# A Review of Geometry in Robotic Locomotion

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**Abstract**

This report presents a limited review on using geometric methods to analysis problems in robotic locomotion. In robotic locomtion, the net displacement of the whole body is usually obtained through cyclic changes in the shape of the robotic mechanism. A robot's motion is constrianed by its environment. By moving its joints in a periodic way, a robot can exploit the interaction with the environment to generate net motion. There are different kinds of locomotion in the nature, but from a differential geometry point of view, something intrinsic lies beneath. Namely, many locomotion problems can be modeled as a connection on a principal fiber bundle. The associated geometric phase describes the net motion a robot gets by changing its shape. Examples are provided to demonstrate how to construct a connection.

## 1 Background

When studying the locomotion of a robot, the robot's configuration space can be divided into two parts. One part describes the position of the robot. A coordinate frame is attached to the moving robot, which is usually called the body frame. The displacement of the body frame with respect to a fixed reference frame is used to describe the position of the robot. The set of frame displacement is $SE(m)$, $m \leq 3$, or a subgroup of $SE(m)$, which is a Lie group, denoted by $G$. The second part defines the internal configuration of the robot, namely, the shape of the mechanism. Usually the set of all possible shape variables can be described by a manifold, $M$, which is called the shape space. Then the total configuration space of the robot is $Q = G \times M$. The shape and position variables are coupled by the constraints which act on the robot. So, by appropriately changing the shape, we can get desired changes in the position variables. This relationship between the shape and position variables, as we shall see later, can be described by the concept of connection on a principal bundle.

Having set up the configuration space of a mechanical system, the dynamical equations can be derived via calculus of variation. Assume there exist a Lagrangian function, $L(q, \dot{q})$, on the tangent space $TQ$, then the motion of the mechanical system is governed by the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} - \tau_i = 0, \tag{1}$$

where $\tau$ is the forcing funciton.

However, in most cases that we are interested in, there exist constraints on the mechanical system. For example, when we drive a car, we prefer there is no slip between the wheels and the ground and the car can only move along the direction of its wheels. The constraints can take many forms. Let's consider such constraints that are linear in the velocities. Given $k$ such constraints, we can write them in the following form:

$$w^i_j(q)\dot{q}^j = 0, \qquad \text{for} \qquad i = 1, \dots, k.$$

This kind of constraints can be integrable(thus become **holonomic** constraints, which depend only on the poistion variables) or not integrable(i.e. **nonholonomic**).

When constarints are taken into account, the Lagrange's equations are modified using the Lagrange multipliers. Then Eq. 1 becomes:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} + \lambda_j w^j_i - \tau_i = 0, \tag{2}$$

where $\lambda$, the Lagrange multipliers are solved for together with the configuration variables $q$ and $\dot{q}$. However, in this way, the physical intuiation, i.e. the effect of shape changes on the motion of the robot, will not be so easy to see. So reseachers rewrote the system in a more intrinsic way. Namely, there is an inherent geometric structure in the locomotion problems. By making use of this inherent structure, one can model the system in a way which shows the coupling between the shape and position variables.

## 2 Connection on a principal fiber bundle, geometric phase

In this section, we reiew the geometric concepts of a connection on a principal fiber budle. From this point of view, we can model the locomoting system in a intrinsic way clearly describing the physical intuition mentioned before.

Recall that the configuration space of a robot can be divided into two parts, $Q = G \times M$. $G$ is a Lie group describing the positon of the robot, usually $SE(2)$ or $SE(3)$, or a subgroup of them. $M$ is a manifold describing the internal configuration of the mechanical system. Such a congiguration space is in fact a **trivial principal fiber bundle**.

