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Math 189
5.20.95
Term Project

Introduction, Summary & Relevant points Re: this term project

For the purpose of this term project I proposed reading the first few sections from the following paper

> Simo, Marsden & Krishnaprasad
> "The Hamiltonian structure of Non-linear Elasticity: The Material and convective representations of solids, Rods and Plates"
> Archive for Rational mechanics, Vol. 104, No. 2, 1988.

Upon closer examination through numerous read throughs it finally became apparent to me that the paper in question (herein to be referenced by SMK) was beyond my grasp within the time period that could be devoted for this project. It became apparent that much of the paper was based on some fundament concepts that I was either unfamiliar with or uncomfortable about. I therefore seized this opportunity to shore up these "foundations." These fundamental concepts or foundations included, "connections", covariant derivatives, flows, Lie derivatives, objectivity, covariance etc. Using SMK as a guide, the book of Marsden and Hughes 1983 became the crutch. Therein I focused heavily on chapters 1 and 2.

In the course of my travels I came across a few windy and bumpy paths that I tried to straighten and regrade. To save the reader the some effort in reading this report I shall here direct them to what I would consider to be the relavent or "highlights" of this report. There happens to be 3 points.

The first one encountered is what appears to me to be an inconsistency in notation in Marsden and Hughes 1983 concerning Christoffel symbols. The point is considered in detail on pages 14 and 15.

The second point concerns where acceleration vectors live. In SMK the acceleration vectors live in the tangent space but on pages 16-18 I argue that they live in the tangent to the tangent space.

The final "highlight" deals with the definitions or concepts of objectivity and covariance. Through my readings and interpretations I have come to definitions and understandings that differ from Marsden and Hughes. These "differences" are detailed on pages 19 through 23.

In summary, I have concluded that in order to adequately comprehend and appreciate SMK a sufficient familiarity of certain basic concepts is required. In this project I have tried to convey my effort getting acquainted with these concepts. With further effort and a "few" free evenings I believe that the full ramifications of SMK is within grasp.

Configuration Space

Let $(\mathcal{B}, G)$ and $(\mathcal{S}, g)$ be two Riemannian manifolds.

Def: A Riemannian manifold is a manifold equipped with a Riemannian metric.

Def: A Riemannian metric (e.g. G) is a $\binom{0}{2}$-tensor (i.e. covariant) s.t. for each $X \in \mathcal{B}$

i) $G_X(W_1, W_2) = G_X(W_2, W_1)$ for $W_1, W_2 \in T_X\mathcal{B}$
(G is symmetric)

ii) $G_X(W, W) > 0$ for $0 \neq W \in T_X\mathcal{B}$
$G_X(W, W) = 0 \iff 0 = W \in T_X\mathcal{B}$
(G is positive definite)

In other words, G_X is an inner product on $T_X\mathcal{B}$. As a result, the following notation is sometimes used:

$$G_X(W_1, W_2) = \langle W_1, W_2 \rangle_X$$

(see, Marsden & Hughes, p. 68
Kreyszig, p. 129)

More general definitions of Riemannian manifolds and metrics are given in Abraham, Marsden & Ratiu on p. 463 and 287 respectively.

$\mathcal{B}$ = "body". Points in $\mathcal{B}$ are denoted by $X \in \mathcal{B}$.

$\mathcal{S}$ = "ambient space" in which the body moves. Points in $\mathcal{S}$ are denoted $x \in \mathcal{S}$.

Typically, $\mathcal{S} = \mathbb{R}^3$ and g is the standard Euclidean metric.

The configuration space $\mathcal{C}$ (or Q) is the set of orientation preserving embeddings φ

$$\varphi : \mathcal{B} \to \mathcal{S}$$

that map the body $\mathcal{B}$ to the ambient space $\mathcal{S}$.

We write

$$\mathcal{C} = \mathrm{Emb}(\mathcal{B}, \mathcal{S}).$$

The set $\varphi(\mathcal{B})$ is termed the current configuration of the the body $\mathcal{B}$.

The embedding, or mapping, φ, is orientation preserving if both $\mathcal{B}$ and $\varphi(\mathcal{B})$ are orientable with the same orientation.

<u>Def</u>. An n-manifold is orientable if it possesses a nowhere vanishing n-form μ on it.

Connected, orientable manifolds admit precisely two orientations (Marsden & Ratiu, p. 122)

Tangent Space to $\mathcal{C}$ at $\varphi \in \mathcal{C}$

Consider a smooth curve $\varepsilon \mapsto \varphi_\varepsilon$ s.t. $\varphi_\varepsilon|_{\varepsilon=0} = \varphi$.

Thus, by definition $\frac{d}{d\varepsilon}\varphi_\varepsilon(X)\Big|_{\varepsilon=0} \in T_{\varphi(X)}\mathcal{S}$ where $T_{\varphi(X)}\mathcal{S}$ is the tangent space to $\mathcal{S}$ at $\varphi(X)$.

