

Holonomic Quantum Computing and Geometric Quantum Phase CDS205

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Abstract

In this paper we present the theory of quantum holonomy and discuss its application to Holonomic Quantum Computing with a goal of presenting the material as geometrically as possible. In addition, we argue that the adiabatic assumption made in calculations of the connection in quantum holonomy is equivalent to averaging a connection in the sense of the Hannay-Berry connection.

1 Introduction

The realization that cyclic adiabatic variations of a quantum mechanical Hamiltonian (with a concurring cyclic evolution of the state) could lead to an observable phase was made by Berry [1] in 1984. In the past several years, it has been discovered that it is possible to generate, given a suitable family of Hamiltonians, a complete set of unitary operators in a subspace of the full system Hilbert space, allowing for the execution of quantum logic [2]. Whereas traditional quantum computing executes quantum gates by allowing the system to evolve dynamically under a (locally) time independent Hamiltonian, Holonomic Quantum Computation (HQC), as this technique is known, generates quantum gates by driving the system state adiabatically through a loop. Indeed, the implementations suggested in [2][3] perform computations on degenerate subspaces of the Hamiltonian family and so dynamic phase may be neglected entirely.

HQC has several potential advantages over traditional quantum computing. The fidelity of a quantum gate depends on the area of the loop of the holonomy (provided that the loop is restricted to the desired sub-manifold) and not on the specific path taken to achieve the loop, and so a simple scaling argument shows that, for

a circular loop, fluctuations of the Hamiltonian are suppressed as $1/r$ where r is the radius of the loop, defined in suitable parameter coordinates. HQC is insensitive to the time taken to traverse a computational loop provided the adiabatic condition (see below) is met, which also implies that a system with very fast degrees of freedom may be used for computations with slowly varying Hamiltonians. (Note however, that decoherence times often track system timescales, so in order to build a realistic quantum computer, it may be necessary to perform operations rapidly to avoid decoherence.) Finally, HQC offers a means of generating quantum logic through a straightforward manipulation of the system Hamiltonian and does not (for individual one- and two-qubit gates) require the ability to turn part of the Hamiltonian on and off.

HQC is performed by realizing parallel transport through a fiber bundle with a suitable non-abelian connection. This underlying geometric structure was first elucidated in the highly readable paper of Simon in 1983 [4].¹ Here, it is demonstrated that Berry's phase may be interpreted as the holonomy of a line bundle over the parameter space. The realization that holonomy could be used to perform quantum computations came with the study of more complicated connection structures, where the connection takes values in a non-abelian Lie algebra, typically $u(n)$ [2]. Indeed, while there was considerable debate surrounding the physicality of Berry's phase, the ability to generate relative phases and transfer populations between states in the computational sub-manifold makes it clear that the geometric effects in HQC are physical.

After the discovery and explanation of geometrical quantum phase, considerable progress was made in the development of a parallel theory for the classical case (see [5] for a comprehensive summary). In particular, it was realized that a key element of the classical theory is that averaging the connection over the relevant group action to form the Hannay-Berry connection [6]. This incorporates the fast degrees of freedom into the connection as we shall see below.

This paper will be organized as follows. We begin with an brief introduction to the connection on fiber bundles including parallel transport and holonomy. We next discuss the traditional adiabatic approach to quantum holonomy and include a motivating example. We next introduce holonomic quantum computing and briefly discuss the most promising proposal for its implementation [3]. This section will be fairly brief, however, as the major ideas are developed in the section on quantum holonomy. Finally, I introduce the new results of this paper, the relationship between the quantum adiabatic assumption and the averaging theorem of the classical theory.

¹That this was published before Berry's paper [1] is due to length of review processes and conference proceedings.

2 The Connection, Fiber Bundles, and Holonomy

This section is intended to serve as a brief introduction to the relevant portions of holonomy theory including some background in the theory of fiber bundles and the connection. The discussion here is based largely on that of Nakahara [7] who presents a unified and consistent discussion. Additional discussion of these topics as they relate more directly to classical systems can be found in [8].

