Rotation of an Asymmetric, Free Rigid Body about its Middle Axis

Jim Morehead
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Professor Jerry Marsden

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Excellent!

In this short paper we will investigate the motion of an asymmetric top thrown initially rotating approximately about its middle axis. It will be shown that when the body's rotation is again approximately about its middle axis, the body is flipped over 180° about its long axis relative to its initial orientation. Implications for the actual experiment are also discussed.

Motion in the body frame

Since the angular momentum of the free rigid body is conserved, the motion in the six-dimensional phase space may be reduced to a three-dimensional phase space of the angular momentum as seen in the body, $\vec{\Pi} = (\Pi_1, \Pi_2, \Pi_3)$. The equations of motion in this phase space (Euler's equations) can be explicitly integrated, so we will first do this, then specialize the results to our problem. This analysis can be found in Landau and Lifshitz, Section 37; and Marsden and Ratiu, section 2.6 H.

Euler's equations are $\dot{\Pi} = \Pi \times I^{-1}\Pi$, where I is the (diagonalized) moment of inertia tensor. Explicitly,

$$\dot{\Pi}_1 = \left(\frac{1}{I_3} - \frac{1}{I_2}\right)\Pi_2\Pi_3 \equiv a_1 \Pi_2 \Pi_3$$

$$\dot{\Pi}_2 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\Pi_3\Pi_1 \equiv a_2 \Pi_3 \Pi_1$$

$$\dot{\Pi}_3 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right)\Pi_1\Pi_2 \equiv a_3 \Pi_1 \Pi_2$$

we will assume $\underline{I_1 > I_2 > I_3}$, so that $a_2 < 0$; $a_1, a_3 > 0$.

NOTE: This is a misprint in Marsden and Ratiu.

These equations may be integrated with the help of two conserved quantities; the energy, $E = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right)$

and the magnitude of the body angular momentum, $\Pi^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2$.

These two equations enable us to express Π_1 and Π_3 in terms of Π_2 :

$$\Pi_1^2 = \frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}(\alpha^2 - \Pi_2^2), \qquad \text{where} \qquad \alpha^2 = \frac{a I_2 (a - I_3)}{I_2 - I_3} b^2$$

$$\Pi_3^2 = \frac{I_3(I_1-I_2)}{I_2(I_1-I_3)}(\beta^2 - \Pi_2^2) \qquad \beta^2 = \frac{a I_2 (I_1 - a) b^2}{I_1 - I_2}$$

and $a = \frac{\Pi^2}{2E}$, $b = \frac{2E}{\Pi}$

(The notation follows Marsden and Ratiu)

Substituting these expressions into the $\dot{\Pi}_2$ equation, we get

$$\dot{\Pi}_2 = \pm a_2 \sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}(\alpha^2-\Pi_2^2)}\sqrt{\frac{I_3(I_1-I_2)}{I_2(I_1-I_3)}(\beta^2-\Pi_2^2)}$$

$$= \mp \sqrt{a_1 a_3 (\alpha^2 - \Pi_2^2)(\beta^2 - \Pi_2^2)}, \quad \text{using } a_1 = \frac{I_2 - I_3}{I_2 I_3},\ a_2 = \frac{I_3 - I_1}{I_3 I_1},\ a_3 = \frac{I_1 - I_2}{I_1 I_2}.$$

Let's assume the bottom sign for reasons which will become clear later.

Now if we let $\Pi_2 = \alpha u$, we have

$$\beta \frac{du}{dt} = \sqrt{a_1 a_3 (\alpha^2 - \alpha^2 u^2)(\beta^2 - \alpha^2 u^2)}$$

$$\rightarrow \frac{du}{dt} = \alpha \sqrt{a_1 a_3}\sqrt{(1-k^2u^2)(1-u^2)}, \quad k = \frac{\alpha}{\beta} \ (\leq 1).$$

$$\int_0^u \frac{du}{\sqrt{(1-k^2u^2)(1-u^2)}} = \alpha\sqrt{a_1 a_3}\, t \equiv \tau$$

where it is assumed $u=0 \rightarrow \Pi_2 = 0$ at $t=0$.

