1988
(?)

Florence Lin

Very Good!

Symplectic Reduction and Geometric Prequantization of Cotangent Bundles

1. Introduction

As the symplectic manifold is central in the geometric description of the classical dynamics of many systems, the symplectic reduction theorem (Marsden and Weinstein, 1974) can often be applied to reduce the dimension of the momentum phase space describing the system. For example, as mentioned by Souriau (1970, p. 324), the phase space of each of the following systems can be described by a symplectic manifold: a nonrelativistic particle, the harmonic oscillator, the Kepler problem, and the n-body problem. In fact, each of these systems also has the property that its symplectic form is exact. In addition, a system satisfying Maxwell's equations in a vacuum can also be so described (Marsden and Weinstein, 1982; Puta, 1987).

Generalization of the symplectic reduction theorem to the Poisson reduction theorem (Marsden and Ratiu, 1986) allows the treatment of yet a larger number of systems not previously covered, including the free rigid body, the heavy top, and incompressible fluids (Marsden, Ratiu, and Weinstein, 1984).

However, in this paper attention will be restricted to the case where the momentum phase space is a symplectic manifold, in particular, a cotangent bundle whose reduction is also a cotangent bundle (whose symplectic structure in general might not be canonical). Such a symplectic manifold is quantizable in the Kostant-Souriau sense and, furthermore, so is its reduction.

In general, quantization assigns a Hilbert space of quantum states to a classical phase space and quantum operators to functions on the classical phase space.

Kostant-Souriau geometric quantization (Kostant, 1970; Souriau, 1970; Souriau, 1966; Auslander and Kostant, 1971; Kazhdan et al., 1978; Sniatycki, 1980; Woodhouse, 1980) is carried out in two steps: (i) prequantization (Kostant, 1970; Souriau, 1970) and (ii) quantization (Auslander and Kostant, 1971).

This paper will discuss the commutativity of symplectic reduction and prequantization with regard to the work of Puta (1987).

2. Commutativity of symplectic reduction and prequantization

Puta (1987) uses Theorem 1, a consequence (Satzer, 1977, Thm 14; Gotay, 1986, Prop (2.4)) of the symplectic reduction theorem (Marsden and Weinstein, 1974) and an explicit expression for the prequantum operator to prove Theorem 2 regarding the commutativity of symplectic reduction and prequantization.

2. A. Theorem 1.

Let Q be the configuration space and let G be a Lie group acting symplectically on the symplectic manifold (T^*Q, ω) where T^*Q is the cotangent bundle with symplectic structure ω given by $\omega = d\theta$ for some one-form θ. Let $J : T^*Q \to \mathfrak{g}^*$ be an equivariant momentum map. Let $(T^*Q)_0 := J^{-1}(0)/G$ and $Q_0 := Q/G$. The reduced phase space $((T^*Q)_0, \omega_0)$ of the extended phase space (T^*Q, ω) is symplectic diffeomorphic to $(T^*(Q_0), d(\theta_0))$, where $\omega_0 = d(\theta_0)$ is the canonical symplectic structure on $T^*(Q_0)$. Moreover, $\pi_0^* \theta_0 = i_0^* \theta$, where $\pi_0 : J^{-1}(0) \to J^{-1}(0)/G$ is the canonical projection and $i_0 : J^{-1}(0) \to T^*Q$ is inclusion.

<u>Proof.</u> (Puta (1987), Gotay (1986)).
We have

$$\pi_Q^*(T^*Q_0) = J^{-1}(0) \qquad (1)$$

Thus

$$\pi_Q^*(T^*Q_0)/G = J^{-1}(0)/G = (T^*Q)_0$$

so $(T^*Q)_0$ is diffeomorphic to T^*Q_0.
The following commutative diagram holds:

$$\begin{tikzcd} J^{-1}(0) \arrow[r, "i_0"] \arrow[d, "\pi_0"'] & T^*Q \arrow[d, "\pi_Q \circ \tau_Q"] \\ T^*(Q_0) \arrow[r, "\pi_{Q_0}"] & Q_0 \end{tikzcd}$$

i.e.,

$$\pi_Q \circ \tau_Q \circ i_0 = \tau_{Q_0} \circ \pi_0 \qquad (2)$$

Thus

$$\begin{aligned} (i_0^* \theta)_\alpha (v) &= \theta_{i_0 \alpha} (T i_0 (v)) \\ &= (i_0^* \alpha)(T \tau_Q (T i_0 (v))) \\ &\overset{(1)}{=} \pi_Q^* \alpha_0 (T \tau_Q (T i_0 (v))) \\ &= \alpha_0 (T \pi_Q (T \tau_Q (T i_0 (v)))) \end{aligned}$$

(cont'd.)

$$(i_0^* \theta)_\alpha (v) = \alpha_0 \left(T(\pi_Q \circ \tau_Q \circ i_0)(v) \right)$$
$$\overset{(2)}{=} \alpha_0 \left(T(\tau_{Q_0} \circ \pi_0)(v) \right)$$
$$= \alpha_0 \left(T\tau_{Q_0} (T\pi_0 (v)) \right)$$
$$= (\theta_0)_\alpha (T\pi_0(v))$$
$$= (\pi_0^* \theta_0)_\alpha (v) .$$

Therefore, $\pi_0^* \theta_0 = i_0^* \theta$. Hence $\pi_0^* d\theta_0 = i_0^* d\theta$. By the uniqueness of the reduced symplectic structure in the symplectic reduction theorem, $\omega_0 = d\theta_0$.

q.e.d.

