Rotation of an Asymmetric,

Free Rigid Body about its Middle Axis

Jim Morehead
Math 121, Mechanics and Geometry
Professor Jerry Marsden
May 12, 1989

Excellent!
In this short paper we will investigate the motion of an asymmetric top thrown initially rotating approximately about its middle axis. It will be shown that when the body's rotation is again approximately about its middle axis, the body is flipped over 180° about its long axis relative to its initial orientation. Implications for the actual experiment are also discussed.

Motion in the body frame

Since the angular momentum of the free rigid body is conserved, the motion in the six-dimensional phase space may be reduced to a three-dimensional phase space of the angular momentum as seen in the body, \( \overrightarrow{\Pi} = (\Pi_1, \Pi_2, \Pi_3) \). The equations of motion in this phase space (Euler's equations) can be explicitly integrated, so we will first do this, then specialize the results to our problem. This analysis can be found in Landau and Lifshitz, section 37; and Marsden and Ratiu, section 2.6 H.

Euler's equations are \( \dot{\overrightarrow{\Pi}} = \overrightarrow{\Pi} \times I^{-1} \overrightarrow{\Pi} \), where \( I \) is the (diagonalized) moment of inertia tensor. Explicitly,

\[
\begin{align*}
\dot{\Pi}_1 &= \left( \frac{1}{I_3} - \frac{1}{I_2} \right) \Pi_2 \Pi_3 \\
\dot{\Pi}_2 &= \left( \frac{1}{I_1} - \frac{1}{I_3} \right) \Pi_1 \Pi_3 \\
\dot{\Pi}_3 &= \left( \frac{1}{I_2} - \frac{1}{I_1} \right) \Pi_1 \Pi_2
\end{align*}
\]

we will assume \( I_1 > I_2 > I_3 \), so that \( a_2 < 0; a_3 > 0 \).

NOTE: This is a misprint in Marsden and Ratiu.
These equations may be integrated with the help of two conserved quantities, the energy, \( E = \frac{1}{2} \left( \frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3} \right) \)
and the magnitude of the body angular momentum, \( \Pi^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2 \).

These two equations enable us to express \( \Pi_1 \) and \( \Pi_3 \) in terms of \( \Pi_2 \):

\[
\Pi_1^2 = \frac{I_1(I_2-I_3)}{I_2(I_1-I_3)} (\alpha^2 - \Pi_2^2), \quad \alpha = \frac{aI_2(a-I_3)}{I_2-I_3} \frac{b^2}{1}
\]

\[
\Pi_3^2 = \frac{I_3(I_1-I_2)}{I_1(I_1-I_3)} (\beta^2 - \Pi_2^2), \quad \beta = \frac{aI_2(I_1-a)}{I_1-I_3} \frac{b^2}{1}
\]

and \( a = \frac{\Pi_2^2}{2E}, \quad b = \frac{2E}{\Pi} \)

(The notation follows Marsden and Ratiu)

Substituting these expressions into the \( \Pi_2^2 \) equation, we get

\[
\Pi_2^2 = \pm a_2 \sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)} (\alpha^2 - \Pi_2^2)} \sqrt{\frac{I_3(I_1-I_2)}{I_2(I_1-I_3)} (\beta^2 - \Pi_2^2)}
\]

\[
= \pm a_2 a_3 (\alpha^2 - \Pi_2^2) (\beta^2 - \Pi_2^2), \quad \text{using} \quad a_1 = \frac{I_2-I_3}{I_2I_3}, \quad a_2 = \frac{I_3-I_1}{I_3I_1}, \quad a_3 = \frac{I_1-I_2}{I_1I_2}
\]

Let's assume the bottom sign for reasons which will become clear later.

Now if we let \( \Pi_2 = \alpha \kappa u \), we have

\[ \beta \frac{du}{dt} = \sqrt{a_1 a_3 (\kappa^2 - \alpha^2 u^2)} (\beta^2 - \alpha^2 u^2) \]

\[ \frac{du}{dt} = \kappa \sqrt{a_1 a_3} \sqrt{(1-k^2 u^2)(1-u^2)} \]

\[ k = \frac{\alpha}{\beta} \quad (\leq 1). \]
\[
\int_0^u \frac{du}{\sqrt{(1-k^2 u^2)(1-u^2)}} = \alpha \sqrt{a_1 a_2} t = \tau
\]

where it is assumed \( u = 0 \to \Pi_1 = 0 \) \( \pm t = 0 \).

