Nonuniqueness of the Lagrangian Function

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Abstract

We investigate nonuniqueness in the Lagrangian function associated with given Newtonian equations of motion. We describe what it means for a Newtonian system to be self-adjoint and review the proof that this property is necessary and sufficient for the system to possess a Lagrangian formulation. A method of construction of a Lagrangian function from the equations of motion is described, and the freedom that arises in the construction is discussed. This freedom, or nonuniqueness, in the Lagrangian raises important questions about the canonical quantization procedure.

Contents

1 Introduction

The formalism of Lagrangian mechanics is a standard tool of classical dynamics. Perhaps it is most often regarded as a means of deriving equations of motion in generalized coordinates. The formalism, however, also provides a convenient setting in which to study equations of motion after they have been derived. For example, with knowledge of the Lagrangian function, we can utilize Noether's theorem to seek conservation laws. We can also construct the Hamiltonian function and use canonical transformation theory or Hamilton-Jacobi theory.

Of course, these techniques all require that we know the Lagrangian of a system, not just the equations of motion. Consequently, if we hope to apply them to problems in which we begin with the equations of motion, we must first look back and seek out a Lagrangian that could have generated them. This problem of going backwards to find a Lagrangian associated with known equations of motion is termed the Inverse Problem of Newtonian Mechanics [9] or the Inverse Problem of the Calculus of Variations [2, 7].

Along with the practical gains associated with finding the Lagrangian function of a system, there is another, more fundamental, reason to study the Inverse Problem. Quantum mechanics begins with classical Hamiltonian dynamics. In order to canonically quantize a system, we need to know its Lagrangian function first. To find this function, we may require the techniques of the Inverse Problem.

Now that we are convinced of the need to study the Inverse Problem, a difficulty presents itself. It is clear that we will never be able to find a unique Lagrangian suitable for given equations of motion. For instance, if two Lagrangians differ by a total time derivative, then they will be associated with the same equations of motion (in more modern treatments of Lagrangian dynamics, the Lagrangians differ by a closed one-form on the coordinate manifold [1]). And this is just one aspect of the essential nonuniqueness of the Lagrangian.

In this paper, we investigate this nonuniqueness of the Lagrangian. We begin with a description of the major results of the Inverse Problem. These results provide necessary and sufficient conditions that given Newtonian equations of motion possess an associated Lagrangian function. The results also supply a means of explicitly constructing such an associated Lagrangian. Much of our discussion of this subject follows Santilli's book [9]. Santilli himself follows an approach originated by Helmholtz [3], then developed by later researchers (e.g. [2]). After laying out the results of the Inverse Problem, we study the nonuniqueness of the Lagrangian. We discuss the causes of this nonuniqueness and its ramifications. In particular, we consider some of the dramatic consequences that nonuniqueness has for canonical quantization.

2 The Inverse Problem of Newtonian Mechanics

Suppose that coordinates $q \in R^n$ evolve according to given equations of motion

$$A_{ij}(t,q,\dot{q})\ddot{q}^j + B_i(t,q,\dot{q}) = 0 \quad i,j = 1,...,n \tag{1}$$

where we adopt the summation convention on repeated indices. Our goal in the Inverse Problem is to represent these Newtonian equations as Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0. \tag{2}$$

Let us therefore seek a Lagrangian function $L(t,q,\dot{q})$ satisfying

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = A_{ij}(t,q,\dot{q})\ddot{q}^j + B_i(t,q,\dot{q}). \tag{3}$$

When we expand the total time derivative, this equation becomes

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial t} - \frac{\partial L}{\partial q^i} = A_{ij}(t,q,\dot{q})\ddot{q}^j + B_i(t,q,\dot{q}). \tag{4}$$

If this relation is to hold true for all q,$\dot{q}$, and $\ddot{q}$ then it follows that $L(t,q,\dot{q})$ must be a solution of the partial differential equations

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = A_{ij} \tag{5}$$

and

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial t} - \frac{\partial L}{\partial q^i} = B_i. \tag{6}$$

These are $n^2 + n$ partial differential equations in one unknown, $L(t, q, \dot{q})$. They are overdetermined and therefore need not possess a solution. However, it is possible to describe when a solution L will exist and, when it does exist, to construct it explicitly.