2.B. The prequantum operator

The following expression for the prequantum operator δ_f corresponding to a classical observable $f \in C^\infty(M)$ holds for the special case where $M = T^*Q$ (Woodhouse, 1980, eqs. (5.4.2), (5.5.1)):

$$\delta_f = -i\hbar X_f + X_f \lrcorner \theta + f$$

where $\hbar$ is Planck's constant divided by 2π, θ is the canonical one-form, and X_f is the vector field determined by

$$X_f \lrcorner \omega + df = 0 .$$

Analogously, the prequantum operator for the reduced space is

$$\delta^0_{f_0} = -i\hbar X_{f_0} + X_{f_0} \lrcorner \theta_0 + f_0 .$$

2.C. Theorem 2.

Let Q be the configuration space. Let G be a Lie group acting symplectically on the symplectic manifold (T^*Q, ω). Let $Q_0 = Q/G_0$. Let $f, g : T^*Q \to \mathbb{R}$ be smooth G-invariant functions on T^*Q, and let $f_0, g_0 : T^*Q_0 \to \mathbb{R}$ be the induced functions on T^*Q_0. Let δ_f be the prequantum operator on the extended phase space T^*Q for the function f. Let $\delta^0_{f_0}$ be the prequantum operator on the reduced phase space T^*Q_0 for the function f_0. Then (i) $\delta_f(g)$ is a G-invariant function on T^*Q and (ii) $[\delta_f(g)]_0 = \delta^0_{f_0}(g_0)$, i.e., geometric prequantization of a function (on T^*Q or $J^{-1}(0)/G$) and symplectic reduction of T^*Q to $J^{-1}(0)/G$ are commuting operations, where $J: T^*Q \to \mathfrak{g}^*$ is an equivariant momentum map.

Proof. (Puta (1987))

(i) follows from a straightforward calculation.

(ii) (a)
$$\begin{aligned} X_{f_0}(g_0)(\pi_0(x)) &= (X_{f_0})_{\pi_0(x)}(g_0) \\ &= T_x\pi_0\,(X_f)_x(g_0) \\ &= (X_f)_x(g_0\circ\pi_0) \\ &= (X_f)_x(g\circ i_0) \\ &= (X_f(g))(i_0(x)) \end{aligned}$$

(b)
$$\begin{aligned} [\theta_0(X_{f_0})\cdot g_0](\pi_0(x)) &= [\theta_0(X_{f_0})(\pi_0(x))][g_0(\pi_0(x))] \\ &= [(\theta_0)_{\pi_0(x)}(X_{f_0})_{\pi_0(x)}]\cdot[g\circ i_0(x)] \\ &= [(\theta_0)_{\pi_0(x)}(T_x\pi_0(X_f)_x)]\cdot[g(i_0(x))] \\ &= [\pi_0^*(\theta_0)_{\pi_0(x)}(X_f)_x]\cdot[g(i_0(x))] \\ &= [\theta_{i_0(x)}(X_f)_{i_0(x)}]\cdot[g(i_0(x))] \\ &= [\theta(X_f)(i_0(x))]\cdot[g(i_0(x))] \\ &= [\theta(X_f)\cdot g](i_0(x)) \end{aligned}$$

(c)
$$\begin{aligned} (f_0 g_0)(\pi_0(x)) &= f_0(\pi_0(x))\cdot g_0(\pi_0(x)) \\ &= f(i_0(x))\cdot g(i_0(x)) \\ &= (f\cdot g)(i_0(x)) \end{aligned}$$

F. Lin

Therefore,

$$-i\hbar X_{f_0}(g_0)(\pi_0(x)) + [\theta_0(X_{f_0})\cdot g_0](\pi_0(x)) + (f_0 g_0)(\pi_0(x)) = -i\hbar (X_f(g))(i_0(x)) + [\theta(X_f)\cdot g](i_0(x)) + (f\cdot g)(i_0(x))$$

Hence,

$$[\delta_f(g)]_0 = \delta^0_{f_0}(g_0).$$

q.e.d.

3. Generalization to $\mu \neq 0$

Theorems 1 and 2 can be generalized to $\mu \neq 0$ in certain cases (cf. Puta (1984)).

3.A. Theorem 1′.