This integral is an elliptic integral of the first kind. Inverting it we have,

$u = \operatorname{sn} \tau$, a Jacobi elliptic function, order $k = \frac{\alpha}{\beta}$

$$\rightarrow \Pi_2 = \alpha \operatorname{sn} \tau$$

With Π_2 we can solve for Π_1 and Π_3:

$$\Pi_1 = \pm\sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}(\alpha^2 - \alpha^2 \operatorname{sn}^2\tau)}$$

$$= \pm\sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}\alpha^2(1- \operatorname{sn}^2\tau)}$$

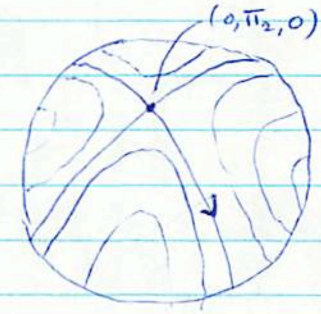
$$= \pm\sqrt{\frac{I_1(I_2-I_3)}{I_2(I_1-I_3)}\alpha^2}\ \operatorname{cn}\tau$$

And $\Pi_3 = \pm\sqrt{\frac{I_3(I_1-I_2)}{I_2(I_1-I_3)}(\beta^2 - \alpha^2 \operatorname{sn}^2\tau)}$

$\operatorname{cn}\tau$, $\operatorname{dn}\tau$: the other Jacobi elliptic functions

$$= \pm\sqrt{\frac{I_3(I_1-I_2)}{I_2(I_1-I_3)}\beta^2}\sqrt{1-k^2\operatorname{sn}^2\tau} = \pm\sqrt{\frac{I_3(I_1-I_2)}{I_2(I_1-I_3)}\beta^2}\ \operatorname{dn}\tau$$

Now let's specialize to the case we're interested in. Let's assume the motion is on the separatrix of the motion. This is somewhat but not too unrealistic for the experiment, which will be discussed later.



On the separtrix, $\vec{\Pi} = (0, -\Pi, 0)$ at $\tau = -\infty$ and evolves to $(0, +\Pi, 0)$ at $\tau = +\infty$. (where, of course, $\Pi = \|\vec{\pi}\|$).

In that case the constants of the motion are

$$\Pi = \Pi \quad \text{and} \quad E = \frac{\Pi^2}{2I_2}$$

The previously defined constants are thus

$$a = \frac{\Pi^2}{2E} = \underline{I_2}$$

$$b = \frac{2E}{\Pi} = \underline{\frac{\Pi}{I_2}}$$

$$\alpha = \sqrt{\frac{a I_2 (a - I_3)}{I_2 - I_3} b^2} = \underline{\Pi}$$

$$\beta = \sqrt{\frac{a I_2 (I_1 - a) b^2}{I_1 - I_2}} = \underline{\Pi}$$

$\to k = \frac{\alpha}{\beta} = 1$. The elliptic functions for $k = 1$ reduce to hyperbolic functions

$sn \to \tanh$; $cn, dn \to \operatorname{sech}$

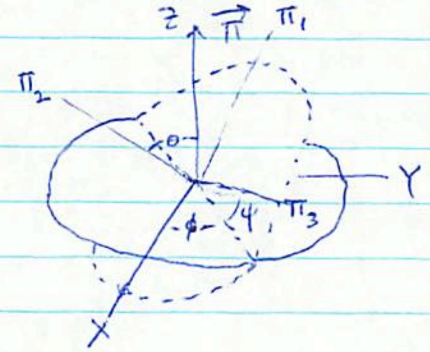
The general solution just found then reduces to

$$\Pi_1 = \Pi \sqrt{\frac{I_1 (I_2 - I_3)}{I_2 (I_1 - I_3)}} \operatorname{sech} \tau$$

$$\Pi_2 = \Pi \tanh \tau \quad , \quad \tau = \Pi \sqrt{\frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_2^2 I_3}} \, t$$

$$\Pi_3 = \Pi \sqrt{\frac{I_3 (I_1 - I_2)}{I_2 (I_1 - I_3)}} \operatorname{sech} \tau$$

