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Good! (Maybe should include Puta too)

↑ Sorry, I see it is done separately

The free rigid body: reduction, quantization condition, and prequantization

1. Introduction

This paper discusses various aspects of the prequantization of the classical free rigid body as a first step toward the eventual development of an extension of Puta's (1987) theorem regarding the commutativity of symplectic reduction and prequantization. In order to do this, this paper first discusses the reductions involved in the free rigid body problem and the quantization condition required in order to perform a prequantization.

2. Phase space of the classical free rigid body

The velocity phase space of the free rigid body is the tangent bundle $TSO(3)$. The momentum phase space of the free rigid body is the cotangent bundle $T^*SO(3)$. This is a 6-dimensional symplectic manifold parameterized by the Euler angles and their conjugate momenta. As a symplectic manifold, it is also a Poisson manifold.

3. Reduction theorems and reduced spaces

The Lie-Poisson reduction theorem can be applied to reduce the phase space of the rigid body from the 6-dimensional space $T^*SO(3)$ in material coordinates to the 3-dimensional space $so(3)^*$ in body coordinates. Both the extended and reduced spaces are Poisson manifolds. In this case the reduced space is not a symplectic manifold (it can not be symplectic because its dimension is not even).

By the Symplectic Stratification Theorem, a Poisson manifold is the disjoint union of its symplectic leaves (Marsden and Ratiu, 1987; Marsden and Weinstein, 1983; Weinstein, 1983; Kirillov, 1976).

A consequence of the Kirillov-Kostant-Souriau theorem (Abraham and Marsden, 1978, p. 302) is that for a dual of a Lie algebra with the Lie-Poisson structure, the symplectic leaves are the coadjoint orbits.

By Poisson reduction, $T^*SO(3)/SO(3) \cong so(3)^*$, a Poisson manifold. By symplectic reduction, $T^*SO(3)/SO(3) \cong S^2$, a symplectic manifold, which is a symplectic leaf of $so(3)^*$.

4. Quantization condition

A symplectic manifold is quantizable if the integral of the symplectic form ω over any closed surface is an integral multiple of 2π (Guillemin and Sternberg, 1980, p. 251; Kazhdan et al., 1978, p. 499). This is Kostant's (1970) integrality condition. If the symplectic manifold is a cotangent bundle, then $\omega = d\theta$ for some one-form. Hence, by Stokes' theorem, the integral of ω over any closed surface is zero, so a cotangent bundle is quantizable.

The sphere S^2 is a symplectic manifold. By Souriau (1970, p. 324), the sphere S^2 is quantizable if $2r$ is an integer, where r is the radius of the sphere. (The sphere does

what symplectic form? (The coadjoint orbit one is not just the area element) not have an exact symplectic structure; otherwise, it would be quantizable for all values of r.) This leads to a quantization condition. One would like to see a condition resembling (see, e.g., Edmonds (1960))

$$L^2 = \ell(\ell+1)\hbar^2 \qquad \ell = 0, 1, 2, \ldots$$

Quantizability has been discussed with respect to a physical interpretation as the Bohr-Sommerfeld-Wilson quantization condition by Sniatycki (1980), who cites Sniatycki (1975), and Simms (1973), who cites Simms (1972).

The geometric quantization of the related classical problem of the liquid drop, has been discussed by Guillemin and Sternberg (1980) and Rosensteel and Ihrig (1980).

5. Possible commutativity of reduction and prequantization

5.A. Restriction to quantizable symplectic leaves of reduced manifold.

First of all, prequantization can only be performed on a quantizable symplectic manifold. Thus it can only be applied to $T^*SO(3)$ and to the quantizable symplectic leaves of $so(3)^*$, i.e., the spheres S^2 with radius r such that $2r$ is an integer.

5.B. Restriction of Puta's theorem to symplectic reduction

The reduction of $T^*SO(3)$ to $so(3)^*$ is not a symplectic reduction, but a Lie-Poisson reduction, so the theorem of Puta (1987, Theorem 2) does not apply to the determination of the possible commutativity of the reduction and prequantization operations for a system satisfying the free rigid body equations. Furthermore, not all of the symplectic leaves of $so(3)^*$ are quantizable, so one can only consider the symplectic reduction of $T^*SO(3)$ to one of the spheres S^2 with radius r such that $2r$ is an integer to see if Puta's theorem applies to the determination of whether reduction commutes with prequantization.

<u>5.C.</u> Extension of Puta's theorem to Poisson manifolds and extension of the definition of the prequantum operator.

So far in Puta's work the operation of prequantization has been discussed in terms of symplectic manifolds and reduction has been restricted to symplectic reduction. In order to extend the ideas to cover the rigid body the procedere must be extended to Poisson manifolds either with or without the application of the Symplectic Stratification Theorem. See Marsden and Ratiu (1986) for a discussion of Poisson reduction and Weinstein (1983) for a discussion of Poisson manifolds.

First one needs to extend the definition of a quantizable manifold to Poisson manifolds.

Further, one needs the Poisson analog of an exact symplectic manifold in order to have a definition for the prequantum operator on a Poisson manifold.

Then it may be possible to show under certain conditions that Poisson reduction and geometric prequantization are commuting operations.

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See also the related discussions of
Gotay (1986) and Guillemin and Sternberg (1982).

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