The Vlasov-Poisson Equation and the Energy Casimir Method

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I. Introduction

The continuum formulation for a collisionless plasma is, in some sense, the limit of a multi-particle system interacting through electro-magnetic forces. In the process of passing to the continuum limit, though, the canonical structure of the system becomes muddled, and the equations of motion take a form very different from those of discrete particles. The theory of stability in plasmas is, however, very well developed and curiously different from similar treatments of Hamiltonian systems with a finite number of degrees of freedom; the latter treatments allow linear (spectral) stability as well as non-linear criteria to be established. Here I will show the explicit development of the Hamiltonian nature of the Vlasov-Poisson equations and consider the application of the energy-casimir and energy-momentum methods to determine stability properties. Successful application of these methods would enable non-linear stability of plasma equilibria to be examined.

The most general formalism for Hamiltonian dynamics is that of the Poisson manifold, so to begin with a phase space and Poisson bracket will be introduced for electrostatic excitations of a plasma. Properties of this bracket and attempts to find its casimirs will be discussed. Applications of the energy-casimir method is limited by the completeness of the set of casimirs available, so the outlook for extending them will be considered. The Poisson manifold can be viewed as the result of a symplectic reduction from a larger space, this space will be reconstructed and used for the energy-momentum method. Although this formulation does not reveal any new results, it sheds considerable light on the underlying physics behind a plasma and the mathematics of the Vlasov-Poisson equation

II. The Vlasov-Poisson Equation

We will consider, here, a one species (electron) plasma neutralized by a background of infinitely massive ions. In the absence of two particle collisions (and correlations,) the state of the plasma is entirely described by a one particle distribution function $f(\mathbf{x},\mathbf{v})$. The configuration space for each particle will be denoted Q, in most of the examples this will be taken as R^1, or possibly S^1. The phase space for the plasma, denoted P, is the set of all such distribution functions which satisfy the appropriate boundary conditions $P \subset \mathcal{F}(T^*Q)$. The space P is linear and forms a Lie algebra, $\mathbf{g}$, when supplemented with the canonical bracket from T^*Q,

$$[f,g] = \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial g}{\partial \mathbf{v}} - \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{\partial g}{\partial \mathbf{x}} \quad . \tag{1}$$

The evolution of a plasma with the distribution $f(\mathbf{x},\mathbf{v})$ is given by the Vlasov-Poisson equation which plays a role in P analogous to Hamilton's equations. In terms of the T^*Q bracket the Vlasov-Poisson equation has the form

$$\frac{\partial}{\partial t} f = -[\, f,\ \mathcal{E}[f]\,] \quad , \tag{2}$$

where

$$\begin{aligned} \mathcal{E}[f](\mathbf{x},\mathbf{v}) &= \frac{1}{2}|\mathbf{v}|^2 + e\phi[f](\mathbf{x}) \\ &= \frac{1}{2}|\mathbf{v}|^2 + e\int d\mathbf{x}'d\mathbf{v}'\ \frac{f(\mathbf{x}',\mathbf{v}')}{|\mathbf{x}-\mathbf{x}'|^2} \quad . \end{aligned} \tag{3}$$

e

Since P is a linear space there is an isomorphism between T_fP, and P; in addition the natural metric,

$$\langle\langle f,g \rangle\rangle = \int_{T^*Q} d\mathbf{x}d\mathbf{v}\ f(\mathbf{x},\mathbf{v})\ g(\mathbf{x},\mathbf{v})$$

induces an isomorphism between $\mathbf{g}$ and $\mathbf{g}^*$ which will be denoted by $\flat : \mathbf{g} \to \mathbf{g}^*$ and $\sharp : \mathbf{g}^* \to \mathbf{g}$. Therefore the right hand side of (2) is a tangent vector from T_fP, and the solution of (2) is a parametrized curve $f(t) : \mathrm{R} \to P$. There is a parallel point of view whereby (2) can be solved in terms of a canonical diffeomorphism $\eta_t \in \mathcal{D}\mathrm{iff}_{\mathrm{can}}(T^*Q)$:

$$f(t) = f(0) \circ \eta_t^{-1} = (\eta_t^{-1})^* f(0) \quad . \tag{4}$$

Were $\mathcal{E}$ independent of f it would be a simple generating function (Hamiltonian) of the diffeomorphism, and f would be passively pulled back, however the interdependence makes η_t also a function of f. None-the-less, equation (4) shows that f evolves by a canonical diffeomorphism; this is an important fact about the Hamiltonian flows in P.