This integral is an elliptic integral of the first kind. Inverting it we have,

\[ U = \text{Sn} \tau, \quad \text{a Jacobii elliptic function, order } k = \frac{\alpha}{\beta} \]

\[ \rightarrow \Pi_2 = \alpha \text{sn} \tau \]

with \( \Pi_2 \) we can solve for \( \Pi_1 \) and \( \Pi_3 \):

\[
\Pi_1 = \pm \sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}} \left( \alpha^2 - \alpha^2 \text{sn}^2 \tau \right)
\]

\[
= \pm \sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}} \alpha^2 \left( 1 - \text{sn}^2 \tau \right)
\]

\[ = \pm \sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}} \alpha^2 \text{cn} \tau \]

And \( \Pi_3 = \pm \sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}} \left( \beta^2 - \alpha^2 \text{sn}^2 \tau \right) \]

\[ = \pm \sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}} \beta^2 \text{dn} \tau \]

\[ \text{the other Jacobii elliptic functions} \]
Now let's specialize to the case we're interested in. Let's assume the motion is on the separatrix of the motion. This is somewhat unrealistic for the experiment, which will be discussed later. On the separatrix, \( \mathbf{r} = (0, -\Pi, 0) \) at \( t = -\infty \) and evolves to \((0, \Pi, 0)\) at \( t = +\infty \). (where of course \( \Pi = ||\mathbf{r}|| \)).

In that case the constants of the motion are

\[
\mathbf{r} = \Pi \quad \text{and} \quad E = \frac{\Pi^2}{2I_2}
\]

The previously defined constants are thus

\[
a = \frac{\Pi^2}{2E} = \frac{I_2}{b^2}
\]

\[
b = \frac{2E}{\Pi} = \frac{\Pi}{I_2}
\]

\[
\alpha = \sqrt{\frac{a I_2 (a - I_2)}{I_2 - I_2}} = \frac{\Pi}{b^2}
\]

\[
\beta = \sqrt{\frac{a I_2 (I_2 - a)}{I_2 - I_2}} = \frac{\Pi}{b^2} \quad \rightarrow k = \frac{\alpha}{\beta} = 1 \quad \text{. The elliptic functions for } k = 1 \text{ reduce to hyperbolic functions}
\]

\[5 \rightarrow \tanh j; \ c, d \rightarrow \sech\]

The general solution just found then reduces to

\[
\Pi_1 = \Pi \sqrt{\frac{I_1 (I_2 - I_3)}{I_2 (I_2 - I_3)}} \sech \tau
\]

\[
\Pi_2 = \Pi \tanh \tau, \quad \tau = \Pi \sqrt{\frac{(I_2 - I_3)(I_1 - I_3)}{I_1 I_3 I_3}}
\]

\[
\Pi_3 = \Pi \sqrt{\frac{I_3 (I_1 - I_3)}{I_1 (I_1 - I_3)}} \sech \tau
\]
Now we need to reconstruct the solution in the space frame. Let's use Euler angles with the \( \Pi_2 \)-axis corresponding to the fixed \( z \)-axis, along which \( \Pi_1 \) points.

Then taking the projections of the constant vector \( \Pi \) on the \( (\Pi_1, \Pi_2, \Pi_3) \) axes we have:

\[
\begin{align*}
\Pi_1 \sin \theta \cos \psi &= \Pi_1 \\
\Pi_1 \cos \theta &= \Pi_2 \\
\Pi_1 \sin \theta \sin \psi &= \Pi_3
\end{align*}
\]

unknown functions \text{ known functions}

\[\cos \theta = \frac{\Pi_2}{\Pi_1} = \tanh \tau\]

\[\tan \psi = \frac{\Pi_3}{\Pi_1} = \sqrt{\frac{I_3(I_1-I_2)}{I_1(I_2-I_3)}}\]

\( \phi \) must be obtained by a quadrature. From the relations

\[
\frac{\Pi_2}{I_3} = \phi \sin \theta \sin \psi + \dot{\theta} \cos \psi, \quad \frac{\Pi_1}{I_1} = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi
\]

in moving frame. Not momentum.

we get

\[\dot{\phi} = \frac{\Pi_2 \sin \psi + \Pi_1 \cos \psi}{I_3 \sin \theta}\]

And using the relations for \( \theta, \psi \) in terms of momenta, we get
\[
\frac{d\phi}{dt} = \frac{\Pi_1^2}{I_3} + \frac{\Pi_1^2}{I_1} - \frac{\Pi}{\Pi_3^2 + \Pi_1^2}
\]

Again \( \Pi_1 = \Pi \sqrt[3]{I_1(I_2-I_3)} \) s.e.h \( \theta \), \( \Pi_3^2 = \Pi \sqrt[3]{I_2(I_3-I_2)} \) s.e.h \( \theta \).

Plugging these in we get tremendous cancellation,

\[
-\frac{d\phi}{dt} = \frac{\Pi}{I_2} = \omega_0,
\]
the frequency of rotational about the middle axis before and after.