If (i) $\mu \in \mathfrak{g}^*$ is a regular value of J, (ii) $\mathfrak{g} = \mathfrak{g}_\mu$, and (iii) all the other hypotheses are the same as in Theorem 1 (above), then the reduced phase space $((T^*Q)_\mu, \omega_\mu)$, where $(T^*Q)_\mu := (J^{-1}(\mu)/G_\mu)$ is symplectic diffeomorphic to (T^*Q_μ, Ω_μ) where $Q_\mu := Q/G_\mu$ and Ω_μ is in general not canonical.

Theorem 1′ appears in Gotay (1976) as the Kummer-Marsden-Satzer Theorem. See also Abraham and Marsden (1978, Thm 4.3.3).

Theorem 1′ and the additional assumption

$$\pi_\mu^* \theta_\mu = i_\mu^* \theta$$

imply Theorem 2′.

3.B. Theorem 2′.

Let Q be the configuration space. Let G be a Lie group acting symplectically on the symplectic manifold (T^*Q, ω). Let $\mu \in \mathfrak{g}^*$ be a regular value of J, where $J : T^*Q \to \mathfrak{g}^*$ is an equivariant momentum map. Let $\mathfrak{g} = \mathfrak{g}_\mu$. Let $\pi_\mu^* \theta_\mu = i_\mu^* \theta$.

Let $f, g: T^*Q \to \mathbb{R}$ be smooth G-invariant functions on T^*Q, and let $f_\mu, g_\mu: T^*Q_\mu \to \mathbb{R}$ be the induced functions on T^*Q_μ. Let δ_f be the prequantum operator on the extended phase space T^*Q for the function f. Let $\delta^\mu_{f_\mu}$ be the prequantum operator on the reduced phase space T^*Q_μ for the function f_μ. Then (i) $\delta_f(g)$ is a G-invariant function on T^*Q and (ii) $[\delta_f(g)]_\mu = \delta^\mu_{f_\mu}(g_\mu)$, i.e., geometric prequantization of a function (on T^*Q or $J^{-1}(\mu)/G_\mu$) and symplectic reduction of T^*Q to $J^{-1}(\mu)/G_\mu$ are commuting operations.

These issues will be discussed further in the future. Related issues have been discussed by Puta (1984), Gotay (1986), and Guillemin and Sternberg (1982).

References

R. Abraham and J. E. Marsden, Foundations of Mechanics, 2nd ed., Benjamin/Cummings, Reading, MA, 1978.

L. Auslander and B. Kostant, "Polarization and unitary representations of solvable Lie groups," Invent. Math. 14, 255-354 (1971).

M. J. Gotay, "Constraints, reduction, and quantization," J. Math. Phys. 27, 2051-2066 (1986).

V. Guillemin and S. Sternberg, "The moment map and collective motion," Ann. Phys. 127, 220-253 (1980).

D. Kazhdan, B. Kostant, and S. Sternberg, "Hamiltonian group actions and dynamical systems of Calogero type," Comm. Pure Appl. Math. 31, 481-507 (1978).

B. Kostant, "Quantization and unitary representations, Part I: Prequantization," Lecture Notes in Mathematics, Vol. 170, Springer-Verlag, New York, 1970, pp. 87-208.

M. Kummer, "On the construction of the reduced phase space of a Hamiltonian system with symmetry," Indiana Univ. Math. J. 30, 281-291 (1981).

J. Marsden and T. Ratiu, "Reduction of Poisson manifolds," Lett. Math. Phys. 11, 161-169 (1986).

J.E. Marsden, T. Ratiu, and A. Weinstein, "Reduction and Hamiltonian structures on duals of semidirect product Lie algebras," Cont. Math. AMS 28, 55-100 (1984).

J. Marsden and A. Weinstein, "Reduction of symplectic manifolds with symmetry," Rep. Math. Phys. 5, 121-130 (1974).

J.E. Marsden and A. Weinstein, "The Hamiltonian structure of the Maxwell-Vlasov equations," Physica 4D, 394-406 (1982).

M. Puta, "On the reduced phase space of a cotangent bundle," Lett. Math. Phys. 8, 189-194 (1984).

M. Puta, "On the geometric prequantization of Maxwell equations," Lett. Math. Phys. 13, 99-103 (1987).

W.J. Satzer, Jr., "Canonical reduction of mechanical systems invariant under Abelian group actions with an application to celestial mechanics," Indiana Univ. Math. J. 26, 951-976 (1977).

J. Sniatycki, Geometric Quantization and Quantum Mechanics, Springer-Verlag, New York, 1980, pp 11-12.

F. Lin

J. M. Souriau, "Quantification géométrique," Commun. Math. Phys. **1**, 374-398 (1966.

J. M. Souriau, *Structure des Systèmes Dynamiques*, Dunod, Paris, 1970.

N. Woodhouse, *Geometric Quantization*, Clarendon Press, Oxford, 1980.