2.1 Construction of Lagrangian Function

It is not difficult to see that we cannot solve equations (5) and (6) for arbitrary Newtonian equations (1). For example, relation (5) can never be solved unless we have $A_{ij} = A_{ji}$. It turns out that the integrability conditions of equations (5) and (6) - necessary and sufficient conditions for the existence of a solution L – depend on a property of the Newtonian equations called self-adjointness.

2.1.1 Self-Adjointness

Consider a system of differential equations $F_i(t, q, \dot{q}, \ddot{q}) = 0$, where $i = 1, ..., n$ and $q \in R^n$. Suppose that we choose some particular path $q(t) = q_o(t)$, not necessarily satisfying these equations, and evaluate the functions F_i along q:

$$\mathcal{F}_i(t) = F_i(t, q(t), \dot{q}(t), \ddot{q}(t)). \tag{7}$$

Now we introduce small variations upon the path by setting $q(t, \epsilon) \equiv q_o(t) + \epsilon\eta(t)$, where $\eta(t)$ is some fixed function of t and where $\epsilon \in R$. To measure the resulting change in the $\mathcal{F}_i(t)$, we define the *variational forms* M_i by

$$\begin{aligned} M_i(\eta) &\equiv \frac{\partial \mathcal{F}_i}{\partial \epsilon}|_{\epsilon=0} = \frac{\partial F_i(t, q(t,\epsilon), \dot{q}(t,\epsilon), \ddot{q}(t,\epsilon))}{\partial \epsilon}|_{\epsilon=0} \\ &= \frac{\partial F_i}{\partial q^j}|_{\epsilon=0}\eta^j + \frac{\partial F_i}{\partial \dot{q}^j}|_{\epsilon=0}\dot{\eta}^j + \frac{\partial F_i}{\partial \ddot{q}^j}|_{\epsilon=0}\ddot{\eta}^j. \end{aligned} \tag{8}$$

Note that the variational forms are always linear in the variation η. The coefficients that appear in equation (8) are known functions of time. To emphasize this fact, we write

$$M_i(\eta) = a_{ij}(t)\eta^j + b_{ij}(t)\dot{\eta}^j + c_{ij}(t)\ddot{\eta}^j \tag{9}$$

where

$$a_{ij}(t) = \frac{\partial F_i}{\partial q^j}(t, q_o(t), \dot{q}_o(t), \ddot{q}_o(t)) \quad b_{ij}(t) = \frac{\partial F_i}{\partial \dot{q}^j}(t, q_o(t), \dot{q}_o(t), \ddot{q}_o(t)) \quad c_{ij}(t) = \frac{\partial F_i}{\partial \ddot{q}^j}(t, q_o(t), \dot{q}_o(t), \ddot{q}_o(t)). \tag{10}$$

So far, we have computed a system of variational forms that describes how $\mathcal{F}_i$ changes when we alter the path $q(t)$ away from $q_o(t)$.

Now, we associate with the system of variational forms an *adjoint system*. The adjoint system is another system of variational forms, $\tilde{M}_i$, satisfying

$$\tilde{\eta}^i M_i(\eta) - \eta^i \tilde{M}_i(\tilde{\eta}) = \frac{dQ(\eta, \tilde{\eta})}{dt} \tag{11}$$

for some fixed function $Q(\eta, \tilde{\eta})$ and for all variations η and $\tilde{\eta}$. Let us calculate the adjoint system of M_i using the notation of equation (9):

$$\begin{aligned} \tilde{\eta}^i M_i(\eta) &= \tilde{\eta}^i a_{ij}\eta^j + \tilde{\eta}^i b_{ij}\dot{\eta}^j + \tilde{\eta}^i c_{ij}\ddot{\eta}^j \\ &= \eta^j[\tilde{\eta}^i(a_{ij} - \dot{b}_{ij} + \ddot{c}_{ij}) + \dot{\tilde{\eta}}^i(-b_{ij} + 2\dot{c}_{ij}) + \ddot{\tilde{\eta}}^i c_{ij}] \\ &\quad + \frac{d}{dt}[\tilde{\eta}^i b_{ij}\eta^j + \tilde{\eta}^i c_{ij}\dot{\eta}^j - \eta^j \frac{d}{dt}(\tilde{\eta}^i c_{ij})] \end{aligned} \tag{12}$$