We begin with a principal G bundle P with a projection $\pi : P \rightarrow M$ such that G is the fiber, the bundle looks locally like $G \times M$, and the natural group action which acts fiber-wise is compatible with the bundle structure. A *connection* is a division of TP into a vertical (tangent to the fibers) V_p and a horizontal part H_p such that $T_pP = V_pP \oplus H_pP$, any vector field can be naturally decomposed into $X = X^V + X^H$, and the horizontal structure is invariant under the group action. V_p is defined intrinsically as the space of tangents to curves that lie in a fiber, that is, curves $c(t)$ such that $c : [0, 1] \rightarrow P$ s.t. $c(t) \in \pi^{-1}(m)$. A general fiber bundle lacks sufficient structure, however, to uniquely define the horizontal direction. A connection provides the additional structure.² An important point is that the vertical subspace is naturally identified with the Lie algebra \mathfrak{g} of G since $V_{\pi^{-1}(m)}P$ is the tangent space to a fiber and hence isomorphic to the tangent space of G . An element of $\xi \in \mathfrak{g}$ naturally defines a v.f. $\xi^\# \in \mathfrak{X}(P)$ by $\xi^\#[f(u)] = d/dt|_{t=0} f(u \exp(\xi t))$.

As defined, the connection is an abstract factorization of the TP and is of little use in calculation. To deal with the connection more concretely, we introduce the *connection valued one form* $\omega : TP \rightarrow \mathfrak{g}$. This form reconstructs the connection in the following manner: ω is defined so that $\omega(\xi^\#) = \xi$ and $R_g^*\omega = \text{Ad}_{g^{-1}}\omega$. Given such a form, the horizontal subspace is defined to be the kernel of ω , so that $H_uP = \ker \omega(X)|_{X \in T_uP}$.

In many cases, we will be working on the base M . We can naturally define a family of local connection one forms using sections $\sigma_i : U_i \rightarrow P, U_i \in M$ such that $\pi(\sigma(m)) = m$. This then naturally induces a connection one form $\mathcal{A}_i : T_{U_i}M \rightarrow \mathfrak{g}$ by $\mathcal{A}_i = \sigma_i^*\omega$.³ From our perspective, the most important aspect of the connection is that it leads naturally to *parallel transport* and consequently to the *covariant*

²The notion of “vertical” is very natural. For example $\pi_*X = 0$ when $X \in V$ so the vector can be thought of as perpendicular to the base of the bundle. Care must be taken in interpreting this, however, as the base M is not naturally embedded into P since general bundles do not admit global sections.

³Although only peripherally relevant to the current discussion, the choice of sections σ_i corresponds to the choice of a gauge. \mathcal{A}_i is the *gauge potential*. We see that gauge invariance is a consequence of the fact that the overall structure of the dynamics is governed by ω and the choice of a specific section doesn’t effect the dynamics.

derivative, D. To deal with parallel transport, we introduce the *horizontal lift*. Let $\gamma : [0, 1] \rightarrow M$ be a curve in the base manifold. A horizontal lift $\tilde{\gamma}$ is any continuous curve such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$ and $\dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)}P$. That is, the tangent to the curve is always in the horizontal subspace. If we choose a section $\sigma_i : U_i \rightarrow P$ where $\gamma \in U_i$ the horizontal lift of the curve is described by curve in G , $g(t) : [0, 1] \rightarrow G$ where $\tilde{\gamma}(t) = \sigma_i(\gamma(t))g(t)$. The differential equation for this curve is

$$\frac{dg(t)}{dt} = -\omega(\sigma_{i*}\dot{\gamma}(t))g(t) = -\mathcal{A}_i(\dot{\gamma}(t))g(t). \quad (1)$$

