ is given by (8):

Theorem 2. *Let $\mu \in \mathcal{G}^*$ be G-invariant. Then, $(\mathbf{J}^{-1}(\mu), T^*X, \psi_{A,\mu}, G)$ is a principal fiber bundle, where $\psi_{A,\mu} := \pi_0 \circ t_{A_\mu}$ and $\overline{A}$ is any connection on the principal fiber bundle (Q, X, π, G).*
Moreover, the connection $\overline{A}$ induces a diffeomorphism $\widetilde{\psi}_{A,\mu}$ between the reduced phase space $\mathbf{J}^{-1}(\mu)/G$ and the cotangent bundle T^*X, the former being endowed with the symplectic form $\Omega_0 + \pi_X^* B_\mu$ where Ω_0 is a (canonical) symplectic form on T^*X.

According to this result, we have the following diagram for a G-invariant $\mu \in \mathcal{G}^*$:

$$\begin{tikzcd}
 & \mathbf{J}^{-1}(\mu) \arrow[r, "\pi_\mu"] \arrow[dl, "i"'] \arrow[d, leftrightarrow, "t_{A_\mu}"] & (\mathbf{J}^{-1}(\mu)/G, \Omega_\mu) \arrow[d, leftrightarrow, "\widetilde{\psi}_{A,\mu}"] \\
(T^*Q, \Omega) \arrow[r, "\mathrm{hor}_A"] \arrow[dr, "\pi_Q"'] & \mathbf{J}^{-1}(0) \arrow[r, "\pi_0"] \arrow[d, "\pi_{Q|0}"] & (\mathbf{J}^{-1}(0)/G \cong T^*X, \Omega_0) \arrow[d, "\pi_X"] \\
 & Q \arrow[r, "\pi"] & X
\end{tikzcd} \tag{13}$$

The above discussion indicates that if a reduced Hamiltonian structure on one reduced manifold, say $P_0 \cong T^*X$ is known, then one can find corresponding Hamiltonian and symplectic forms on every reduced manifold P_μ for all $\mu \in \mathcal{G}^*$. Note that in this case, we do not require the knowledge of the Hamiltonian H and symplectic form Ω on all of T^*Q.

4. Hamilton Equations in the Magnetic Monopole Field

Now, I will use the discussion of the last section for the special case of particle motion in presence of a magnetic monopole.

For this system, the "shape space" is $X := Q/G = \mathcal{R}^3 \setminus \{0\}$. Let (x, r) denote the elements of T^*X. As explained in §2, the particle motion is the flow of the Hamiltonian vector field X_{H_0} with respect to the symplectic form $\Omega_0 = \Omega_{\mathrm{can}} - \frac{e}{c}\pi_X^* B$, where $H_0 = \frac{1}{2m}\|r\|^2$, $\pi_X : T^*X \mapsto X$ and B is given by (5).
(6a) gives an open covering of X, $U_1 \bigcup U_2$, such that one can let $B = dA_i$ on each open set. Consider the Abelian group $G = U(1) \cong S^1$ and the trivial local fiber bundles $U_i \times G$. Clearly, there exist principal "circle" fiber bundles (Q, X, π, S^1) whose local trivialization is $U_i \times S^1$. Here, we let the left action of $G = S^1$ on Q be the simple multiplication along

the fibers, *i.e.* in complex representation and local trivialization, $\Phi_{g_0}(x,g) = (x, g_0 g)$ where $g = e^{i\beta}$ for some $\beta \in \mathcal{R}$. Clearly, the corresponding Lie algebras can be identified with the real numbers, $\mathcal{G} \cong \mathcal{G}^* \cong \mathcal{R}$. Also, for this Lie group, $\mathrm{Ad}_g^* = \mathrm{id}$ for all $g \in S^1$.

Let $\Psi_i : \pi^{-1}(U_i) \mapsto U_i \times S^1$. Considering the existing natural homeomorphism between each fiber G_x ($x \in X$) and S^1, we have the following homeomorphism for every $x \in U_1 \bigcap U_2$ induced by Ψ_1, Ψ_2:

$$g_{12}(x) : S^1 \longmapsto G_x \subset U_2 \times S^1 \longmapsto G_x \subset U_1 \times S^1 \longmapsto S^1.$$

Clearly, $g_{12}(x) \in S^1$ for all $x \in U_1 \bigcap U_2$ and can be presented as

$$g_{12}(x) := e^{ih(x)} \qquad \text{for some} \quad h \in \mathcal{F}(U_1 \bigcap U_2). \tag{14}$$