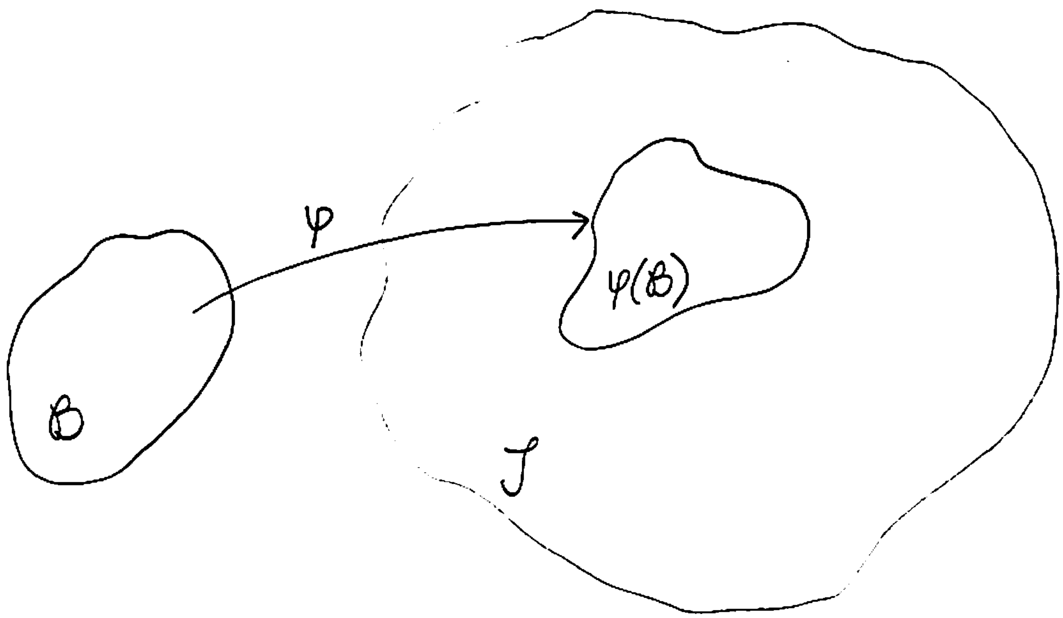
The map

$$X \in \mathcal{B} \mapsto \frac{d}{d\varepsilon}\varphi_\varepsilon(X)\Big|_{\varepsilon=0} \in T_{\varphi(X)}\mathcal{S}$$

is a vector field over $\varphi : \mathcal{B} \to \mathcal{S}$ since for each $X \in \mathcal{B}$ the map assigns a vector in the tangent space $T_{\varphi(X)}\mathcal{S}$.

The tangent space to $\mathcal{C}$ is thus defined by

$$T_\varphi \mathcal{C} = \left\{ V_\varphi : \mathcal{B} \to T\mathcal{S} \;\middle|\; V_\varphi(X) \in T_{\varphi(X)}\mathcal{S} \quad \forall X \in \mathcal{B} \right\}$$



$X \in \mathcal{B}$ $\qquad\qquad$ $x = \varphi(X) \in \varphi(\mathcal{B}) \subset \mathcal{S}$

Kinematics

A motion of the body $\mathcal{B}$ is represented by a curve of mappings $\varphi_t \in \mathcal{C}$. The variable $t \in \mathcal{I} \subset \mathbb{R}$ parameterizes the curve of mappings and is termed time, since we'll let $\varphi_t(\mathcal{B})$ be the configuration of $\mathcal{B}$ at time t.

More specifically, $\varphi_t(X) = \varphi(X,t)$.

<u>Note</u>: In SMK p.129 line 1, replace φ_t with $\varphi_t(\mathcal{B})$

Given a motion φ_t, the following quantities are defined:

(i) material velocity: $\underset{\sim}{V}_t \in T_{\varphi_t}\mathcal{C}$

$$V_t(X) := \frac{\partial}{\partial t}\varphi_t(X) = \frac{\partial}{\partial t}\varphi(X,t)$$

(ii) spatial velocity: $v_t \in T\mathcal{S}$

$\mathcal{X}(\varphi_t(\mathcal{B}))$ = space of vector fields on $\varphi_t(\mathcal{B})$

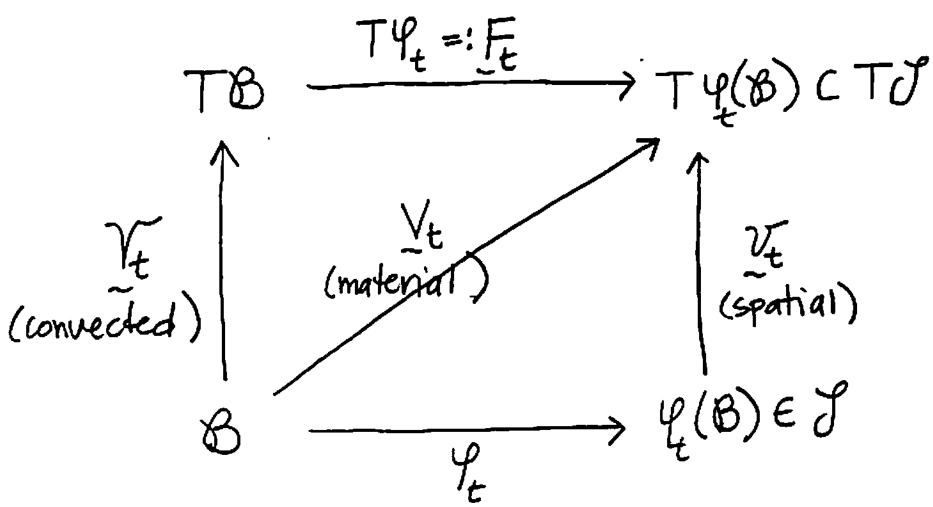
$$v_t(x) := \underset{\sim}{V}_t \circ \varphi_t^{-1}$$

(iii) Convected velocity: $\mathcal{V}_t \in T\mathcal{B}$

$$\mathcal{V}_t(X) = \varphi_t^*(\underset{\sim}{v}_t)$$

$$:= T\varphi_t^{-1} \circ \underset{\sim}{v}_t \circ \varphi_t$$

$$= T\varphi_t^{-1} \circ \underset{\sim}{V}_t$$



Note that there is a potential 4th velocity that could be defined. Namely

$$\mathbb{V}_t(x) = T\varphi_t^{-1} \circ \underset{\sim}{V}_t \circ \varphi_t^{-1} = \underset{\sim}{\mathcal{V}}_t \circ \varphi_t^{-1} = T\varphi_t^{-1} \circ \underset{\sim}{v}_t$$