Now we need to reconstruct the solution in the space frame. Let's use Euler angles with the Π_2-axis corresponding to the fixed Z axis, along which $\vec{\Pi}$ points



Then taking the projections of the constant vector $\vec{\Pi}$ on the (Π_1, Π_2, Π_3) axes we have

$$\Pi \sin\theta \cos\psi = \Pi_1$$
$$\Pi \cos\theta = \Pi_2$$
$$\Pi \sin\theta \sin\psi = \Pi_3$$

unknown functions known functions

$$\rightarrow \quad \cos\theta = \frac{\Pi_2}{\Pi} = \tanh\tau$$

$$\tan\psi = \frac{\Pi_3}{\Pi_1} = \sqrt{\frac{I_3(I_1 - I_2)}{I_1(I_2 - I_3)}}$$

ϕ must be obtained by a quadrature. From the relations

$\frac{\Pi_3}{I_3} = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$, $\frac{\Pi_1}{I_1} = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$ (angular velocity in moving frame. Not momentum)

we get $\dot\phi = \dfrac{\frac{\Pi_3}{I_3}\sin\psi + \frac{\Pi_1}{I_1}\cos\psi}{\sin\theta}$

And using the relations for θ, ψ in terms of momenta, we get

$$\frac{d\phi}{dt} = \frac{\frac{\Pi_3^2}{I_3} + \frac{\Pi_1^2}{I_1}}{\Pi_3^2 + \Pi_1^2} \cdot \Pi$$

Again $\Pi_1 = \Pi \sqrt{\frac{I_1(I_2 - I_3)}{I_2(I_1 - I_3)}} \operatorname{sech} \tau$, $\Pi_3 = \Pi \sqrt{\frac{I_3(I_1 - I_2)}{I_2(I_1 - I_3)}} \operatorname{sech} \tau$

Plugging these in we get tremendous cancellation,

$\rightarrow \frac{d\phi}{dt} = \frac{\Pi}{I_2} = \underline{\omega_0}$, the frequency of rotational about the middle axis before and after.

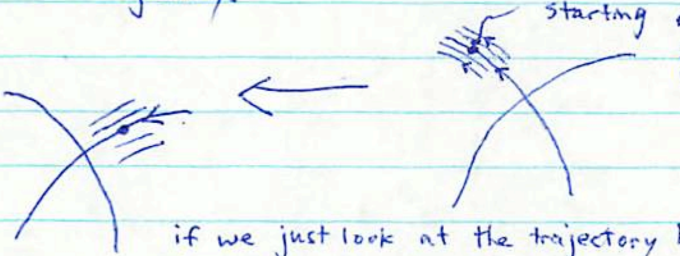
From these formulas we can get a picture of what happens. Initially $(\tau \to -\infty)$ the body is spinning at a rate $\frac{\Pi}{I_2}$ about its middle axis, with $\cos\theta = \tanh(\tau \to -\infty) = -1$
$\rightarrow$ middle axis along $-Z$

As τ increases, the body keeps spinning and its own middle axis, with polar angle θ, goes from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ as $\tau \to \infty$.
$\rightarrow$ middle axis along $+Z$.

The middle axis rotating by $180°$ is exactly equivalent to <u>a $180°$ rotation about the long (Π_3) axis</u>, as promised.

This is not, however, exactly the situation in a little experiment of tossing an asymmetric object. We can't throw it well enough or wait long enough to watch it spin many times about its middle axis before flipping over. So we should try to get a feeling for what effect small deviations have.

For that purpose let's examine what happens if the initial condition is on the separatrix, so that we can use the formulas just derived, but not at the fixed point. The justification for using the separatrix as "almost generic" is that away from the fixed point, the separatrix is locally just like any other trajectory.



starting on the separatrix will be qualitatively just like starting near the separatrix

if we just look at the trajectory before the other trajectories diverge away from it at the opposite fixed point, the separatrix motion is like the nearby motions.