so that the adjoint system takes the form

$$\tilde{M}_i(\tilde{\eta}) = (a_{ji} - \dot{b}_{ji} + \ddot{c}_{ji})\tilde{\eta}^j + (-b_{ji} + 2\dot{c}_{ji})\dot{\tilde{\eta}}^j + c_{ji}\ddot{\tilde{\eta}}^j \tag{13}$$

and the function Q is given by

$$Q(\eta, \tilde{\eta}) = \frac{d}{dt}[\tilde{\eta}^i b_{ij}\eta^j + \tilde{\eta}^i c_{ij}\dot{\eta}^j - \eta^j \frac{d}{dt}(\tilde{\eta}^i c_{ij})]. \tag{14}$$

The adjoint system $\tilde{M}_i$ is unique [9].

A system of variational forms is called *self-adjoint* if $M_i(\eta) = \tilde{M}_i(\eta)$, $i = 1, ..., n$, for all variations $\eta(t)$. Inspecting our expressions (9) and (13), we see that the system M_i is self-adjoint if

$$\begin{aligned} c_{ij} &= c_{ji} && (15) \\ b_{ij} + b_{ji} &= 2\dot{c}_{ji} && (16) \\ a_{ij} - a_{ji} &= \ddot{c}_{ji} - \dot{b}_{ji}. && (17) \end{aligned}$$

Referring back to the definitions of the coefficients (10), these conditions imply

$$\begin{aligned} \frac{\partial F_i}{\partial \ddot{q}^j} &= \frac{\partial F_j}{\partial \ddot{q}^i} && (18) \\ \frac{\partial F_i}{\partial \dot{q}^j} + \frac{\partial F_j}{\partial \dot{q}^i} &= 2\frac{d}{dt}\frac{\partial F_i}{\partial \ddot{q}^j} \\ &= \frac{d}{dt}\left(\frac{\partial F_j}{\partial \ddot{q}^i} + \frac{\partial F_i}{\partial \ddot{q}^j}\right) && (19) \\ \frac{\partial F_i}{\partial q^j} - \frac{\partial F_j}{\partial q^i} &= \frac{d}{dt}\left[\frac{d}{dt}\left(\frac{\partial F_j}{\partial \ddot{q}^i}\right) - \frac{\partial F_j}{\partial \dot{q}^i}\right] \\ &= \frac{1}{2}\frac{d}{dt}\left(\frac{\partial F_i}{\partial \dot{q}^j} - \frac{\partial F_j}{\partial \dot{q}^i}\right). && (20) \end{aligned}$$

We say that a system of ordinary differential equations $F_i(t, q, \dot{q}, \ddot{q}) = 0$, $i = 1, ..., n$, is self-adjoint if it satisfies these equations.

2.1.2 Self-Adjointness of Newtonian Equations

If we apply the self-adjointness conditions to Newtonian differential equations (1),

$$A_{ij}(t, q, \dot{q})\ddot{q}^j + B_i(t, q, \dot{q}) = 0,$$

then the functions F_i take the form

$$F_i(t, q, \dot{q}, \ddot{q}) = A_{ij}(t, q, \dot{q})\ddot{q}^j + B_i(t, q, \dot{q}). \quad (21)$$

Equations (18) - (20) become

$$\begin{aligned}
A_{ij} &= A_{ji} \\
\frac{\partial A_{ik}}{\partial \dot{q}^j}\ddot{q}^k + \frac{\partial A_{jk}}{\partial \dot{q}^i}\ddot{q}^k + \frac{\partial B_i}{\partial \dot{q}^j} + \frac{\partial B_j}{\partial \dot{q}^i} &= \frac{d}{dt}(A_{ij} + A_{ji}) \\
\frac{\partial A_{ik}}{\partial q^j}\ddot{q}^k - \frac{\partial A_{jk}}{\partial q^i}\ddot{q}^k + \frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} &= \frac{1}{2}\frac{d}{dt}\left(\frac{\partial A_{ik}}{\partial \dot{q}^j}\ddot{q}^k - \frac{\partial A_{jk}}{\partial \dot{q}^i}\ddot{q}^k + \frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right).
\end{aligned}$$