Proposition 2.1: $\mathcal{V}_t = -\dfrac{\partial \varphi_t^{-1}}{\partial t} \circ \varphi_t$

Proof:

$$X = \varphi_t^{-1} \circ \varphi_t$$

$$0 = \frac{\partial X}{\partial t} = \frac{\partial \varphi_t^{-1}}{\partial t} \circ \varphi_t + \frac{\partial \varphi^{-1}}{\partial x} \circ \frac{\partial \varphi_t}{\partial t}$$

$$= \frac{\partial \varphi_t^{-1}}{\partial t} \circ \varphi_t + T\varphi_t^{-1} \circ V_t$$

$$= \frac{\partial \varphi_t^{-1}}{\partial t} \circ \varphi_t + \mathcal{V}_t$$

$$\therefore \mathcal{V}_t = -\frac{\partial \varphi_t^{-1}}{\partial t} \circ \varphi_t \qquad \square$$

The deformation gradient F_t associated with the motion φ_t is defined to be the tangent map of φ_t.

$$F_t = T\varphi_t$$

This makes sense since the deformation gradient takes "line elements" (i.e. vectors in $T_X\mathcal{B}$) in the ref. conf. or body $\mathcal{B}$ into the corresponding line elements (i.e. vectors in $T_{\varphi(x)}\mathcal{S}$) of the current configuration $\varphi_t(\mathcal{B})$.

Metric and convected metric tensors.

Recall that g is the metric tensor on the ambient space $\mathcal{S}$ in which the body $\mathcal{B}$ moves.

The convected metric is defined by the pull back relation

$$C_t = \varphi_t^*(g)$$

or

$$C_{AB}\, d\underset{\sim}{X}^A \otimes d\underset{\sim}{X}^B = \varphi_t^*\left(g_{ab} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b}\right)$$

$$= g_{ab} \circ \varphi_t \; F^a{}_A d\underset{\sim}{X}^A \otimes F^b{}_B d\underset{\sim}{X}^B$$

$$= F^a{}_A F^b{}_B (g_{ab} \circ \varphi_t)\; d\underset{\sim}{X}^A \otimes d\underset{\sim}{X}^B$$

$$\therefore \quad C_{AB} = F^a{}_A F^b{}_B (g_{ab} \circ \varphi_t)$$

C_t is termed the right Cauchy-Green strain tensor.

If Cartesian coordinates are used for $\mathcal{S}$ then $g_{ab} = \delta_{ab}$. Thus,

$$C_{AB} = F^a{}_A F^b{}_B \,\delta_{ab} = F^a{}_A F^a{}_B$$

or

$$C_t = F_t^T F_t$$

where F_t^T is the transpose of F_t.

Connections

A connection on $\mathcal{B}$ is an operation ∇ that associates to the vector fields $\underset{\sim}{W}$ and $\underset{\sim}{Y}$ on $\mathcal{B}$ a third vector field $\nabla_{\underset{\sim}{W}}\underset{\sim}{Y}$ which is termed the covariant derivative of $\underset{\sim}{Y}$ in the direction $\underset{\sim}{W}$ such that

(i) $\nabla_{\underset{\sim}{W}}\underset{\sim}{Y}$ is linear in W and Y

(ii) $\nabla_{f\underset{\sim}{W}}\underset{\sim}{Y} = f\,\nabla_{\underset{\sim}{W}}\underset{\sim}{Y}$, f = scalar function

(iii) $\nabla_{\underset{\sim}{W}}(f\underset{\sim}{Y}) = f\,\nabla_{\underset{\sim}{W}}\underset{\sim}{Y} + (\underset{\sim}{W}[f])\,\underset{\sim}{Y}$

where

$W[f]$ is the directional derivative of f in the direction W.

$$W[f] = W^A \frac{\partial f}{\partial X^A}$$

The Christoffel symbols Γ^A_{BC} of the connection ∇ on $\mathcal{B}$ are defined (on a coord. system $\{X^A\}$) by

$$\Gamma^A_{BC}(X)\,\underset{\sim}{E}_A(X) = (\nabla_{\underset{\sim}{E}_B}\underset{\sim}{E}_C)(X)$$

Thus

$$\begin{aligned}
\nabla_{\underset{\sim}{W}}\underset{\sim}{Y} &= \nabla_{W^A\underset{\sim}{E}_A} Y^B\underset{\sim}{E}_B \\
&= W^A\,\nabla_{\underset{\sim}{E}_A} Y^B\underset{\sim}{E}_B \qquad \text{by (ii)} \\
&= W^A\left(Y^B\,\nabla_{\underset{\sim}{E}_A}\underset{\sim}{E}_B + \underset{\sim}{E}_A[Y^B]\,\underset{\sim}{E}_B\right) \qquad \text{by (iii)} \\
&= W^A\left(Y^B\,\Gamma^C_{AB}\,\underset{\sim}{E}_C + \frac{\partial Y^B}{\partial X^A}\,\underset{\sim}{E}_B\right) \\
&= W^A\left(\Gamma^C_{AB}\,Y^B + \frac{\partial Y^C}{\partial X^A}\right)\underset{\sim}{E}_C
\end{aligned}$$

Typically it is denoted that

$$Y^C|_A = \frac{\partial Y^C}{\partial X^A} + \Gamma^C_{AB} Y^B$$

Thus,

$$\nabla_{\underset{\sim}{W}} \underset{\sim}{Y} = Y^C|_A W^A \underset{\sim}{E}_C$$