So for this purpose we use the derived expressions with the initial condition at a <u>finite</u> time in the past:

$$\cos\theta = \tanh\tau$$

$$\phi = \frac{\Pi}{I_2} t .$$

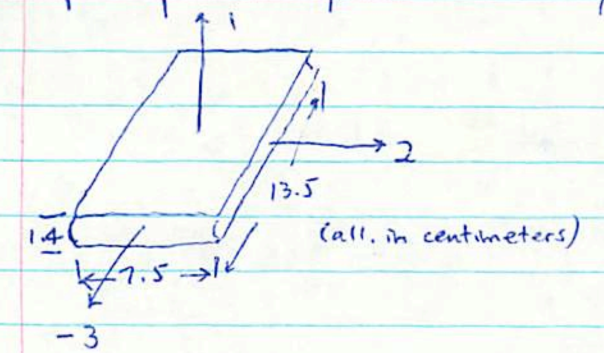
Again call $\frac{\Pi}{I_2} = \omega_0$; then $\tau = \Pi\sqrt{\frac{(I_2-I_3)(I_1-I_2)}{I_1 I_2^2 I_3}}\, t = \omega_0\sqrt{\frac{(I_2-I_3)(I_1-I_2)}{I_1 I_3}}\, t$

So $\tau = \sqrt{\frac{(I_2-I_3)(I_1-I_2)}{I_1 I_3}}\,\phi$

$$\longrightarrow \cos\theta = \tanh\left(\sqrt{\frac{(I_2-I_3)(I_1-I_2)}{I_1 I_3}}\,\phi\right)$$ along the trajectory.

We see that as $\theta \to \frac{\pi}{2}$, $\cos\theta \to 1$ and thus $\sqrt{\quad}\,\phi \to \infty$. But if the initial and final θ's are near, but not on the poles, then ϕ goes around some finite number of times.

Let's apply this to the experiment I've been doing: tossing up a pocket Spanish dictionary.



A moment of inertia of a rectangular box is proportional to the sum of the squares of the two sides perpindicular to its axis. Only the ratios of the moments of inertia matter, so after a quick calculation we can set:

$$I_1 \approx 4 \qquad I_2 \approx 3.2 \qquad I_1 = 1 \quad \rightarrow \quad \sqrt{\frac{(I_2-I_3)(I_1-I_2)}{I_1 I_3}} \approx \frac{2}{3}.$$

so $\cos\theta \approx \tanh\left(\frac{2}{3}\phi\right)$

What is often, but not always, observed in the experiment is that the book flips over in about one rotation of ϕ. To examine this more closely let's assume for simplicity that the initial and final θ are the same (relative to opposite axes; $\theta_f = \pi - \theta_i$). That is, the angular velocity's deviation from the middle body axis when thrown is the same as when caught. With θ_i and θ_f so related, ϕ_i and ϕ_f will be similarly related and we can work with just one limit.

Let's see what range of initial and final θ correspond to rotating approximately once. Exactly one rotation of ϕ corresponds to $\phi_i = -\pi$, $\phi_f = \pi$. So let's assume $\phi_f \in \left[\frac{3}{4}\pi, \frac{5}{4}\pi\right]$, say, Then applying $\cos\theta \approx \tanh\left(\frac{2}{3}\phi\right)$ we find that the initial and final polar angles are in the interval

$$9^\circ < \theta < 22^\circ .$$

That is, if the book is thrown with its rotation axis between 9° and 22° from the book's middle axis, then it will approximately flip over in about one rotation. If the initial rotation axis is nearer to the middle axis, then the book will less than flip over in one rotation. If the initial axis is farther away than 22° it will flip more quickly, but our analysis which assumed the trajectory is near the separatrix and can be approximated by the separatrix motion is less and less valid in that region.

This range of angles corresponds reasonably with what is observed. Usually the book flips over in one rotation, but often the throw is good enough that it doesn't completely flip over in one rotation. So the analysis based on the separatrix motion <u>does</u> have some applicability to the real experiment.