A **left action** of a Lie group $G$ on a manifold $Q$ is a smooth map $\Phi : G \times Q \to Q$ which satisfies the following two properties:

(1) $\Phi(e, q) = q$ for all $q \in Q$, where $e$ is the identity element of $G$;

$$(2)\ \Phi(g_1, \Phi(g_2, q)) = \Phi(g_1 g_2, q) \text{ for every } g_1, g_2 \in G \text{ and } q \in Q.$$

$\Phi(g, q)$ can also be viewed as a map from $Q$ to $Q$ with $g \in G$ fixed. That is, $\Phi_g : Q \to Q$ given by $\Phi_g(q) = \Phi(g, q)$. $\Phi$ is said to be a free left action if $\Phi(g, q) = q$ implies $g = e$. Let $\xi \in \mathfrak{g}$ the Lie algebra of $G$, the **infinitesimal generator** of $\Phi$ corresponding to $\xi$ is the vector field defined by:

$$\xi_Q(q) = \frac{d}{dt}(\Phi(\exp(\xi t), q))|_{t=0}.$$

In the case of locomotion, let $G$ denote the position group of a robot. If the robot's position is $h$ and it is displaced by an amount $g$, then the final position is $gh$. This is actually a left action of $G$, the position space of the robot, on itself. For every $g \in G$, the associated left action is $L_g : G \to G$ given by $L_g(h) = gh$.

Let $G$ be a Lie group and $M$ a manifold, then the manifold $Q = G \times M$ together with a free left action of $G$ on $Q$ given by: $\Phi_h(g, x) = (hg, x)$, where $h \in G$, $g \in G$ and $x \in M$, is called a **trivial principal fiber bundle.** $M$ is the base and $G$ the structure group. Given $(g, x) \in Q = G \times M$, two natural projections are defined on the bundle. They are $\pi_1 : Q \to G : (g, x) \mapsto g$ and $\pi_2 : Q \to M : (g, x) \mapsto x$.

Given a particular point $x_0$ in $M$, the set of point $(g, x_0) \in Q$, i.e. $\pi_2^{-1}(x_0)$, is called the **fiber over** $x_0$. The **vertical subspace** of the principal fiber bundle $Q$ at a point $q$ is defined to be

$$V_q Q = \{v_q \in T_q Q : v_q = \xi_Q(q) \text{ for some } \xi \in \mathfrak{g}\}.$$

From the definition we can see that a vector in $V_q Q$ is tangent to the orbit of $q$ under the action of $G$. It's said to be vertical since it's tangent to the fiber direction, i.e. for a trivial principal bundle $Q = G \times M$, the elements of $V_q Q$ have the form $(\xi_G(g), 0)$, for some $\xi \in \mathfrak{g}$

Now the concept of **connection** on a principal fiber bundle can be introduced. A connection $A$ on a principle bundle $Q = G \times M$ is a Lie algebra valued one form on $Q$, i.e. $A(q) : T_q Q \to \mathfrak{g}$ where $\mathfrak{g}$ is the Lie Algebra of $G$. The connection has the following properties:

$$(1)\ A(q) \cdot \xi_Q = \xi, \text{ for } \xi \in \mathfrak{g}$$

$$(2)\ A(\Phi_g(q)) \cdot D_q \Phi_g(\dot{q}) = Ad_g A(q) \cdot \dot{q}.$$

A connection $A$ on a principal fiber bundle assigns to each point $q \in Q$ a subspace of $T_q Q$, which is called a **horizontal subspace**:

$$H_q Q = ker(A(q)) = \{v_q \in T_q Q : A(v_q) = 0\}.$$

Then it follows that $T_q Q = H_q Q \oplus V_q Q$ and $H_{\Phi_g(q)} = T_q \Phi_g H_q Q$, for $g \in G$. And conversely, if given a subspace $H_q Q$ of $T_q Q$ which depends smoothly on $q$ and it's true that $T_q Q = H_q Q \oplus V_q Q$, $H_{\Phi_g(q)} = T_q \Phi_g H_q Q$, there in general exists a connection $A$ such that $H_q Q = ker(A(q))$. In some sense, a connection can be think of as a projection that splits the tangent space at a point $q$ into a horizontal and vertical part. And the horizontal subspace can be alternatively defined as the set of tangent vectors upon which the connection form vanishes. Note that the vertial subspace at $q$ is just the tangent space to the vertical fiber over $q$.