Expanding out the total time derivatives and requiring that equalities hold for all choices of $\dot{q}$ and $\ddot{q}$, we arrive the equations:

$$A_{ij} = A_{ji} \tag{22}$$

$$\frac{\partial A_{ik}}{\partial \dot{q}^j} = \frac{\partial A_{jk}}{\partial \dot{q}^i} \tag{23}$$

$$\frac{\partial B_i}{\partial \dot{q}^j} + \frac{\partial B_j}{\partial \dot{q}^i} = 2\left\{\frac{\partial}{\partial t} + \dot{q}^k\frac{\partial}{\partial q^k}\right\}A_{ij} \tag{24}$$

$$\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k\frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right). \tag{25}$$

These are the conditions of self-adjointness for Newtonian equations. It turns out that these conditions are precisely the integrability conditions of equations (5) and (6).

2.1.3 Explicit Construction of Lagrangian Function

We are now prepared to state a key theorem of the Inverse Problem. We will show that self-adjointness is a necessary and sufficient condition for a Newtonian system (1), with invertible matrix A_{ij}, to possess a Lagrangian L satisfying

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \equiv A_{ij}(t,q,\dot{q})\ddot{q}^j + B_i(t,q,\dot{q}). \tag{26}$$

The proof goes as follows. We showed earlier that equation (26) is leads to the relations (5) and (6). So, first let us suppose that a Lagrangian satisfying equation (26) exists. Then, using identifications (5) and (6), it is straightforward to show that the Newtonian system satisfies the conditions of self-adjointness (22) - (25). This demonstrates that self-adjointness is a necessary condition.

Now suppose that the Newtonian system satisfies the conditions of self-adjointness (22) - (25). Then we explicitly construct the following solution to equations (5) and (6):

$$L(t,q,\dot{q}) = K(t,q,\dot{q}) + D_i(t,q)\dot{q}^i + C(t,q) \tag{27}$$

where

$$\begin{aligned}
K(t,q,\dot{q}) &= \dot{q}^i \int_0^1 d\tau' \left\{ \dot{q}^j \int_0^1 d\tau A_{ij}(t,q,\tau\dot{q}) \right\} (t,q,\tau'\dot{q}) && (28)\\
D_i(t,q) &= q^j \int_0^1 d\tau Z_{ij}(t,\tau q)\tau && (29)\\
C(t,q) &= q^i \int_0^1 d\tau W_i(t,\tau q) && (30)\\
Z_{ij}(t,q) &= \frac{1}{2}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right) + \left(\frac{\partial^2 K}{\partial q^i \partial \dot{q}^j} - \frac{\partial^2 K}{\partial q^j \partial \dot{q}^i}\right) && (31)\\
W_i(t,q) &= \frac{\partial D_i}{\partial t} - B_i - \frac{\partial K}{\partial q^i} + \frac{\partial^2 K}{\partial \dot{q}^i \partial t} + \left[\frac{\partial^2 K}{\partial q^i \partial \dot{q}^j} + \frac{1}{2}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right)\right]\dot{q}^j. && (32)
\end{aligned}$$

For the derivation of this rather involved solution from equations (5) and (6), we refer the reader to Santilli's book [9]. For the purposes of this proof, it is enough simply to verify that the solution (27) does indeed satisfy equation (26). We leave this relatively straightforward calculation to the interested reader and now give some illustrations of the theorem. Because the notation in formulae (28) - (32) can be somewhat confusing, our examples include detailed calculations.

2.2 Examples

2.2.1 Particles in External Potential

Consider the case of n particles of mass m in an external potential $V(q)$. The Newtonian equations of motion are

$$m\ddot{q}^i + \tfrac{\partial V}{\partial q^i} = 0 \quad i = 1, ..., 3n. \tag{33}$$

Comparing this expression to the form (1), we identify

$$A_{ij} = m\delta_{i,j} \qquad B_i = \tfrac{\partial V}{\partial q^i}. \tag{34}$$