The solution for the horizontal lift starting at $u_0 \in \pi^{-1}(\gamma(t))$ is given by the path ordered exponential

$$\tilde{\gamma}(t)(u_0) = u_0 \mathbb{P} \exp \left[- \int_0^t (\mathcal{A}_i(\gamma(s)))_{\mu} dx^{\mu} \dot{\gamma}(s) ds \right] \quad (2)$$

$$= u_0 \mathbb{P} \exp \left[- \int_{\gamma(0)}^{\gamma(t)} (\mathcal{A}_i(\gamma(s)))_{\mu} dx^{\mu} \right], \quad (3)$$

where the path ordering is required in the exponential because we are interested in non-abelian Lie algebras. The final point of the curve $\tilde{\gamma}(1)$ is the *parallel transport* of u_0 by the connection and the path γ . This coincides with the intuitive notion of parallel transport. Restricting $\tilde{\gamma}$ to the horizontal subspace ensures that we don't make any changes "along the fibers". Parallel transport is naturally associated with a covariant derivative⁴ which is defined locally by

$$\mathbf{D}\alpha(X_0, \dots, X_N) = \mathbf{d}\alpha(X_0^{hor}, \dots, X_N^{hor}), \quad (4)$$

where \mathbf{d} is the normal exterior derivative, X^{hor} denotes the horizontal component of X and α is an N -form.

We are now prepared to introduce the most important construction in this section. Let us now consider closed paths such that $\gamma(0) = \gamma(1)$. What is important is that in general $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$. We can, however, write these in terms of each other. We define the holonomy of the path $\gamma(t)$ to be g_{γ} where $\tilde{\gamma}(0)g_{\gamma} = \tilde{\gamma}(1)$. From eq. (3) it is clear that the holonomy is independent of the initial point on the horizontal lift. The holonomies clearly form a subgroup of G , known as the holonomy group of P . When P is connected, the holonomy group is easily seen to be the same for the entire bundle.

When we are dealing with a closed path eq. (3) becomes a loop integral and can be reduced to a surface integral using Stokes's theorem, when the hypotheses

⁴In many texts, parallel transport is introduced largely in order to derive the covariant derivative which requires the comparison of vectors that live on different fibers.

of Stoke's theorem are met,

$$g_\gamma = \mathbb{P} \exp \left[- \oint_\gamma (\mathcal{A}_i(\gamma(s)))_\mu dx^\mu \right] \quad (5)$$

$$= \mathbb{P} \exp \left[- \iint_S \mathbf{d}\mathcal{A}_{i\mu\nu} dx^\mu \wedge dx^\nu \right], \quad (6)$$

where S is an oriented surface bounded by the loop. It is this latter expression that will lead to the area dependence of the quantum holonomy. The quantity $\mathbf{d}\mathcal{A}$ is the *field strength* of the connection one-form and coincides with the field bilinears which are familiar in physics. In coordinates, the field strength is written

$$\mathbf{d}\mathcal{A}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \quad (7)$$

For our purposes, the field strength is important for two reasons. First, it is crucial for calculating the holonomy of a given loop. Second, the field strength provides a means of determining whether the holonomy group is or is not a proper subgroup of G . When the elements of the (\mathfrak{g} -valued) field strength span the \mathfrak{g} , the holonomy group is the complete group G (this is the Ambrose-Singer Theorem). This will allow us, when performing HQC, to generate a universal set of quantum gates [2].

These are the basic mathematical facts which will be required to understand quantum holonomies and HQC. A more complete treatment can be found in e.g. [5][7][8].

3 Quantum Holonomies

We will now specialize to quantum mechanical systems. We begin with a formal derivation of holonomies for adiabatic changes in the system Hamiltonian. We will then back up and do a simple “brute force” calculation in an attempt to understand the nature of the connection which arises in the adiabatic case. The formal derivation will take elements from [2], [4] and [7] while the latter will follow the discussion in [7]. An interesting discussion of a more general theory which requires neither the adiabatic assumption or the assumption of a degenerate subspace is presented in [9]. This theory is more general than is required for our purposes and so will not be discussed further.