The intuition of a connection on a principal bundle is depicted in Figure 1.

Recall in the context of locomotion, we have constraints of the form $w(q)\cdot\dot{q} = 0$. In some cases, this actually defines a horizontal subspace of a connection, as we will cover later.

Given a principal fiber bundle $Q = G \times M$ and assume we have defined a connection on it. For a point $q \in Q$, the tangent map associated to the natural projection $\pi_2 : Q \rightarrow M$, $T_q\pi_2 : T_qQ \rightarrow T_{\pi_2(q)}M$, maps the horizontal subspace at $q$ isomorphically onto $T_{\pi_2(q)}M$. Or we can say, given a vector $v_x \in T_xM$ and a point $q$ in the fiber over $x$, there is a unique vector in $H_qQ$ whose image is $v_x$ under the projection $T_q\pi_2$. We call this vector the **horizontal lift** of $v_x$.

Let $c(t)$ be a curve in $M$ which passes through $c(0) = x_0 \in M$, we can now define the horizontal lift of $c$. Given a point $q_0$ in the fiber over $x_0$, the horizontal lift of $c$ is a cuver $c^*(t)$ with the following properties:

$$(1)\, c^*(0) = q_0;$$

$$(2)\, \pi_2(c^*(t)) = c(t);$$

$$(3)\, \frac{d}{dt}c^*(t) \in H_{c^*(t)}Q, \text{i.e. } c^*(t) \text{ is everywhere horizontal.}$$

Given two points $x_1 = c(t_1)$ and $x_2 = c(t_2)$, the horizontal lift of $c(t)$ defines a map from $\pi_2^{-1}(x_1)$ to $\pi_2^{-1}(x_2)$. Namely, given a point $q_1$ in the fiber over $x_1$, the horizontal lift $c^*(t)$ of $c(t)$ will map $q_1$ to a point $q_2 = c^*(t_2)$ in the fiber over $x_2$. The map can be called a **parellel displacement** along the curve $c(t)$ in the sense that it's the horizontal lift of $c(t)$. It should be noted that horizontal lift and parellel displacement depend on the choice of the connection.

Now let $c : [0,1] \rightarrow M$ be a closed curve such that $c(0) = c(1) = x \in M$ and $q \in \pi_2^{-1}(x)$. If the parellel displacement along $c$ maps the point $q$ to $q'$, then $q'$ is also in the fiber over $x$, which can be identified as $G \times \{x\}$, where $G$ is the struture group of the pricinpal bundle $Q$.

The **geometric phase** or **holonomy** of the closed curve $c : [0,1] \rightarrow M$ is defined as the change in the group variable when $q$ is mapped to $q'$ by the horizontal lift of $c$. Again, the geometric phase is dependent upon the choice of the connection $A$ on the principal bundle. Futher more, one can define the **holonomy group** of $A$ with reference point $q$ as the group components of all the points in $\pi_2^{-1}(x)$ that can be reached via horizontal lifts of the loops in $M$ passing through $x$. The **holonomy bundle** of $A$ with the reference point $q$ is then the set of points in $Q$ that can be joined with $q$ by horizontal lifts of loops in $M$ passing through $x$. The geometric phase is independent of the choice of the initial point in the fiber since the parellel translation is always horizontal, i.e. independent of the group variable. See Figure 2 for the intuition of horizontal lift and geometric phase.