Based on the observation that P is isomorphic to $\mathbf{g}^*$, the dual of a Lie algebra, we can look for a bracket in the form of the Lie-Poisson bracket. Given functions $F, G : P \to \mathbf{R}$ this bracket will have the form

$$\begin{aligned}\{F,G\}(f) &= \langle\langle\, f,\ \Big[\Big(\frac{\delta F}{\delta f}\Big)^\sharp,\ \Big(\frac{\delta G}{\delta f}\Big)^\sharp\,\Big]\,\rangle\rangle \\ &= \int d\mathbf{x}d\mathbf{v}\ f(\mathbf{x},\mathbf{v})\Big[\Big(\frac{\delta F}{\delta f}\Big)^\sharp,\ \Big(\frac{\delta G}{\delta f}\Big)^\sharp\,\Big]\end{aligned} \tag{5}$$

Integrating by parts and using the boundary conditions we can rewrite this as

$$\begin{aligned}\{F,G\}(f) &= \int d\mathbf{x}d\mathbf{v}\ \Big(\frac{\delta F}{\delta f}\Big)^\sharp\Big[\,\Big(\frac{\delta G}{\delta f}\Big)^\sharp,\ f\,\Big] \\ &= \langle\ \frac{\delta F}{\delta f},\ \Big[\,\Big(\frac{\delta G}{\delta f}\Big)^\sharp,\ f\,\Big]\ \rangle\ .\end{aligned} \tag{6}$$

Recalling the definition of the Hamiltonian vector field X_G,

$$\{F,G\}(f) = X_G[F](f)\ \ ,$$

we see that this bracket evolves points, f, through canonical diffeomorphisms of T^*Q. Furthermore, by requiring that the time evolution of a passive scalar function F be dictated by the equation

$$\frac{\partial}{\partial t}F = \langle\ \frac{\delta F}{\delta f},\ \frac{\partial f}{\partial t}\ \rangle = \{F,H\}\ \ , \tag{7}$$

we can derive the form of the Hamiltonian for the Vlasov-Poisson equation:

$$H[f] = \int d\mathbf{x}d\mathbf{v}\ \mathcal{E}[f](\mathbf{x},\mathbf{v})\ f(\mathbf{x},\mathbf{v})\ \ .$$

The manifold P with the bracket $\{\cdot,\cdot\}$ forms a poisson manifold; as such it is the union of symplectic leaves which take the form of co-adjoint orbits. In particular the leaves are the co-adjoint orbits of the group of canonical diffeomorphisms $G = \mathcal{D}\mathrm{iff}_{\mathrm{can}}(T^*Q)$ which has been written [1]

$$\mathcal{O}rb(f) = \{\ f \circ \eta \mid \eta \in \mathcal{D}\mathrm{iff}_{\mathrm{can}}(T^*Q)\ \}\ \ .$$

One-forms perpendicular to these leaves can be found as the exterior derivatives of casimirs, $C \in \mathcal{F}(P)$. This is an obvious consequence of the definition of a casimir:

$$\{C,F\} = \langle\ \frac{\delta C}{\delta f},\ X_F\ \rangle = 0\ \ ,\ \ \forall F \in \mathcal{F}(P)\ \ .$$

These one-forms are isomorphic to those elements of $T_e\mathcal{D}\mathrm{iff}_{\mathrm{can}}(T^*Q)$ which leave f invariant. It can further be seen that infinitesimal transformations of this form are generated by any function, $B \in \mathcal{F}(T^*Q)$, which has gradients nowhere perpendicular to those of f. Thus in some sense we know the directions perpendicular to the symplectic leaves; as we will see, however, this does not lead in a simple way to knowledge of the casimirs.