Let ∇ be the Levi-Civita connection associated with the spatial Riemannian metric g. γ^a_{bc} are the associated Christoffel symbols

$$g_{ab} = \langle \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle = \left\langle \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right\rangle$$

In Euclidean coords $g_{ab} = \delta_{ij}$ thus

$$g_{ab}(x^c) = \frac{\partial z^i}{\partial x^a} \frac{\partial z^j}{\partial x^b} \delta_{ij} = \frac{\partial z^i}{\partial x^a} \frac{\partial z^i}{\partial x^b}$$

Taking the deriv.

$$\frac{\partial g_{ab}}{\partial x^c} = \frac{\partial z^i}{\partial x^a} \frac{\partial^2 z^i}{\partial x^c \partial x^b} + \frac{\partial^2 z^i}{\partial x^c \partial x^a} \frac{\partial z^i}{\partial x^b}$$

$$= \frac{\partial z^i}{\partial x^a} \delta^i_j \frac{\partial^2 z^j}{\partial x^c \partial x^b} + \delta^i_j \frac{\partial^2 z^j}{\partial x^c \partial x^a} \frac{\partial z^i}{\partial x^b}$$

$$= \underbrace{\frac{\partial z^i}{\partial x^a} \frac{\partial z^i}{\partial x^d}}_{g_{ad}} \underbrace{\frac{\partial x^d}{\partial z^j} \frac{\partial^2 z^j}{\partial x^c \partial x^b}}_{\gamma^d_{cb}} + \frac{\partial z^i}{\partial x^d} \underbrace{\frac{\partial x^d}{\partial z^j} \frac{\partial^2 z^j}{\partial x^c \partial x^a}}_{\gamma^d_{ca}} \frac{\partial z^i}{\partial x^b}$$

(with $\frac{\partial z^i}{\partial x^d}$ and $\frac{\partial z^i}{\partial x^b}$ linked by an arrow labelled g_{db})

$$= g_{ad} \gamma^d_{cb} + g_{db} \gamma^d_{ca} \qquad (1)$$

or

$$\frac{\partial g_{cb}}{\partial x^a} = g_{cd} \gamma^d_{ab} + g_{db} \gamma^d_{ac} \qquad (2)$$

$$\frac{\partial g_{ac}}{\partial x^b} = g_{ad} \gamma^d_{bc} + g_{dc} \gamma^d_{ab} \qquad (3)$$

Adding (1) and (2)

$$\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{cb}}{\partial x^a} = g_{ad}\gamma^d_{cb} + g_{db}\gamma^d_{ca} + g_{cd}\gamma^d_{ab} + g_{db}\gamma^d_{ac}$$

Recall that Riemannian metrics are symm (i.e. $g_{ab} = g_{ba}$) and for torsion-less connections $\gamma^d_{ab} = \gamma^d_{ba}$.

$$= 2g_{db}\gamma^d_{ac} + g_{ad}\gamma^d_{cb} + g_{cd}\gamma^d_{ab} \qquad (4)$$

Now subtract (3) from (4)

$$\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{cb}}{\partial x^a} - \frac{\partial g_{ac}}{\partial x^b} = 2g_{db}\gamma^d_{ac} + \cancel{g_{ad}\gamma^d_{cb}} + \cancel{g_{cd}\gamma^d_{ab}} - \cancel{g_{ad}\gamma^d_{bc}} - \cancel{g_{dc}\gamma^d_{ab}}$$

$$= 2g_{db}\gamma^d_{ac}$$

Or $$2g_{db}\gamma^d_{ac} = \frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{cb}}{\partial x^a} - \frac{\partial g_{ac}}{\partial x^b}$$

which is equivalent to equation (2.11)

The Riemannian connection $\tilde{\nabla}$ associated with the convected metric $C_t = \varphi_t^* g$ is defined by

$$\tilde{\nabla}_{\underset{\sim}{W}} \underset{\sim}{Y} := \varphi_t^*\left(\nabla_{\varphi_{t*}\underset{\sim}{W}}\, \varphi_{t*}\underset{\sim}{Y}\right)$$

for the convected vector fields $\underset{\sim}{W}$ and $\underset{\sim}{Y} \in \mathcal{X}(\mathcal{B})$

We now derive the component expression for $\tilde{\nabla}_{\underset{\sim}{W}} \underset{\sim}{Y}$ similarly to how we did $\nabla_{\underset{\sim}{W}} \underset{\sim}{Y}$

$$\tilde{\nabla}_{\tilde{W}} \underset{\sim}{V} := \varphi_t^* \left(\nabla_{\varphi_{t*} W} \varphi_{t*} \underset{\sim}{V} \right)$$

$$= \varphi_t^* \left(\nabla_{F^a_A W^A \underset{\sim}{e}_a} F^b_B V^B \underset{\sim}{e}_b \right)$$

$$= \varphi_t^* \left(F^a_A W^A \, \nabla_{\underset{\sim}{e}_a} F^b_B V^B \underset{\sim}{e}_b \right)$$

$$= \varphi_t^* \left\{ F^a_A W^A \left\{ F^b_B V^B \, \nabla_{\underset{\sim}{e}_a} \underset{\sim}{e}_b + \underset{\sim}{e}_a \left[F^b_B V^B \right] \underset{\sim}{e}_b \right\} \right\}$$

$$= \varphi_t^* \left\{ F^a_A W^A \left\{ F^b_B V^B \gamma^c_{ab} \underset{\sim}{e}_c + \left\{ \frac{\partial^2 \varphi_t^b}{\partial X^B \partial X^C} \frac{\partial X^C}{\partial x^a} V^B + F^b_B \frac{\partial V^B}{\partial X^C} \frac{\partial X^C}{\partial x^a} \right\} \underset{\sim}{e}_b \right\} \right\}$$

$$= \varphi_t^* \left\{ F^a_A F^b_B \gamma^c_{ab} W^A V^B \underset{\sim}{e}_c + F^a_A \frac{\partial X^C}{\partial x^a} \frac{\partial^2 \varphi_t^b}{\partial X^B \partial X^C} W^A V^B \underset{\sim}{e}_b + F^a_A \frac{\partial X^C}{\partial x^a} \frac{\partial V^B}{\partial X^C} F^b_B W^A \underset{\sim}{e}_b \right\}$$