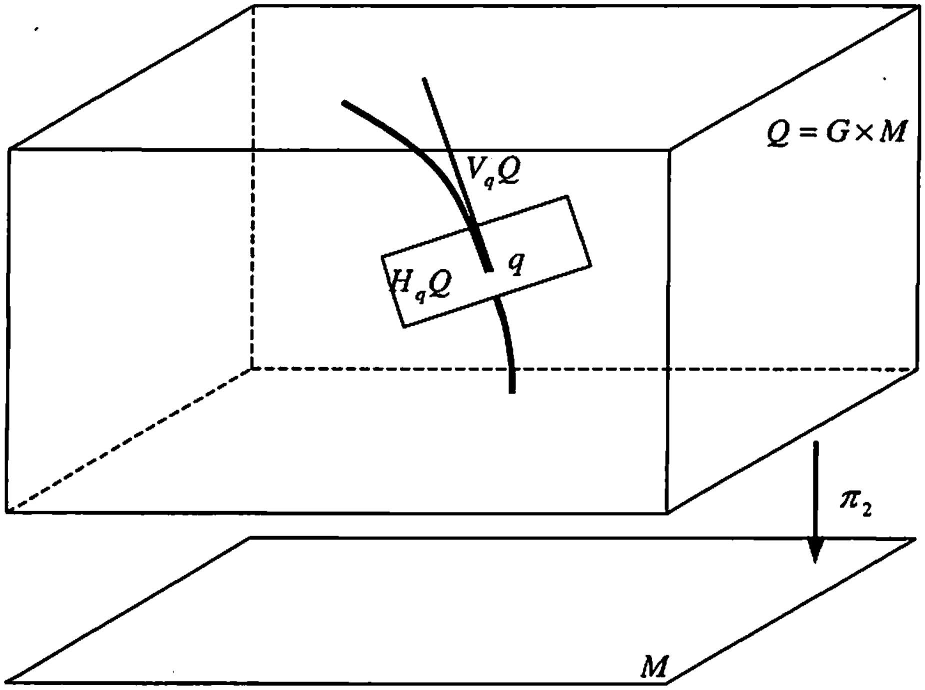


**Figure 1 A connection on a principle bundle**

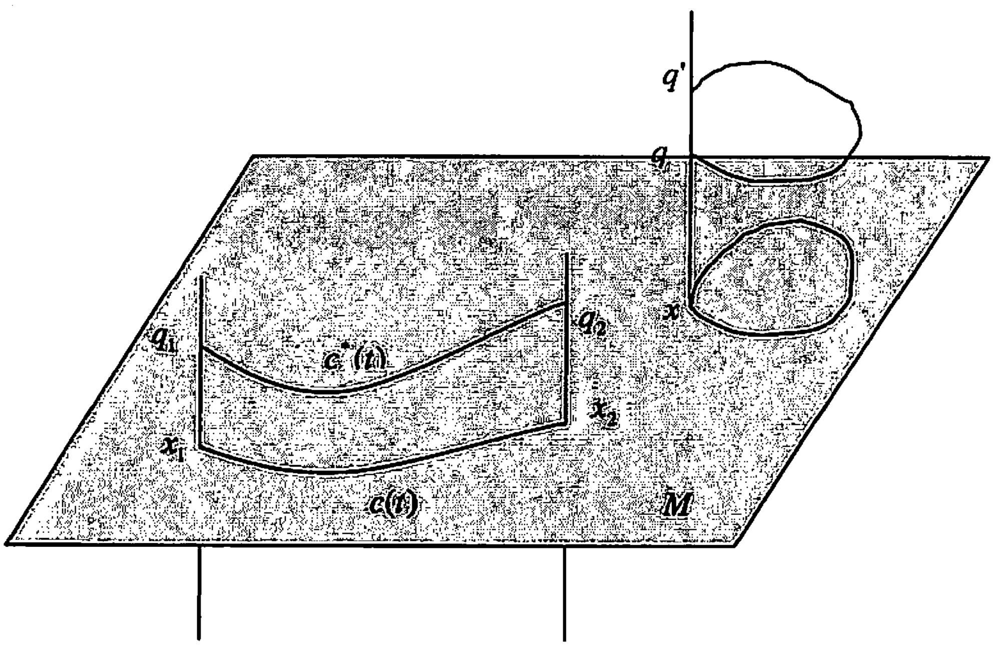


**Figure 2 Horizontal lift and geometric phase**

In locomotion problems, we have seen that the configuration space of the mechanical system can be divided into two parts, $Q = G \times M$, where $G$ is a Lie Group or Lie subgroup describing the poistion of the system and $M$ a manifold that identifies the possible internal configuration of the mechanism. Now we can utilize the above geometric concepts to model the system in a way consistent with the physical intuition.