From these formulas we can get a picture of what happens. Initially \( (t \to -\infty) \) the body is spinning at a rate \( \Pi \) about its middle axis, with \( \omega_1 \theta = \tanh (\varphi \to -\infty) = -1 \), middle axis along \(-z\).

As \( \varphi \) increases, the body keeps spinning and its own middle axis, with polar angle \( \theta \), goes from \( \frac{\pi}{2} \) to \( \frac{\pi}{2} \) as \( \varphi \to \infty \), middle axis along \( +z \).

The middle axis rotating by 180° is exactly equivalent to a 180° rotation about the long \( (\Pi_1) \) axis, as promised.

This is not, however, exactly the situation in a little experiment of tossing an asymmetric object. We can't throw it well enough or wait long enough to watch it spin many times about its middle axis before flipping over. So we should try to get a feeling for what effect small deviations have.

For that purpose let's examine what happens if the
initial condition is on the separatrix, so that we can use the formulas just derived, but not at the fixed point. The justification for using the separatrix as "almost generic" is that away from the fixed point, the separatrix is locally just like any other trajectory.

if we just look at the trajectory before the other trajectories diverge away from it at the opposite fixed point, the separatrix motion is like to nearby motions.

So for this purpose we use the derived expressions with the initial condition at a finite time in the past:

\[ \cos \theta = \tanh \tau \]

\[ \phi = \frac{\pi}{I_2} t. \]

Again call \( \frac{\pi}{I_2} = \omega_0 \); then \( \tau = \frac{\pi I_1 I_2}{I_1 I_2 I_3} t = \omega_0 \sqrt{\frac{(I_2 - \bar{I}_2)(I_1 - \bar{I}_1)}{I_1 I_2 I_3}} \)

So \( \tau = \sqrt{\frac{(I_2 - \bar{I}_2)(I_1 - \bar{I}_1)}{I_1 I_2}} \phi \)

\[ \rightarrow \cos \theta = \tanh \left( \sqrt{\frac{(I_2 - \bar{I}_2)(I_1 - \bar{I}_1)}{I_1 I_2}} \phi \right) \text{ along the trajectory.} \]

We see that as \( \theta \rightarrow \frac{\pi}{2} \), \( \cos \theta \rightarrow 1 \) and thus \( \sqrt{\frac{(I_2 - \bar{I}_2)(I_1 - \bar{I}_1)}{I_1 I_2}} \phi \rightarrow \infty \). But if the initial and final \( \theta \)'s are near, but not on the poles, then \( \phi \) goes around some finite number of times.
Let's apply this to the experiment I've been doing: tossing up a pocket Spanish dictionary.

A moment of inertia of a rectangular box is proportional to the sum of the squares of the two sides perpendicular to its axis. Only the ratios of the moments of inertia matter, so after a quick calculation we can set:

\[ I_1 \approx 4 \]
\[ I_2 \approx 3.2 \quad \Rightarrow \quad \sqrt{(I_2-I_3)(I_1-I_2)} \approx \frac{2}{3} \]
\[ I_1 = 1 \]

so \( \cos \theta \approx \tanh \left( \frac{2}{3} \phi \right) \)

What is often, but not always, observed in the experiment is that the book flips over in about one rotation of \( \phi \). To examine this more closely, let's assume for simplicity that the initial and final \( \theta \) are the same (relative to opposite axes; \( \theta_f = \pi - \theta_i \)). That is, the angular velocity's deviation from the middle body axis when thrown is the same as when caught.

with \( \theta_i \) and \( \theta_f \) so related, \( \phi_i \) and \( \phi_f \) will be similarly related and we can work with just one limit.

Let's see what range of initial and final \( \theta \) correspond to rotating approximately once. Exactly one rotation of \( \phi \) corresponds to \( \phi = -\pi \), \( \phi = \pi \). So let's assume \( \phi \in \left[ \frac{\pi}{4}, \frac{3\pi}{4} \right] \), say.
Then applying \( \cos \Theta = \tanh \left( \frac{2}{3} \phi \right) \) we find that the initial end
end polar angles are in the interval

\[ 9^\circ < \Theta < 22^\circ \]

That is, if the book is thrown with its rotation axis between
9° and 22° from the book's middle axis, then it will approximately
flip over in about one rotation. If the initial rotation axis
is nearer to the middle axis, then the book will less than flip
over in one rotation. If the initial axis is farther away than
22° it will flip more quickly, but our analysis which assumed
the trajectory is near the separatrix can be approximated by
the separatrix motion is less and less valid in that region.

This range of angles corresponds reasonably with
what is observed. Usually the book flips over in one
rotation, but often the throw is good enough that it doesn't
completely flip over in one rotation. So the analysis based
on the separatrix motion does have some applicability to
the real experiment.