It is easy to verify that A_{ij} and B_i satisfy the self-adjointness conditions (22) - (25). So, a Lagrangian exists, and we construct it by calculating

$$\begin{aligned}
K(t,q,\dot{q}) &= \dot{q}^i \int_0^1 d\tau' \left\{ \dot{q}^j \int_0^1 d\tau A_{ij}(t,q,\tau\dot{q}) \right\} (t,q,\tau'\dot{q}) \\
&= \dot{q}^i \int_0^1 d\tau' \left\{ \dot{q}^j \int_0^1 d\tau (m\delta_{i,j}) \right\} (t,q,\tau'\dot{q}) \\
&= \dot{q}^i \int_0^1 d\tau' \{ m\dot{q}^i \} (t,q,\tau'\dot{q}) \\
&= \dot{q}^i \int_0^1 d\tau' m\tau'\dot{q}^i = \frac{1}{2} m\dot{q}^i\dot{q}^i = \frac{1}{2} m\dot{q}^2 &(35)\\
D_i(t,q) &= q^j \int_0^1 d\tau Z_{ij}(t,\tau q)\tau = 0 &(36)\\
C(t,q) &= q^i \int_0^1 d\tau W_i(t,\tau q) \\
&= q^i \int_0^1 d\tau \left\{ -\frac{\partial V}{\partial q^i} \right\} (t,\tau q) \\
&= -q^i \int_0^1 d\tau \frac{\partial V(\tau q)}{\partial \tau q^i} \\
&= -\int_0^1 d\tau \frac{dV(\tau q)}{d\tau} = -V(q). &(37)
\end{aligned}$$

Inserting these calculations into equation (27), we recover the familiar result

$$L(t,q,\dot{q}) = \frac{1}{2} m\dot{q}^2 - V(q). \tag{38}$$

2.2.2 Relativistic Particle

Consider a free relativistic particle obeying the equation of motion

$$\frac{d}{dt} \frac{m\dot{q}^i}{\sqrt{1-(\frac{\dot{q}}{c})^2}} = 0. \tag{39}$$

Carrying out the time differentiation, we obtain

$$\frac{m}{\sqrt{1-(\frac{\dot{q}}{c})^2}} \ddot{q}^i + \frac{m\dot{q}^i}{(1-(\frac{\dot{q}}{c})^2)^{\frac{3}{2}}} \frac{\dot{q}^j}{c^2} \ddot{q}^j = 0, \tag{40}$$

which gives

$$A_{ij} = \frac{m}{\sqrt{1-(\frac{\dot{q}}{c})^2}}\delta_{i,j} + \frac{m\dot{q}^i}{(1-(\frac{\dot{q}}{c})^2)^{\frac{3}{2}}}\frac{\dot{q}^j}{c^2} \qquad B_i = 0. \tag{41}$$

These functions clearly satisfy the self-adjointness conditions (22) - (25), so we proceed to compute the Lagrangian.

$$\begin{aligned}
K(t,q,\dot{q}) &= \dot{q}^i \int_0^1 d\tau' \left\{ \dot{q}^j \int_0^1 d\tau A_{ij}(t,q,\tau\dot{q}) \right\} (t,q,\tau'\dot{q}) \\
&= \dot{q}^i \int_0^1 d\tau' \left\{ \dot{q}^j \int_0^1 d\tau \left(\frac{m}{\sqrt{1-(\frac{\tau\dot{q}}{c})^2}}\delta_{i,j} + \frac{m\tau\dot{q}^i}{(1-(\frac{\tau\dot{q}}{c})^2)^{\frac{3}{2}}}\frac{\tau\dot{q}^j}{c^2} \right) \right\} (t,q,\tau'\dot{q}) \\
&= \dot{q}^i \int_0^1 d\tau' \left\{ \int_0^1 d\tau \frac{d}{d\tau} \left(\frac{m\tau\dot{q}^i}{\sqrt{1-(\frac{\tau\dot{q}}{c})^2}} \right) \right\} (t,q,\tau'\dot{q}) \\
&= \dot{q}^i \int_0^1 d\tau' \left(\frac{m\tau'\dot{q}^i}{\sqrt{1-(\frac{\tau'\dot{q}}{c})^2}} \right) = -mc^2\sqrt{1-\left(\frac{\dot{q}}{c}\right)^2} \qquad (42) \\
D_i(t,q) &= q^j \int_0^1 d\tau Z_{ij}(t,\tau q)\tau = 0 \qquad (43) \\
C(t,q) &= q^i \int_0^1 d\tau W_i(t,\tau q) = 0. \qquad (44)
\end{aligned}$$