Note: $F^a_A \dfrac{\partial X^C}{\partial x^a} = \dfrac{\partial x^a}{\partial X^A} \dfrac{\partial X^C}{\partial x^a} = \dfrac{\partial X^C}{\partial X^A} = \delta^C_A$

$$= \varphi_t^* \left\{ F^a_A F^b_B \gamma^c_{ab} W^A V^B \underset{\sim}{e}_c + \frac{\partial^2 \varphi_t^c}{\partial X^B \partial X^A} W^A V^B \underset{\sim}{e}_c + \frac{\partial V^B}{\partial X^A} F^c_B W^A \underset{\sim}{e}_c \right\}$$

$$= \left[F^a_A F^b_B \gamma^c_{ab} (F^{-1})^C_c + \frac{\partial^2 \varphi_t^c}{\partial X^B \partial X^A} (F^{-1})^C_c \right] W^A V^B \underset{\sim}{E}_C + \frac{\partial V^B}{\partial X^A} W^A \underbrace{F^c_B (F^{-1})^C_c}_{\delta^C_B} \underset{\sim}{E}_C$$

$$= \left[\frac{\partial V^C}{\partial X^A} + \left\{ F^a{}_A F^b{}_B \gamma^c_{ab} (F^{-1})^C{}_c + \frac{\partial^2 \varphi^c_t}{\partial X^B \partial X^A} (F^{-1})^C{}_c \right\} V^B \right] W^A \underset{\sim}{E}_C$$

Thus,

$$\Gamma^C_{AB} = F^a_A F^b{}_B \gamma^c_{ab} (F^{-1})^C{}_c + \frac{\partial^2 \varphi^c_t}{\partial X^B \partial X^A} (F^{-1})^C{}_c$$

These are the Christoffel symbols for the $\tilde{\nabla}$ connection associated with the Convected metric C_t.

This equation also expresses how the Christoffel symbols transform under a change of coordinate system. Due to the last term, i.e.,

$$\frac{\partial^2 \varphi^c_t}{\partial X^B \partial X^A} (F^{-1})^C{}_c$$

it is clear that the Christoffel symbols are <u>NOT</u> tensors.

The above equation is equation (2.13)

To derive eqn (2.14) let $\mathcal{S}$ have Cartesian coord system

Then

$$C_{AB} = F^c_A F^c_B$$

The proof is going to follow exactly that for γ^e_{ab} thus

$$2 C_{DB} \Gamma^D_{AC} = \frac{\partial C_{AB}}{\partial X^C} + \frac{\partial C_{CB}}{\partial X^A} - \frac{\partial C_{AC}}{\partial X^B}$$

which is eqn (2.14).

Description of possible inconsistency in Notation convention in Marsden and Hughes (1983)

In this short aside I present what I believe to be an inconsistency in notation in Marsden and Hughes concerning the Christoffel symbols. I came across this while studying connections, covariant derivative and the Christoffel symbols.

We may begin on p. 31 where the Christoffel symbols are defined as

$$\gamma^c_{ab} = \frac{\partial^2 z^i}{\partial x^a \partial x^b} \frac{\partial x^c}{\partial z^i}$$

Then on p. 32 the coordinate expression for $\nabla_{\underset{\sim}{w}} \underset{\sim}{v}$ is derived as

$$\nabla_{\underset{\sim}{w}} \underset{\sim}{v} = \Big[\underbrace{\frac{\partial v^a}{\partial x^b} + \gamma^a_{bc} v^c}_{= v^a|_b} \Big] w^b \underset{\sim}{e}_a$$

Note that in the term containing the Christoffel symbol γ^a_{bc}, v^c is contracted with the second lower index of γ^a_{bc}.

I have duplicated the proof presented in Marsden and Hughes for the above eqn. and I agree with its form given the definition for γ^c_{ab}. (My proof is not included)

A torsion-less connection, gives rise to Christoffel symbols which are symmetric : $\gamma^c_{ab} = \gamma^c_{ba}$. However, in general, the Christoffel symbols are not symmetric (Marsden and Hughes, p. 75, proposition 4.23) and thus it is important to distinguish between γ^c_{ab} and γ^c_{ba}.

On p. 73 we have

$$\nabla_{\underset{\sim}{W}} \underset{\sim}{Y} = \Big[\frac{\partial Y^A}{\partial X^B} + \Gamma^A_{BC} Y^C \Big] W^B \underset{\sim}{E}_A$$

which is consistent with the above, so we're still okay.

However, if we now move to proposition 4.26 on p. 77 we appear to have an inconsistency.

writing out the expression in proposition 4.26 for a contravariant vector yields

$$V^A|_B = \frac{\partial V^A}{\partial X^B} + \Gamma^A_{CB} V^C$$

Notice that here V^C is contracted with the first lower index on Γ^A_{CB} and not the second as before.

It appears that the Christoffel symbols in prop. 4.26 should be replaced by their "transpose."