Consider as an example the case where $Q = \mathrm{R}^1$; there is a unique symplectic leaf passing through the point

$$f_0(x,v) = \exp\left(-\frac{v^2}{2T}\right)\ \ .$$

(Issues of normalization can be safely ignored as this constitutes a very trivial set of leaves.) Any transformation of the form $B(x,v) = \beta(v)$ will leave f_0 invariant. From there we can pullback any $B(x,v)$ to any point on $\mathcal{O}rb(f_0)$ to give a one-form field defined over the entire symplectic leaf and pointing perpendicular to it:

$$B[f] = \eta^*\beta\ \ ,\ \ f = \eta^* f_0\ \ .$$

To construct a casimir $C[f]$ one faces the problem of extending $B[f]$ to all other symplectic leaves in just such a fashion that the equation

$$\left(\frac{\delta C}{\delta f}\right)^{\sharp} = B \qquad (7)$$

has a solution. Since no natural global transformation across leaves exists we cannot hope to drag $B[f]$ to any other leaves. In fact, were those transformations to exist they would be generated by the the casimirs we seek. In the next section we will see a few ways that such casimirs can be constructed and of what use they may be

III. Casimirs and the Energy Casimir Method

One sub-class of functions $B[f](\mathbf{x},\mathbf{v})$ whose gradients are nowhere perpendicular to those f are the functions of the form

$$B[f](\mathbf{x},\mathbf{v}) = \mathcal{B}\big(f(\mathbf{x},\mathbf{v})\big) \ . \tag{8}$$

By using the same $\mathcal{B}$ on every symplectic leaf we guarantee that (7) is integrable and has the solution,

$$C[f] = \int d\mathbf{x}d\mathbf{v}\ \mathcal{C}\big(f(\mathbf{x},\mathbf{v})\big) \ , \tag{9}$$

where

$$\frac{\partial}{\partial f}\mathcal{C} = \mathcal{B} \ .$$

I'm not sure I believe this.

These casimirs do not, by any means, comprise all casimirs of the system; nor are they the most desirable set for use in the energy- casimir method.

In analogy to finite-dimensional canonical dynamics one would like to express the stability of an equilibrium point of the system in terms of the second variation of the Hamiltonian. In non-symplectic cases one must contend with the fact that only those components of the variation along the symplectic leaves will determine dynamics. So an equilibrium point is not necessarily a point where the exterior derivative vanishes, rather it is a point where the derivative is perpendicular to the symplectic leaf. To "subtract off" that perpendicular component one subtracts from the Hamiltonian a casimir C such that

You don't literally mean this

$$d(H-C)(f_e) = \frac{\delta}{\delta f}(H-C)(f_e) = 0 \ . \tag{10}$$

The functional $F = H - C$ is called the free energy or the Lyaponov functional, and the pint f_e is an equilibrium since

What do you think of Morrison's justification of using this term?

$$X_H(f_e) = X_F(f_e) \cong \left[\left(\frac{\delta F}{\delta f}\right)^{\sharp}[f_e],\ f_e \right] = 0 \ .$$

If the second variation of F at the equilibrium is definite over some neighborhood of f_e then Lyaponov stability can be established. Thus the non-linear stability of an equilibrium can be characterized provided a casimir can be found which satisfies (10) at the equilibrium. To do this in a general case one will need a casimir for every direction (one-form) perpendicular to the symplectic leaf.

A great number of equilibria of the Vlasov-Poisson equation have been found; the casimirs given by (9) correspond to only the most trivial of these. By substituting the casimir (9) into equation (10) we get the equation

$$\mathcal{E}[f_e](\mathbf{x},\mathbf{v}) - \frac{\partial C}{\partial f}\Big(f_e(\mathbf{x},\mathbf{v})\Big) = 0 \ . \tag{11}$$

Inverting the function C', (assuming it is monotonic,) gives

$$f_e = f_e(\mathcal{E}) \ . \tag{12}$$

Of the class of spatially uniform equilibrium solutions, $f_e = f_e(\mathbf{v})$, this casimir gives only those which are radially symmetric and have a single hump; it has long been known that these are stable [2].