Assembling these formulae, we arrive at the usual relativistic Lagrangian

$$L = -mc^2\sqrt{1-\left(\frac{\dot{q}}{c}\right)^2}. \tag{45}$$

2.2.3 Particle Experiencing Drag Force

Although it is conventional to introduce a Rayleigh dissipation function when applying the Lagrangian formalism to nonconservative systems [5, 6], our theorem shows that this may be unnecessary in many circumstances. As long as Newtonian equations are self-adjoint, they will admit a Lagrangian formalism regardless of whether or not they are conservative. To illustrate this point, we consider the case of a particle experiencing a drag force according to

$$\ddot{q}^i + \gamma\dot{q}^i = 0. \tag{46}$$

This Newtonian equation has

$$A_{ij} = \delta_{i,j} \qquad B_i = \gamma \dot{q}^i. \tag{47}$$

It turns out that these functions violate condition (24), so equation (46) is not self-adjoint. However, if we rewrite the equation in the physically equivalent form

$$e^{\gamma t}\ddot{q}^i + e^{\gamma t}\gamma\dot{q}^i = 0, \tag{48}$$

so that

$$A_{ij} = e^{\gamma t}\delta_{i,j} \qquad B_i = e^{\gamma t}\gamma\dot{q}^i, \tag{49}$$

then the equation is self-adjoint. We calculate

$$\begin{aligned} K(t,q,\dot{q}) &= \dot{q}^i \int_0^1 d\tau' \left\{ \dot{q}^j \int_0^1 d\tau A_{ij}(t,q,\tau\dot{q}) \right\} (t,q,\tau'\dot{q}) \\ &= \dot{q}^i \int_0^1 d\tau' \left\{ \dot{q}^j \int_0^1 d\tau (e^{\gamma t}\delta_{i,j}) \right\} (t,q,\tau'\dot{q}) \\ &= \frac{1}{2}\dot{q}^i\dot{q}^i e^{\gamma t} = \frac{1}{2}\dot{q}^2 e^{\gamma t} \end{aligned} \tag{50}$$

$$D_i(t,q) = q^j \int_0^1 d\tau Z_{ij}(t,\tau q)\tau = 0 \tag{51}$$

$$C(t,q) = q^i \int_0^1 d\tau W_i(t,\tau q) = 0 \tag{52}$$

so that the Lagrangian takes the form

$$L = \frac{1}{2}e^{\gamma t}\dot{q}^2. \tag{53}$$

This example exhibits two important features of our theorem. First, it shows that it may be possible to put a nonconservative system in Lagrangian form without the use of a dissipation function. Second, it shows that the property of self-adjointness is somewhat superficial from a physical standpoint. Although equation (46) is not self-adjoint, equation (48), which is clearly physically equivalent, is self-adjoint. This situation forces us to ask how seemingly trivial changes in an equation of motion influence the Lagrangian formalism. This question is one aspect of the problem of the nonuniqueness of the Lagrangian function.

3 Nonuniqueness of Lagrangian Function

It is a familiar fact that some changes in the Lagrangian function do not significantly change the Euler-Lagrange equations. Physics textbooks typically emphasize our freedom to multiply the Lagrangian by a constant or to add a total time derivative to the Lagrangian [5]. Here, we study the nonuniqueness of the Lagrangian more generally. We discuss the causes of this nonuniqueness and some of its consequences.

3.1 Nonuniqueness in Construction of Lagrangian

Assume that we are given self-adjoint Newtonian equations of motion (1). In our previous discussion, we provided a recipe to calculate a Lagrangian L that satisfies the identity (26)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \equiv A_{ij}(t,q,\dot{q})\ddot{q}^j + B_i(t,q,\dot{q}).$$

Suppose that another Lagrangian L' satisfies the same identity,

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}^i} - \frac{\partial L'}{\partial q^i} \equiv A_{ij}(t,q,\dot{q})\ddot{q}^j + B_i(t,q,\dot{q}).$$