The configuration space $Q$ of a mechanical system has an inherent structure of a principle fiber bundle. A connection $A$ can be defined on it by appropriately choosing the horizontal subspace since the connection vanishes on the horizontal space. In fact the horizontal subspace is chosen so that it describes the constraints on the system and symmetries in the mechanism. Having established the coonnection, by changing the shape variables in the base space in a cyclic way, geometric phase, net changes in the group variables, can be obtained via the horizontal lift. This is exactly the way how real systems, animals or robots, gain their locomotion.

# 3 Kinematic constraints and connection

Consider a mechanical system on a configuration space $Q = G \times M$ with $k$ constraints

$$w^i(q) \cdot \dot{q} = 0, \qquad \text{for} \qquad i = 1, \ldots, k.$$

where each $w^i$ is a one form.

Define

$$H_qQ = \{v_q \in T_qQ : w^i(q)\dot{q} = 0, \text{for}\, i = 1, \ldots, k\},$$
$$V_qQ = \{\xi_Q \in T_qQ : \xi \in \mathfrak{g}\}.$$

If $T_qQ = H_qQ \oplus V_qQ$ and $H_{\Phi_g(q)} = T_q\Phi_g H_qQ$, for $g \in G$, then a connection $A$ can be defined on $Q$ such that $H_qQ = ker(A(q))$. The connection one form can always be written in the local coodinates as

$$A(q) \cdot \dot{q} = Ad_g(\xi + \mathbb{A}(x) \cdot \dot{x})$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{g}$ and $\mathbb{A} : TM \to \mathfrak{g}$

Then we can write the constraints as

$$\dot{g} = -g(\mathbb{A}(x) \cdot \dot{x}).$$

In this case, the locomotion of the system is fully determined by the one form $\mathbb{A}$.

Below is an example of a two wheeled kinematic car to demonstrated how to construct a connection out of the kinematic constraints.

Consider the two wheeled car moving in the plane in Figure 3. The configuration space here is $Q = SE(2) \times (S^1 \times S^1)$. Then the configuration of the system is $q = (x, y, \theta, \psi_1\psi_2)$. Using the non-slip assumption between the wheels and the ground, we can model the constraints as follows:

$$w^1(q) \cdot \dot{q} = \dot{x}\cos\theta + \dot{y}\sin\theta - \frac{\rho}{2}(\dot{\psi}_1 + \dot{\psi}_2) = 0;$$

$$w^2(q)\cdot\dot{q} = -\dot{x}\sin\theta + \dot{y}\cos\theta = 0;$$

$$w^3(q)\cdot\dot{q} = \dot{\theta} - \frac{\rho}{2w}(\dot{\psi}_1 - \dot{\psi}_2) = 0.$$

The Lagrangian for the system is given by

$$L(q,\dot{q}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_w(\dot{\psi_1}^2 + \dot{\psi_2}^2),$$

where $m$ is the mass of the car and $J$ the moment of inertia, $J_w$ the intertia of the wheels.

Here the group is $G = SE(2)$, the base is $M = (S^1 \times S^1)$. Let $\psi = (\psi_1, \psi_2)$ be a point in the base, then $g = (x, y, \theta)$ is a point in the fiber over $\psi$. The left action of $G$ on $Q$ is given by

$$\Phi_g(x_1, y_1, \theta_1, \psi_1, \psi_2) = \begin{pmatrix} x_1\cos\theta - y_1\sin\theta + x \\ x_1\sin\theta + y_1\cos\theta + y \\ \theta_1 + \theta \\ \psi_1 \\ \psi_2 \end{pmatrix}$$

and the tangent of the left action is

$$D_q\Phi_g(\dot{x}, \dot{y}, \dot{\theta}, \dot{\psi_1}, \dot{\psi_2}) = \begin{pmatrix} \dot{x}\cos\theta - \dot{y}\sin\theta \\ \dot{x}\sin\theta + \dot{y}\cos\theta \\ \dot{\theta} \\ \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix}$$

Let $\xi = (u, v, w) \in se(2)$, then the corresponding infinitesimal generator is $\xi_Q = (u - wx, v + wx, w, 0, 0)$.