The same inconsistency appears in at least the following additional locations

- p. 79
- Box 4.1

I came across this problem, specifically, when I was trying to prove that the partial derivatives in the coordinate expression for the Lie derivative of a tensor can be replaced with covariant derivatives when the connection is torsion-free. See proposition 6.11. on p. 97

Acceleration vector fields

Recall that the material velocity $\underset{\sim}{V}_t$ was an element of the tangent bundle $T_{\varphi_t}\mathcal{C}$.

Consider a particle moving along a path on a manifold M. The velocity of that particle at $x \in X$ is in the tangent space of the manifold at that point, namely $T_x M$. Let v_x denote the velocity of the particle at x, thus $v_x \in T_x M$.

The acceleration of a particle is defined as the rate of change of velocity. Let a_x denote the acceleration of the particle at $x \in M$. By consistent logic we would expect that the acceleration vector lies in the Tangent to the tangent space at x. That is,

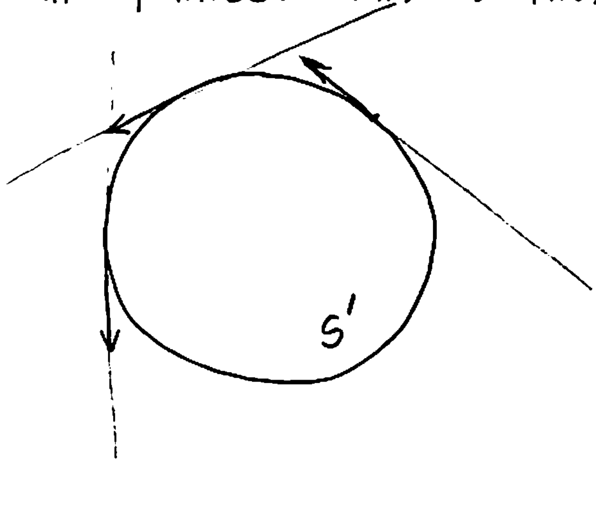
$$a_x \in T(T_x M)$$

In contrast, in the paper SMK, the material acceleration $\underset{\sim}{A}_t$ is stated to exist in $T\mathcal{C}$ rather than $T(T\mathcal{C})$ as would be expected.

It is possible to give an example to illustrate that the acceleration does not necessarily live in the tangent space to a manifold.

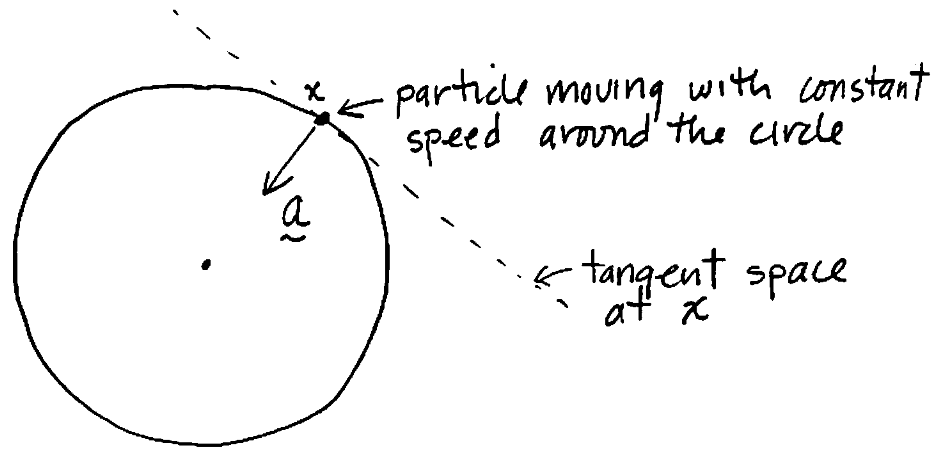
Consider our manifold to be S^1

For a particle moving on S^1, its velocity is tangent to S^1 at all points. This is illustrated below



So, the velocity of a particle at a point $x \in S^1$ as it moves around the circle is contained in the tangent to the circle at x.

Now consider a particle moving with constant velocity around the circle. The acceleration of the particle we know to be perpendicular to the circle directed toward the center of the circle



Clearly $\underset{\sim}{a}$ is not an element of the tangent space at x.

We now define some acceleration vector fields

(i) material acceleration $\underset{\sim}{A}_t \in T(T\mathcal{S})$

$$\underset{\sim}{A}_t = \frac{\partial V_t}{\partial t} = \frac{\partial^2 \varphi_t}{\partial t^2}$$

(ii) spatial acceleration $\underset{\sim}{a}_t \in T(T\mathcal{S})$

$$\underset{\sim}{a}_t = \underset{\sim}{A}_t \circ \varphi_t^{-1}$$

(iii) convected acceleration $\underset{\sim}{\mathcal{A}}_t \in T(T\mathcal{B})$