Then we can show that the two must differ by a total time derivative:

$$L - L' = \frac{df(t,q)}{dt} \tag{54}$$

for some function $f(t,q)$. This follows because the difference of the Lagrangians satisfies

$$\frac{d}{dt}\frac{\partial L-L'}{\partial \dot{q}^i} - \frac{\partial L-L'}{\partial q^i} = \frac{\partial^2 L-L'}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j + \frac{\partial^2 L-L'}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \frac{\partial^2 L-L'}{\partial \dot{q}^i \partial t} - \frac{\partial L-L'}{\partial q^i} \equiv 0. \tag{55}$$

Since this must hold for arbitrary paths, we conclude that

$$\frac{\partial^2 L-L'}{\partial \dot{q}^i \partial \dot{q}^j} = 0 \tag{56}$$

$$\frac{\partial^2 L-L'}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \frac{\partial^2 L-L'}{\partial \dot{q}^i \partial t} = \frac{\partial L-L'}{\partial q^i}. \tag{57}$$

Equation (56) implies that

$$L - L' = F_i(t,q)\dot{q}^i + G(t,q), \tag{58}$$

for some functions $F(t,q)$ and $G(t,q)$. Inserting this into equation (57), we find the relations

$$\frac{\partial F_i}{\partial q^j} - \frac{\partial F_j}{\partial q^i} = 0 \tag{59}$$

$$\frac{\partial F_i}{\partial t} - \frac{\partial G}{\partial q^i} = 0. \tag{60}$$

Expression (59) implies that the one-form $F = F_i dq^i$ is closed, so by the Poincaré Lemma [1], it is exact on some region:

$$F_i = \frac{\partial H(t,q)}{\partial q^i} \tag{61}$$

for some function $H(t,q)$. Inserting this result into (60), we see that

$$G(t,q) - \frac{\partial H}{\partial t} = J(t) \tag{62}$$

for some function $J(t)$. So, we obtain

$$L - L' = \frac{\partial H(t,q)}{\partial q^i}\dot{q}^i + \frac{\partial H(t,q)}{\partial t} + J(t) = \frac{d}{dt}[H(t,q) + \int^t dt' J(t')], \tag{63}$$

which proves our claim. So, we see that the Lagrangian function is specified up to a total time derivative by equation (26). This ambiguity is familiar, and we will not discuss it further. There are more profound ambiguities to consider.

3.2 Nonuniqueness in Equations of Motion

We have seen that identity (26) completely pins down the Lagrangian except for a total time derivative term. With some thought, however, we discover that this identity does not capture the Inverse Problem in complete generality. Instead, the Inverse Problem should involve the search for any Lagrangian satisfying

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \Longleftrightarrow A_{ij}\ddot{q}^j + B_i = 0. \tag{64}$$

The significance of this change is reflected in the following example. Suppose the Lagrangian L satisfies identity (26) for some system of Newtonian equations. Then the Lagrangian $L' = cL$, $c \in R$, will not satisfy the identity unless $c = 1$. However, provided $c \neq 0$, L' will clearly produce an equivalent set of equations and will therefore satisfy (64).

Practically speaking, it is essential that we take this more general perspective on the Inverse Problem. This is because, as we saw in the case of a particle experiencing a drag force, sometimes an equation is not self-adjoint until it gets massaged a bit. So, for equation (46), while no Lagrangian satisfies identity (26) we found a Lagrangian that satisfies (64).

These examples show that the more general version of the Inverse Problem, expressed in relation (64), is the appropriate one for most purposes. On the other hand, although we would like work with the more general Inverse Problem, a complete solution to this problem seems unmanageable. To achieve such a solution directly, we would have to be able to list all the equations equivalent to a given Newtonian system and then find a Lagrangian for each one that happened to be self-adjoint. Such a list of equations would have to include the given system itself, the given system with each equation multiplied by any nonvanishing function, the given system with each equation cubed, and so on. Confronted with the difficulty of assembling such a list, we abandon the pursuit of a complete solution to the more general Inverse Problem (64). Instead, we present several noteworthy examples to convey an impression of the great variety of Lagrangians that can satisfy (64). This variety makes the nonuniqueness of the Lagrangian quite vast for most Newtonian systems.