We consider a quantum mechanical system with a continuous family of Hamiltonians $H(\lambda)$ which depend continuously on $\lambda \in M$ where M is a smooth manifold of parameters. The experimenter has the ability to control these parameters and drive the system through a trajectory in its state space. The state space for the system consists of rays in a complex Hilbert space \mathcal{H} . We will make the following

assumptions regarding the system. The Hamiltonian has, for each λ a degenerate subspace of its (discrete) eigenspace with fixed dimension n . There are no level crossings (the energy of our eigenspace is separated by a finite energy from all other eigenspaces of the Hamiltonians). A holonomy will be generated by traversing a closed loop $\gamma(t) : \mathbb{R} \rightarrow M$ so that the system evolves adiabatically.

Adiabatic evolution of the system occurs when the Hamiltonian varies slowly in comparison to relevant dynamical timescales so that the system remains in a fixed eigenstate of the Hamiltonian in the following sense:⁵ under an adiabatic evolution, a system which begins in the n^{th} eigenstate of the Hamiltonian $H(\gamma(0))$, $|n(\gamma(0))\rangle$ will remain in that state. This eigenstate is well defined because we have assumed that no level crossings take place. This description is valid in the case that the n^{th} eigenstate is degenerate. We thus write

$$H(\gamma(t)) |n(\gamma(t))\rangle = E_n(\gamma(t)) |n(\gamma(t))\rangle. \quad (8)$$

We include the path dependence in the ket to indicate that the eigenstate is a function of the location in parameter space.

We now wish to construct the relevant fiber bundle upon which we will be working. Although the Hamiltonian takes a central role in describing the how the system will be manipulated, it is, in fact in the state space that we will construct our fiber bundle. For a specific $H(\lambda)$ we can choose a state vector $|n(\lambda)\rangle$ in this eigenspace, which corresponds to choosing a section of the bundle. This will lead to a natural projection for a fiber bundle. $\pi : |n(\lambda)\rangle \mapsto \lambda \in M$ is a reasonable initial guess, however is not necessarily a single valued map and so is not a good projection. We do see, however, that the fiber is naturally isomorphic to the set of operators which leave the eigenspace invariant. If the dimension of the eigenspace is n and $\dim \mathcal{H} = N$, then the bundle is [2]

$$P = \frac{U(N)}{U(N-n)}. \quad (9)$$

This is understood as follows, the total set of unitaries acting on \mathcal{H} is $U(N)$. However, we are only interested in those unitaries that leave the subspace invariant, and so take the quotient with $U(N-n)$ which do not. The base of the manifold is now defined by taking the quotient with the group $U(n)$ which acts on the degenerate subspace. Thus, the principal bundle with which we work is defined by

$$\pi : \frac{U(N)}{U(N-n)} \rightarrow \frac{U(N)}{U(N-n) \times U(n)}. \quad (10)$$

⁵We assume that the Hamiltonian has a countable set of eigenvectors, including degeneracies

Notice that the fiber of this bundle is isomorphic to the space of allowable states in the degenerate subspace.

Many interesting cases, including Berry's Phase and the systems that have been proposed for quantum computation, have a total Hilbert space dimension that is one larger than the dimension of the computational subspace, or have a one dimensional computational space. In either case, the base of this manifold is the complex projective space $C\mathbf{P}^n$, and the bundle is S^{n+1} . We shall return to this structure later.

Unlike the abstract bundles discussed in the previous section, this bundle inherits additional Hermitian structure from the underlying Hilbert space. We can use the Hermitian inner product to naturally define a connection on the space. To see how, choose a basis for a specific degenerate subspace, $|n_i\rangle$. The horizontal subspace at $|n\rangle$ is defined to be the vector space spanned by $|m\rangle$ such that $\langle m|n_i\rangle = 0 \forall i$.