$$\underset{\sim}{\mathcal{A}}_t = \varphi_t^*(\underset{\sim}{a}_t)$$

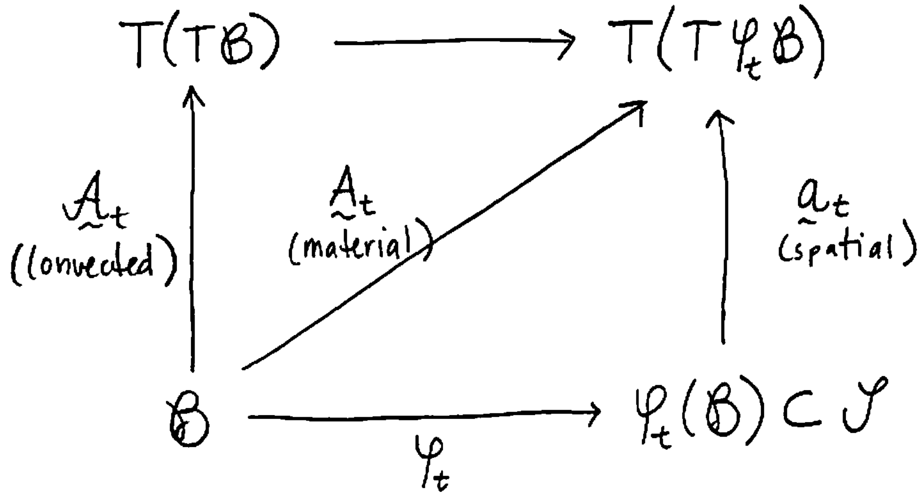
<u>Note</u>: Since the current configuration $\varphi_t(\mathcal{B})$ is <u>embedded</u> in $\mathcal{S}$ it seems plausible to me to be able to make the identifications

$$\left.\begin{array}{l} A_t \in T\mathcal{S} \\ a_t \in T\mathcal{S} \end{array}\right\} \text{ as stated in SMK}$$

however I do <u>not</u> see that the identification

$$\mathcal{A}_t \in T\mathcal{B}$$

can be made in general.



Lie Derivative

Def. Let $\underset{\sim}{w}$ be a time dependent vector field on M

Let t be a possibly time-dependent tensor field on M

The Lie derivative of $\underset{\sim}{t}$ with respect to $\underset{\sim}{w}$ is defined by

$$L_{\underset{\sim}{w}} \underset{\sim}{t} := \left(\frac{d}{dt} \psi^*_{t,s} t_t \right) \Big|_{t=s}$$

where $\psi_{t,s}$ is the flow of the vector field $\underset{\sim}{w}$

Applying this definition we can define the Lie derivative of a convected vector field $W \in \mathfrak{X}(\mathcal{B})$ relative to the convected velocity V_t of the motion φ_t by

$$\mathcal{L}_{\underset{\sim}{V}} \underset{\sim}{W} := \varphi^*_t \left[\mathcal{L}_{\varphi_{t*} V_t} (\varphi_{t*} W) \right]$$

$$= \varphi^*_t \left[\mathcal{L}_{\underset{\sim}{v}_t} (\varphi_{t*} W) \right]$$

$$= \varphi^*_t \left[\mathcal{L}_{\underset{\sim}{v}_t} \underset{\sim}{w} \right]$$

where $\quad \mathcal{L}_{\underset{\sim}{v}_t} \underset{\sim}{w} := \varphi_{t*} \frac{\partial}{\partial t} \varphi^*_t \underset{\sim}{w} \qquad \underset{\sim}{w} \in \mathfrak{X}(\varphi_t(\mathcal{B}))$

Objectivity & Covariance

The concepts of objectivity and covariance are difficult to define since the definitions vary from author to author. In essence they both deal with form invariance of an object or quantity under a change of frame.

Before delving too deeply into into this discussion lets develop some notation.

Consider two frames of reference (or manifolds) K and K'. Let ψ be a mapping from K to K'. That is,

$$\psi : K \to K'$$

ψ is a time dependent mapping.

Let $\underset{\sim}{t}$ be a tensor field on K. Denote the "corresponding" tensor field on K' by $\underset{\sim}{t}'$. To help fix ideas, lets consider an example.

Take $x \in K$, and $x' \in K'$ to be related through the ψ mapping as follows:

$$\underset{\sim}{x}'(t) = \underset{\sim}{Q}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{c}(t)$$

where c is a vector, Q is a proper orthogonal tensor and $\underset{\sim}{x}(t)$ is the position vector describing the motion of a particle in the K frame.

The velocity of the particle in the K frame is given by

$$\underset{\sim}{v}(t) = \frac{\partial \underset{\sim}{x}}{\partial t} =: \dot{\underset{\sim}{x}}$$

The "corresponding" velocity of the particle (i.e. $\underset{\sim}{v}'$) in the K' frame is calculated by

$$\begin{aligned}
\underset{\sim}{v}' &= \dot{\underset{\sim}{x}'} \\
&= \dot{\overline{\underset{\sim}{Q}\underset{\sim}{x} + \underset{\sim}{c}}} \\
&= \underset{\sim}{Q}\dot{\underset{\sim}{x}} + \dot{\underset{\sim}{Q}}\underset{\sim}{x} + \dot{\underset{\sim}{c}} \\
&= \underset{\sim}{Q}\underset{\sim}{v} + \dot{\underset{\sim}{Q}}\underset{\sim}{x} + \dot{\underset{\sim}{c}}
\end{aligned}$$

Note that $\underset{\sim}{v}$ is a vector. Knowing that $\underset{\sim}{v}$ is a vector and having knowledge of the mapping ψ it is possible to push v forward to K' using ψ. Doing so we obtain

$$\psi_* \underset{\sim}{v} = \underset{\sim}{Q} \underset{\sim}{v}$$

Note that $\psi_* \underset{\sim}{v} \neq \underset{\sim}{v}'$.

We're now at a point where we can start talking about the terminology. This is where it gets very confusing in the literature that I've read.

In order to maximize agreement with the largest number of authors (though somewhat limited and certainly baised by the fact that most of them are researchers in continuum mechanics) I shall make the following definitions for covariance and objectivity in the context of a mapping. A more general statement will be given below.

Let $\underset{\sim}{t}$ be a tensor field on K. Let $\underset{\sim}{t}'$ be the "corresponding" tensor field on K'. Let ψ denote a mapping $\psi : K \to K'$.

If ψ is a diffeomorphism and if $\underset{\sim}{t}' = \psi_* \underset{\sim}{t}$ then we say that $\underset{\sim}{t}$ is covariant.