3.3 Examples

3.3.1 Particles in Harmonic Oscillator Potential

Consider two particles in a harmonic oscillator potential, with equations of motion given by

$$\begin{aligned} \ddot{q}_1^i + \omega^2 q_1^i &= 0 \\ \ddot{q}_2^i + \omega^2 q_2^i &= 0, \end{aligned} \tag{65}$$

where $i = 1, \ldots 3$. In these equations, q_1 refers to the position of particle 1, q_2 refers to the position of particle 2, and ω is the harmonic oscillator frequency. Now, if we seek a Lagrangian satisfying the usual identities

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1^i} - \frac{\partial L}{\partial q_1^i} \equiv \ddot{q}_1^i + \omega^2 q_1^i \tag{66}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2^i} - \frac{\partial L}{\partial q_2^i} \equiv \ddot{q}_2^i + \omega^2 q_2^i, \tag{67}$$

then we obtain the familiar Lagrangian

$$L = \frac{1}{2}(\dot{q}_1)^2 + \frac{1}{2}(\dot{q}_2)^2 - \frac{1}{2}\omega^2(q_1)^2 - \frac{1}{2}\omega^2(q_2)^2. \tag{68}$$

Suppose instead that we seek a Lagrangian yielding the equations of motion, exchanged in position:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1^i} - \frac{\partial L}{\partial q_1^i} \equiv \ddot{q}_2^i + \omega^2 q_2^i \tag{69}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2^i} - \frac{\partial L}{\partial q_2^i} \equiv \ddot{q}_1^i + \omega^2 q_1^i. \tag{70}$$

It turns out that a Lagrangian satisfying these relations is

$$L = \dot{q}_1^i \dot{q}_2^i + \omega^2 q_1^i q_2^i. \tag{71}$$

(This Lagrangian is a simplification of the so called Morse-Feshbach Lagrangian [8]). Certainly, it yields the equations of motion (65) just as well as the usual Lagrangian (68) does. So here we have a striking example of an unusual Lagrangian associated with very mundane Newtonian equations.

This example also shows how the nonuniqueness of the Lagrangian can have dramatic ramifications. Suppose that we canonically quantize a system of two harmonic oscillators using the alternate Lagrangian (71). According to canonical quantization, the canonical momentum $p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_2$ does not commute with the coordinate q_1. So, this alternate Lagrangian suggests that it is impossible simultaneously to measure the velocity of particle

2 and the position of particle 1. And according to the alternate Lagrangian, there is no obstacle to making a simultaneous measurement $p_2 = \dot{q}_1$ and q_1. Hence, although the usual Lagrangian (68) and the alternate Lagrangian (71) have essentially equivalent Euler-Lagrange equations, the Lagrangians are radically different from a quantum mechanical perspective. This situation, wherein classically equivalent Lagrangians give rise to non-equivalent quantum mechanical theories, has been studied in several contexts [4].

3.3.2 Free Particle in One Dimension

As another example of a bizarre Lagrangian, let us consider the motion of a free particle in one dimension. The equation of motion is simply $\ddot{q} = 0$, where $q \in R$. Multiplying the equation through by a nonvanishing function of the energy, we obtain the equivalent equation

$$f(\dot{q}^2)\ddot{q} = 0 \quad f \neq 0 \tag{72}$$

It is straightforward to verify that this equation is self-adjoint (in one-dimension the first two conditions of self-adjointness, (22) and (23), are satisfied identically). The constructed Lagrangian is

$$L = \int_0^1 d\tau' \frac{1}{\tau'} \int_0^{(\tau'\dot{q})^2} dx f(x). \tag{73}$$

For the choice $f(x) = 1$, we regain the usual Lagrangian $L = \frac{1}{2}\dot{q}^2$. For another choice, say $f(x) = 1 + 5x^2$, we obtain

$$L = \frac{1}{2}\dot{q}^2 + \dot{q}^6. \tag{74}$$

Clearly, L can be made arbitrarily complicated by obnoxious choices of $f(x)$. The quantum theories arising from these various Lagrangians differ fundamentally.

4 Conclusion

In this paper, we have explored the nonuniqueness of the Lagrangian, some of its sources and some of its ramifications. From a mathematical standpoint, basic questions remain about the solution to the general Inverse Problem (64). From a physical standpoint, issues of nonuniqueness raise basic concerns about the canonical quantization procedure. At this point, physics can only offer an experimental justification for selecting one Lagrangian for quantization out of the many classically viable choices. It is hoped that further research into this question may provide insight into the mysterious efficacy of the canonical quantization procedure itself.

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