What we have developed so far is sufficient to at least write down an equation for parallel transport. We can decompose the time derivative of an arbitrary state into vertical and horizontal components, in the one dimensional case $|\dot{\psi}\rangle = \langle\psi|\dot{\psi}\rangle|\psi\rangle + |\text{hor}_{\psi}\rangle$, where this relationship defines $|\text{hor}_{\psi}\rangle$. The parallel transport is then given by $|\dot{\psi}\rangle = |\text{hor}_{\psi}\rangle$. However, for the general calculations, it is useful to have an explicit expression for the connection one form. To find an explicit expression for the connection, we note that, in coordinates, the covariant derivative may be written

$$\mathbf{D}_{\mu} = \frac{\partial}{\partial\lambda^{\mu}} + \mathcal{A}_{\mu}^i \frac{\partial}{\partial\xi^i}, \quad (11)$$

where ξ is a basis for the Lie algebra of the fiber. It is often more convenient to express \mathcal{A} as a matrix, so that we find [2],

$$(\mathcal{A}_{\mu})_{\alpha\beta} = \langle\psi_{\alpha}(\lambda)| \frac{\partial}{\partial\lambda^{\mu}} |\psi_{\beta}(\lambda)\rangle. \quad (12)$$

That is, we express the $\mathfrak{u}(n)$ valued \mathcal{A} in the matrix basis of $\mathfrak{u}(n)$. $|\psi_{\alpha}(\lambda)\rangle$ form an orthogonal basis of the eigenspace at parameter λ . Given a choice $|\psi_{\alpha}(0)\rangle$, this basis is well defined because the geometric phase will preserve orthogonality. From the connection, we can calculate the curvature and consequently calculate the Lie algebra of the holonomy group using the Ambrose-Singer theorem discussed above.

The holonomy can now be written by simply repeating the general formulas eq. (5) and eq. (6). The group element that we find from these calculations is now an element of $U(n)$ and performs a unitary rotation of the system. Since we are

considering a degenerate eigenspace, we can choose our energy levels so that there is no dynamic phase. Thus, given a system which begins in state $|\psi_0\rangle$, the final state, after the cyclic evolution, will be given by

$$|\psi(T), \gamma\rangle = -\mathbb{P} \exp \left[- \oint_{\gamma} \mathcal{A}_{\mu} d\lambda^{\mu} \right] |\psi_0\rangle \quad (13)$$

$$= -\mathbb{P} \exp \left[- \oint_{\gamma} \langle \psi_{\alpha}(\lambda) | \frac{\partial}{\partial \lambda^{\mu}} | \psi_{\beta}(\lambda) \rangle d\lambda^{\mu} \right] |\psi_0\rangle. \quad (14)$$

This expression shows explicitly how the family of Hamiltonians determines the structure of the connection through the dependence of the eigenstates $|\psi(\lambda)\rangle$ on the specific Hamiltonian $H(\lambda)$.

We are now in a position to put the pieces together and understand geometrically how the holonomy is generated. The fiber bundle in which we work has our parameter space as a base, where we can naturally associate with a particular configuration a degenerate eigenspace of the Hamiltonian. The fiber consists of the set of unitary rotations which leave this subspace invariant in the larger Hilbert space. How the eigenspace changes as we vary the Hamiltonian gives rise naturally to the connection which we have discussed above. So, as we drive the system parameters in a loop through the parameter space, the system is distorted geometrically in such a way that upon return to the initial parameters, the system has been rotated by a unitary. This rotation can be non-trivial in that it will have immediately observable consequences provided the individual states in the degenerate subspace can be probed.

Before moving on to HQC, it is instructive to consider a system which closely models the classical ball-on-a-hoop example of holonomy and to see how the connection arises as a consequence of the fast degrees of freedom. Consider a system with a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{P^2}{2M} + V(r, R), \quad (15)$$

with r a fast degree of freedom and R a slow degree of freedom. p and P are the corresponding conjugate momenta. Let us suppose that the R changes much more slowly than r so that we can consider R to be fixed locally. The state of the system will depend parametrically on the slow degree of freedom and an approximate solution will be $|R\rangle$ where

$$h(R) |R\rangle = \left(\frac{p^2}{2m} + V(r, R) \right) |R\rangle = e_n(R) |R\rangle. \quad (16)$$