Now, corresponding to any local $\mathcal{G}$–valued 1-forms on U_i's, there exists a connection on Q. Since in our case, $\mathcal{G} \cong \mathcal{R}$, henceforth, we regard connections as regular 1-forms.
Conversely, if $\overline{A}$ is a connection on Q, there exist local 1-forms, A_i, on U_i such that $A_i = f_i^* \overline{A}$, where f_i is a cross section of the bundle over U_i ($f_i : U_i \mapsto \pi^{-1}(U_i)$ and $\pi \circ f_i = \mathrm{id}$). In particular, one may chose the zero cross sections of the bundle, $s_i : U_i \mapsto U_i \times \{1\} \overset{\Psi_i^{-1}}{\longmapsto} \pi^{-1}(U_i)$. Then, it can be shown (see [1]) that if $A_i = s_i^* \overline{A}$, where $\overline{A}$ is any connection that satisfies the properties (9a) with respect to an Abelian group such as S^1, we must have on $U_1 \bigcap U_2$

$$\langle A_2(x),\, v\rangle = \langle A_1(x),\, v\rangle \;+\; \underbrace{g_{12}^{-1}(x)\, D_x g_{12}(x)}_{dh(x)} \cdot v \qquad \forall x \in U_1 \bigcap U_2, \quad \forall v \in T_x X.$$

That is to say $A_1 - A_2$ must be an exact form on $U_1 \bigcap U_2$. Moreover, the function $h(x)$ determines the correspondence between local trivializations of the fiber bundle in their overlap region. Similar results is obtained if different sections are used to lift A_i's to a 1-form $\overline{A}$ on Q.

For A_1 and A_2 given by (6b), $h(x) (= g_{12}^{-1}(x)\, D_x g_{12}(x)) = 2g\phi$ for every $x \in U_1 \bigcap U_2 = \{r > 0,\ \alpha < \theta < \pi - \alpha\}$. This defines uniquely the fiber bundle $\pi : Q \mapsto X = \mathcal{R}^3 \setminus \{0\}$. Specially, since A_1 and A_2 and the local trivializations are independent of the radius r and $\mathcal{R}^3 \setminus \{0\} \cong S^2 \times \mathcal{R}^+$, the fiber bundle is evidently of the form $(\pi', \mathrm{id}) : Q = Q' \times \mathcal{R}^+ \mapsto S^2 \times \mathcal{R}^+$. It can be shown that for proper choice of sections, Q' can be identified with S^3, where $\pi' : S^3 \mapsto S^2 \cong \mathcal{CP}^1$, the complex projective space, is the <u>Hopf Fibration</u>.

Next, we note that if $A_i = f_i^* \overline{A}$ for section f_i, then $dA_i = f_i^* d\overline{A}$ on U_i. Further, for Abelian groups such as S^1, we must have $dA_1 = dA_2$. In our case, of course $dA_1 = dA_2 = B$ which induces $d\overline{A} = \overline{B}$ for some 2-form $\overline{B}$ on T^*Q. I add here that for the S^1 group, $\overline{B}$ represents the curvature of the connection $\overline{A}$.

(If we chose the zero-sections s_i to lift A_1 and A_2 to the connection $\overline{A}$, then we may write $\overline{A}$ in local trivialization $U_k \times S^1$, in terms of A_k as

$$\overline{A}(x,g) = \pi^* A_k(x,g) \; + \; \Phi^*_{g^{-1}}\, dg \qquad \forall (x,g) \in U_k \times S^1, \qquad k = 1,2. \tag{15}$$

Check that $\overline{A}$ satisfies (9a) where for this case, $\xi_Q(x,g) = \xi\, g$ for every $\xi \in \mathcal{R} \cong \mathcal{G}^*$.)

Hence, we have found the principal circle bundle $(Q' \times \mathcal{R}^+, S^2 \times \mathcal{R}^+, \pi, S^1)$ and the connection $\overline{A}$ induced from 1-forms A_1 and A_2 given by (6b). Then it is only straightforward to apply theorem 2. Clearly, the group S^1 acts freely, properly and canonically on the fiber bundle:

Proposition. *On $T^*X = T^*(S^2 \times \mathcal{R}^+)$, the Hamiltonian flow of motion of a particle in the magnetic monopole field (5), is generated by X_{H_0} with respect to the symplectic form Ω_0 ($= \Omega_B$), i.e. equations (1) and (2).*
Let $\mu \in \mathcal{R} \cong \mathcal{G}^*$ be equal to $\frac{e}{c}$, then $A_\mu = \frac{e}{c}\overline{A}$, induced from A_1, A_2 as in (15), defines a connection on the S^1–fiber bundle $T^*(Q' \times \mathcal{R}^+)$. $\frac{e}{c}$ here corresponds to the charge constraint.
Let the $\mathbf{J}$ momentum map on the fiber bundle be given by (8). Then on $P_{e/c} = \mathbf{J}^{-1}(\frac{e}{c})/S^1 \cong T^*(S^2 \times \mathcal{R}^+)$, we have the induced Hamiltonian structure $H_{e/c}$, $\Omega_{e/c}$ such that

$$\begin{cases} H_{e/c} = H_0 \circ \widetilde{\psi}_{A,e/c}\,, \\ \Omega_{e/c} = \Omega_0 + \frac{e}{c}\pi_X^* B = \Omega_{\text{can}} - \frac{e}{c}\pi_X^* B + \frac{e}{c}\pi_X^* B = \Omega_{\text{can}}, \end{cases} \tag{16}$$

where $\widetilde{\psi}_{A,e/c}$ is the projection of $t_{A_{e/c}}$, given by (11), on $T^*(S^2 \times \mathcal{R}^+)$ and by the properties of the connection (9a), B is equal to $\overline{B} = d\overline{A}$ passed to the quotient.
(See also diagram (13).)

I add here that the same procedure can be applied to the case of regular magnetic field where the phase space is the trivial principal fiber bundle $(\mathcal{R}^3 \times S^1, \mathcal{R}^3, \pi, S^1)$ and the connection induces a 1-form on all of $\mathcal{R}^3$. The results were given in the introduction.

- The above derivation was carried out in a somewhat different approach by Sternberg in [7, 8]. In [9], Weinstein explained the relationship between Sternberg's approach and the one discussed before. Sternberg approach is as follows:
  He pulls back T^*X with respect to the mapping $\pi : Q \mapsto X$ (or equivalently, pulls back Q with respect to $\pi_X : T^*X \mapsto X$) to get the manifold $\widetilde{Q}$. $(\widetilde{Q}, T^*X, \widetilde{\pi}, S^1)$ is a principal circle bundle where the action of S^1 on $\widetilde{Q}$ is $\widetilde{\Phi}_g = (\Phi_g, \Phi^*_{g^{-1}})$, with Φ_g simple multiplication by g. Note that $\widetilde{Q} \neq T^*Q$ and is not even generally endowed with a symplectic structure. However, Sternberg proceeds to define a symplectic

structure on $\widetilde{Q}/G \cong T^*X$ by using the connection $\widetilde{A} : T\widetilde{Q} \mapsto \mathcal{R}$, which can be found from local 1-forms $\pi_X^* A_1$ and $\pi_X^* A_2$ in a similar fashion as $\overline{A}$. Also, similar to (9b), one can define $\widetilde{A}_\mu = \mu\widetilde{A} : T\widetilde{Q} \mapsto \mathcal{R}$.
Now, hor_A given by (10) defines a mapping from T^*Q to $\mathbf{J}^{-1}(0)$ which in turn is projected onto T^*X (see diagram (13)). This defines a mapping from T^*Q to T^*X, which by definition of $\widetilde{Q}$ as a pullback of T^*M to P, as well as P to T^*M, induces a connection-dependent mapping ρ_A form T^*Q to $\widetilde{A}$. Then, $\rho_{A,\mu}$, the restriction of ρ_A to $\mathbf{J}^{-1}(\mu)$, introduces a diffeomorphism between $\mathbf{J}^{-1}(\mu)$ and $\widetilde{Q}$.
Therefore, one can approach the $\widetilde{Q}$ space similar to the $\mathbf{J}^{-1}(\mu)$ space for any $\mu \in \mathcal{G}^*$ and obtain similar results. In fact, diagram (13) can be modified for this case to

$$\begin{tikzcd}
 & \mathbf{J}^{-1}(\mu) \arrow[r, "\pi_\mu"] \arrow[dl, "i"'] \arrow[d, leftrightarrow, "\rho_{A,\mu}"] & (\mathbf{J}^{-1}(\mu)/G, \Omega_\mu) \arrow[d, leftrightarrow, "\widetilde{\psi}_{A,\mu}"] \\
(T^*Q, \Omega) \arrow[r, "\rho_A"] \arrow[dr, "\pi_Q"'] & \widetilde{Q} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{\pi}_Q"] & (\widetilde{Q}/G \cong T^*X, \Omega_0) \arrow[d, "\pi_X"] \\
 & Q \arrow[r, "\pi"] & X
\end{tikzcd} \tag{17}$$

This concludes my survey of the magnetic monopole example. This example can be generalized to a general Yang-Mills field with non-Abelian group structure.

At the end, I just mention that there exists a parallel reduction procedure for the Lagrangian description of the system as explained in [5]§3.

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