In order to apply casimirs of form analogous to (9) to a wider class of equilibria P.J. Morrison [3] added a passively convected scalar $g(\mathbf{x},\mathbf{v})$, (a dye,) to the dynamics thereby expanding the phase space to $P^2 = P \times P \ni (f,g)$. He expanded the bracket in the following fashion:

$$\begin{aligned}\{F,G\}(f,g) = \Big\langle \frac{\delta F}{\delta f}, \Big[\Big(\frac{\delta G}{\delta f}\Big)^{\sharp}, f \Big] + \Big[\Big(\frac{\delta G}{\delta g}\Big)^{\sharp}, g \Big] \Big\rangle \\ + \Big\langle \frac{\delta F}{\delta g}, \Big[\Big(\frac{\delta G}{\delta f}\Big)^{\sharp}, g \Big] \Big\rangle \ .\end{aligned} \tag{13}$$

This is a semi-direct product bracket.

In the case where the functional, G, is independent of g, as any Hamiltonian would be, both f and g evolve independently via the same canonical transformation, generated by $\left(\frac{\delta G}{\delta f}\right)^{\sharp}$. Functionals with g dependence, however, permit f to be pushed off the sets which were once the symplectic leaves of P. The g distribution always evolves by canonical transformations and casimirs will start from the same class as before; by considering the second term on the right of (13) we see that a casimir of this class must have

$$\left(\frac{\delta C}{\delta f}\right)^{\sharp} = \mathcal{B}\left(g(\mathbf{x},\mathbf{v})\right) \ .$$

Integrating this gives a set of casimirs

$$\mathcal{C}[f,g] = \int d\mathbf{x}d\mathbf{v} \left\{ \mathcal{C}_1\left(g(\mathbf{x},\mathbf{v})\right) f(\mathbf{x},\mathbf{v}) \ + \ \mathcal{C}_2\left(g(\mathbf{x},\mathbf{v})\right) \right\} \ , \tag{14}$$

A simple application of (10) gives relations

$$\mathcal{E}[f_e](\mathbf{x},\mathbf{v}) - \mathcal{C}_1\left(g_e(\mathbf{x},\mathbf{v})\right) = 0 \ , \tag{15}$$

$$\mathcal{C}_2'\left(g_e(\mathbf{x},\mathbf{v})\right) + f_e(\mathbf{x},\mathbf{v})\mathcal{C}_1'\left(g_e(\mathbf{x},\mathbf{v})\right) = 0 \ . \tag{16}$$

Inverting this requires that $\mathcal{C}_1$ be monotonic, but leaves $\mathcal{C}_2$ arbitrary. This gives the equilibria

$$g_e = g_e(\mathcal{E}), \quad \text{and} \quad f_e = f_e(g_e) \ ,$$

for which f_e is arbitrary but still spherically symmetric in the spatially uniform case. Morrison is, however, able to show that non-linear stability does not exist for multihumped distributions.

To be sure, the requirement that a function $B(\mathbf{x},\mathbf{v})$ have gradients nowhere perpendicular to those of f is satisfied by a wider class of functions than those of equation (8). Since the requirement is a local one it should suffice to satisfy (8) locally; the problem comes in extending this piece-wise definition of B to neighboring symplectic leaves where the topology of the contours of f are different. All elements on the same symplectic leaf as f will have contours of the same topology, such is the nature of canonical transformations. For this reason the perpendicular one-forms can be transported over the entire leaf. Neighboring leaves, however, can contain f's with widely differing contour topologies and hence widely differing normal directions. By introducing an auxiliary field g, Morrison was, in some sense, able to introduce a natural relation between leaves. Notice that for constant values of $g(\mathbf{x},\mathbf{v})$ the casimir (14) is a restricted form of (9): one where C is affine in f. The coefficients of the affine function are permitted to vary from leaf to leaf with the value of g; herein lies the generalization. Unfortunately it seems as though it would require an infinite series of such generalizations to find every conceivable inter-relation between leaves, if it were even possible.