$h(R)$ is the effective Hamiltonian. Because the R degrees of freedom are so slow, we will assume that the state is always well described by $|R\rangle$ up to a phase. The

total state becomes

$$|\psi(r, R)\rangle = \Phi(R) |R\rangle. \quad (17)$$

Substituting this into the Hamiltonian eigenvalue equation we find

$$H |\psi\rangle = \left[-\frac{\nabla_r^2}{2m} - \frac{\nabla_R^2}{2M} + V(r, R) \right] |\psi(r, R)\rangle \quad (18)$$

$$\begin{aligned} &= \Phi(R) \left[-\frac{\nabla_r^2}{2m} + V(r, R) \right] |R\rangle \\ &\quad - \frac{1}{2M} [(\nabla_R^2 \Phi(R)) + 2(\nabla_R \Phi(R))\nabla_R + \Phi(R)\nabla_R^2] |R\rangle. \end{aligned} \quad (19)$$

Multiplying both sides by $\langle R|$ on the left we find

$$E_n = e_n + \frac{1}{2M} \left(\frac{\partial}{\partial R} + A_\mu \right)^2, \quad (20)$$

where $A_\mu = \langle R| \nabla_R |R\rangle$ appears as a connection! The fast degrees of freedom have given rise to a non-trivial geometric connection which corresponds to a symmetry of the system. This is similar to the way in which the holonomy arises in the example of the bead on a rotating hoop. Without the fast degree of freedom, the we do not generate the same geometric phase. In order to have a consistent holonomy which takes on global properties, it is necessary to explore the system globally, this is accomplished by the fast degrees of freedom and is expressed in an averaged connection [5]. If, for example, the bead is not moving on the rotating hoop, it only explores the local geometry and the corresponding connection will not reflect the global symmetry of the geometry. These notions will be explored more completely in a later section.

4 Holonomic Quantum Computing

Although HQC has provided the motivation for this paper, it is a straightforward result of the the theory developed above. Consequently, this section will provide only a brief overview of the material and the interested reader is referred to [2] and [10] for additional details and explicit computations of the holonomic and computational structure a general class of Hamiltonian families.

Quantum computation is accomplished by performing a specific sequence (possibly continuous, as in our case) of unitary operations on a the entire Hilbert space of a set of qubits. A quantum computer is referred to as universal if it is possible to approximate an arbitrary unitary on \mathcal{H} to arbitrary accuracy in a time which scales polynomially with the desired accuracy. In this discussion we will limit ourselves

to one and two qubit gates, since it is a well known result that the ability to perform these two operations is sufficient for universality.

We have seen that adiabatic manipulation of the system Hamiltonian can generate unitary transformations on the computational subspace and that the set of unitaries which can be generated is given by the span of the field strength.⁶ A given family $H(\lambda)$ will therefore allow for universal quantum computation exactly when the field strength spans the full Lie algebra $\mathfrak{u}(n)$. However, beyond the fact that the gates are implemented through a holonomy, the remaining machinery of quantum computing is unchanged. Quantum holonomy provides a novel and fundamentally distinct means of implementing quantum logic.

An elegant example of a technique for implementing a holonomic quantum computer was recently proposed [3] using multilevel atoms in a cavity. Consider an atom with n ground states $|g_n\rangle$ and one excited state $|e\rangle$ (a manifold of this form is common in atomic physics) where each individual ground-excited state transition is independently driven by a laser. In the rotating frame, the Hamiltonian for this system can be written

$$H = \sum_i (\Omega_i |e\rangle \langle g_i|) + h.c. \quad (21)$$

$\Omega_i \in \mathbb{C}$ is the Rabi frequency of the transition and is a function of a driving intensity. Here the parameter manifold is \mathbb{R}^n . This Hamiltonian is readily diagonalized (using, e.g. Mathematica) and we find that there is an $n - 1$ dimensional zero energy eigenspace. This space has no population in the excited state and will form the computational subspace.

It is shown in [3] that the connection of this system is irreducible and so the Hamiltonian family can be used to generate a universal set of quantum gates. Note that in this system, the qubits are not distinct two level systems as in many proposed quantum architectures, but instead are encoded in the degenerate eigenspace by choosing a suitable basis. This means that the dimension of the computational subspace is twice the number of qubits. However, realistic atomic systems are limited in that the number of qubits that can be encoded in a single atom is fundamentally limited. For this type of HQC, it will be necessary to perform conventional information transfer between distinct atoms to achieve a quantum computer with more than 2 qubits. Various schemes for making this extension are evaluated in [3]. Various schemes for HQC in semiconductor and NMR systems are explored in [11] and [12].