If ψ is an isomorphism and if $\underset{\sim}{t}' = \psi_* \underset{\sim}{t}$ then we say that $\underset{\sim}{t}$ is objective.

Clearly, if $\underset{\sim}{t}$ is covariant then it is also objective but the converse is <u>not</u> true.

Looking back at the example presented on the previous page and above we conclude that the velocity is not an objective quantity (the mapping ψ used was an isomorphism). (Its obviously not covariant either)

This definition of objective is more stringent than that given by Marsden and Hughes (1983, p. 100, def. 6.18) where $\underset{\sim}{t}$ was objective if $\underline{t}' = \psi_* \underline{t}$ for any diffeomorphism ψ.

The definition given by Marsden & Hughes was the only rigorous definition I came across for objective. It was consisent with the continuum mechanics literature which exclusively deals with isomorphic maps ψ, so it was a long and slow process to decide to change it.

The definitions of objective and covariant given above are consisent with the use of the terms in SMK on p. 134. However, SMK view the mappings ψ as superposed motions on the current configuration and not changes of frame as was presented above. This is another very interesting point which I shall discuss in greater detail below.

Lets now turn our attention to Marsden and Hughes definition of covariance. The def. appears in the footnote on p. 156. In summary, if $\underset{\sim}{u}, \underset{\sim}{v}, \ldots$ are tensor fields on K and equations of the theory take the form

$$F(\underset{\sim}{u}, \underset{\sim}{v}, \ldots) = 0$$

then these equations are called covariant if for any diffeomorphism $\psi : K \to K'$

$$\psi_* F(\underset{\sim}{u}, \underset{\sim}{v}, \ldots) = F(\psi_* \underset{\sim}{u}, \psi_* \underset{\sim}{v}, \ldots) \qquad (*)$$

This definition does not make sense to me (unless $\underset{\sim}{u}, \underset{\sim}{v}, \ldots$ are covariant quantities as defined above). For example, consider $F(\underset{\sim}{u}) = \underset{\sim}{u}$. Eqn (*) thus reads

$$\psi_* \underset{\sim}{u} = \psi_* \underset{\sim}{u}$$

which is an identity. This would appear to be implying that all tensor fields on K are covariant. However, we saw that the velocity vector field is <u>not</u> covariant.

It appears to me that (*) should read

$$\psi_* F(\underset{\sim}{u}, \underset{\sim}{v}, \ldots) = F(\underset{\sim}{u}', \underset{\sim}{v}', \ldots) \qquad (**)$$

where we have replaced, e.g. the push forward of $\underset{\sim}{u}$ ($\psi_* \underset{\sim}{u}$) with $\underset{\sim}{u}'$.

For the example $F(\underset{\sim}{u}) = \underset{\sim}{u}$ considered above, (**) reads

$$\psi_* \underset{\sim}{u} = \underset{\sim}{u}'$$

which is consistent with the definition given above for a covariant tensor field $\underset{\sim}{u}$.

In Eqn. (**), if the independent variables $\underset{\sim}{u}, \underset{\sim}{v}, \ldots$ are covariant then (*) and (**) are equivalent.

Further justification for (**) can be found in the proof of Marsden and Hughes (pp. 164 – 167) concerning the covariant form of the balance of energy and how it follows that

(i) $\underset{\sim}{t} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n}$ $\quad h = -\underset{\sim}{q} \cdot \underset{\sim}{n}$

(ii) conservation of mass

(iii) balance of momentum

(iv) " " moment of momentum

(v) " of energy

(vi) $\underset{\sim}{\sigma} = 2\rho \dfrac{\partial e}{\partial \underset{\sim}{g}}$

The proof initiates by assuming that balance of energy holds in the unprimed and primed frames. Specifically see eqns (1) and (2) of Chapter 2 sect. 4.13, p. 166). This appears to be an application of (**) rather than (*) other ~~wi~~wise (2) would be written in terms of the push forward of the unprimed quantities.

Lets now return to SMK eqn (2.44) on p. 134. It is there stated that the stored energy function is covariant if

$$\eta \circ \bar{W}(\underset{\sim}{g}, \underset{\sim}{F}, \underset{\sim}{G}) = \bar{W}(\eta_* \underset{\sim}{g}, T\eta \circ \underset{\sim}{F}, G) \qquad (2.44)$$

which appears to be the statement of (*) above. However, once we realize that g, F and G are all covariant quantities we see that (2.44) is an equivalent statement of (**).

Let us now return to a previously mentioned point. Above, we have used ψ to denote a change of frame. In contrast, in SMK they use an equivalent of ψ to denoted a superposed motion of the deformed configuration. This is equivalent as to whether we view a transformation as active (the object is physically moved) or passive (the reference frame is rotated but the object does not move).

In the continuum mechanics literature this brings about two different schools of thought as alluded to above. That is, do we take ψ to be a superposed motion as in SMK or do we view it has a change of frame. In either point of view in continuum mechanics the mapping ψ is taken to be an isometry. The people who view ψ as a change of frame will admit all isometries ψ whether orientation preserving or not. The school which views ψ as a superposed motion (e.g. Naghdi) only admit orientation preserving isometries.

These schools thus have different definitions of objective. The frame of reference school views $\underset{\sim}{t}$ to be objective if $t' = \psi_* t$ for all isometries ψ, while the superposed motion school views $\underset{\sim}{t}$ to be objective if $\underset{\sim}{t}' = \psi_* \underset{\sim}{t}$ for all orientation preserving isometries ψ.

The frame of reference school is thus (at least on outward appearance) more restrictive than the superposed motion school. I say "at least on outward appearance" since I believe there must be a way to reconcile these points of view.