IV. Construction and the Energy Momentum Method

In an attempt to use the energy-momentum method in place of its ill-fated casimir counterpart we will reconstruct (or construct) the unreduced dynamics from the reduced dynamics of $\mathfrak{g}^*$. Since P is isomorphic to the space of infinitesimal canonical diffeomorphisms it naturally reconstructs to the space $T^*G = T^*\mathcal{D}\mathrm{iff}_{\mathrm{can}}(T^*Q) \ni (\eta, p_\eta)$. A functional, $F : \mathfrak{g}^* \to \mathrm{R}$, can be mapped onto a right invariant function $F_R : T^*G \to \mathrm{R}$ via right translation from T^*_eG:

$$F_R = F \circ T^*R_\eta \quad .$$

Like ideal fluid dynamics the Vlasov-Poisson equation portrays the plasma in spatial representation; this does not seem surprising. Moreover the base space, G, is a canonical particle placement field, so the underlying symmetry is a canonical particle relabeling.

At this point it is possible to make contact with the variational principle for the Vlasov-Poisson equation proposed by F.E. Low [4]. In its proper form this Lagrangian is not a function on $TG \ni (\eta, \dot{\eta})$, rather it is a function on a submanifold

$TM \ni (\xi, \dot{\xi})$ [5]:

$$\tilde{L}(\xi, \dot{\xi}) = \int d\mathbf{x}d\mathbf{v}\ f_0(\mathbf{x}, \mathbf{v}) \{\ \frac{1}{2}|\dot{\xi}(\mathbf{x}, \mathbf{v})|^2 - e\phi[f_0(\xi(\mathbf{x}, \mathbf{v}))]\ \}\ , \quad (17)$$

where $f_0(\mathbf{x}, \mathbf{v})$ is an initial distribution. The space M is the set of all canonical diffeomorphisms, η, composed with the fiber projection on T^*Q, $\pi : T^*Q \to Q$; $(\mathbf{x}, \mathbf{v}) \mapsto \mathbf{x}$; in other words M is the space of mappings $T^*Q \to Q$ which can be generated by canonical transformations. Correspondence between TM and P is made through the following momentum map: $J : T^*M \to \mathbf{g}^*$; $(\xi, \dot{\xi}) \mapsto f$, given by

$$f(\mathbf{x}, \mathbf{v}) = \int d\mathbf{x}_0 d\mathbf{v}_0\ f_0(\mathbf{x}_0, \mathbf{v}_0)\delta(\mathbf{x} - \xi(\mathbf{x}_0, \mathbf{v}_0))\delta(\mathbf{v} - \dot{\xi}(\mathbf{x}_0, \mathbf{v}_0))\ . \quad (18)$$

The Lagrangian, (17), can be converted by the standard Legendre transform to a Hamiltonian on T^*M which we shall call $\tilde{H}(\xi, p_\xi)$. Marsden and Morrison then go on to point out that $G \subset T^*M$ and thus the Hamiltonian may be pulled back onto G by the inclusion to give a function $\tilde{H}_G(\eta)$. Adding this to a kinetic energy which is linear in $\dot{\eta}$ gives a Lagrangian on TG for which the Legendre transform to T^*G is degenerate. An action principle with such a Lagrangian naturally confines orbits to a sub-manifold called the manifold of principle constraint by the Dirac-Bergmann theory of constraints [6]. Thus the Vlasov-Poisson equation on $\mathbf{g}^*$ is compatible with an action priciple on TG, but with the complication of a degenerate Lagrangian; and the momentum map (18) is a map from the primary constraint manifold to $\mathbf{g}$.

To implement the energy-momentum method using the momentum map $J : T^*M \to \mathbf{g}^*$ given by (18) we would construct the submanifold

$$J^{-1}(f) = \{\ (\xi, \dot{\xi}) \mid J(\xi, \dot{\xi}) = f\ \}\ .$$

These would be finite diffeomorphisms generated by those very same generating functions whose gradients are nowhere perpendicular to those of $f(\mathbf{x}, \mathbf{v})$. This is very plausible since $T^*_{\xi,\dot{\xi}}J^{-1}(f)$ pulls back to a one-form perpendicular to the symplectic leaf in g^*. Charactering these manifolds would seem to have no intrinsic advantage over characterizing the symplectic leaves of g^*.

References

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