⁶In fact, additional restraints will be imposed by physical restrictions on the variations of the Hamiltonian which are possible and other physical constraints. This will not concern us.

5 Averaging and Adiabaticity

Classically, two types of holonomy are encountered. Purely geometric holonomy, which is a result of a natural global structure is encountered, occurs for example when a coordinate system is transported on a curved Riemann manifold along a non-geodesic path. Dynamical holonomy is encountered when a system with both fast degrees of freedom dynamically generates a non-trivial connection which is explored by a slow degree of freedom. The standard example of this is the ball-on-a-hoop holonomy which is explored in detail in e.g. [8].

The question which naturally arises when considering quantum holonomy is which, if either, of these two frameworks does quantum holonomy fit into. At first glance, it is not immediately clear. Since we are working in a degenerate eigenspace of the Hamiltonian, it seems as though fast dynamics can, in a certain sense, be neglected. At the same time, the adiabatic condition is strikingly similar to the condition that the hoop rotate slowly in comparison to the speed of the bead.

The connection that has been developed in this paper, however, *is* the Hannay-Berry connection for the system.⁷ This is seen by considering what it is that the Hannay-Berry connection accomplishes. The connection defines a connection which maintains the system in a G -invariant subset (in the classical literature this set is a level set of the momentum map induced by the Hamiltonian family of actions defined by G on the bundle). Classically, the Hannay-Berry connection is calculated explicitly from the connection on the underlying bundle by averaging the connection over the group G [6]. Quantum mechanically, we develop the connection implicitly by making the adiabatic assumption. In section 3, we developed the connection explicitly within the restricted subspace. However, it is quite conceivable to begin in the fiber bundle defined over the entire Hilbert space and then calculate the Hannay-Berry connection by averaging over coupled, non-degenerate states. This *should* yield the same connection. We see hints that this is true when we considered the simple model at the end of section 3. There, the overall dynamics are averaged over the fast degrees of freedom while the evolution is restricted to eigenstates of the reduced Hamiltonian. The averaging in this case is over the orbit of the rapid variables which may be a complicated evolution.

⁷While I present this as a novel conclusion, I am hard pressed to believe that it is, especially considering that it is in part attributed to Berry. I have not found this idea developed in the literature, however. Anandan and Aharonov take steps in this direction in [13], but they predate the geometry developed in [5]. Regardless, I think that it's elaboration is important and leads to a deeper understanding of the structure of Quantum Holonomy.

6 Conclusion

In this paper we have presented the theory of quantum holonomies and HQC and presented at least a qualitative argument for how quantum holonomy fits into the rigorous framework developed for working with classical systems. It is clear that the quantum case must fit into this framework in some sense, because quantum mechanics is a special case of general Hamiltonian dynamics. However, the approach typically taken to studying quantum systems does not illuminate the underlying parallel structure. As was argued in the final section, the classical theory for working with averaged holonomies is directly applicable to the quantum case, which is not *a priori* the case.

There are a number of loose ends left by this paper, the largest of which would be a rigorous statement of the ideas in section 5. There are a number of other ideas which I believe are also of interest. First, it would be useful to develop a theory of quantum holonomy which works with the C^* algebra of operators on the Hilbert space instead of the Hilbert space itself. This would be necessary to incorporate holonomy into the stochastic theories of filtering and measurement which would be important for studying the noise characteristics of the system. It is not immediately clear, however, what the natural geometric framework would be. It would also be interesting to study if and how holonomy can be used to generate non-classical states between non-degenerate subspaces. While it is clear that the adiabatic nature would destroy the overall phase coherence of the two subspaces, it is not clear that it should be impossible to create other correlations. Finally, there are only a few cases where a realistic proposal has been made to take advantage of quantum holonomy. It would be fruitful to study new model systems for HQC.

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