It can be verified that

$$H_qQ = span\begin{bmatrix} \rho\cos\theta & 0 \\ \rho\sin\theta & 0 \\ 0 & \frac{\rho}{w} \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is just the set of velocities satisfying the 3 constraints mentioned above. And the vertical subspace(tangent to the group orbit)

$$V_qQ = \{\xi_Q \in T_qQ : \xi \in \mathfrak{g} = se(2)\}$$

can be identified with

$$V_qQ = span\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

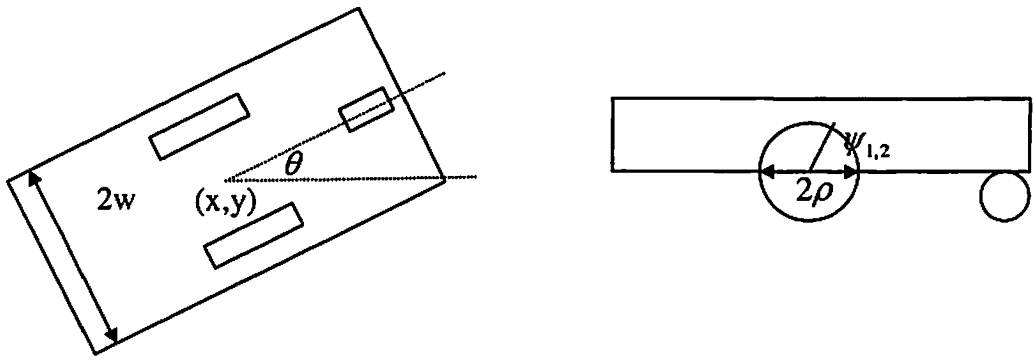


**Figure 3 Two wheeled kinematic car**

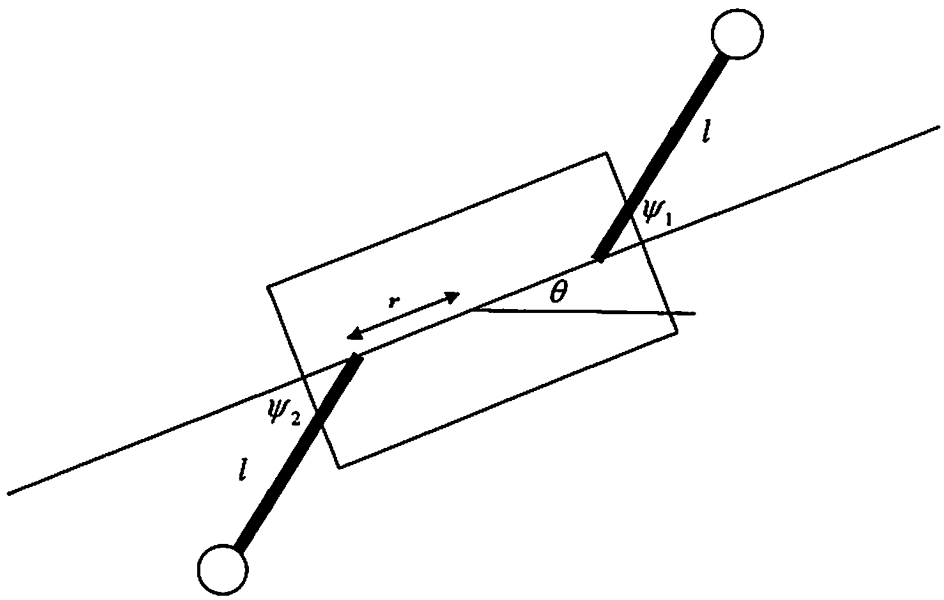


**Figure 4 A simplified model planar space robot**

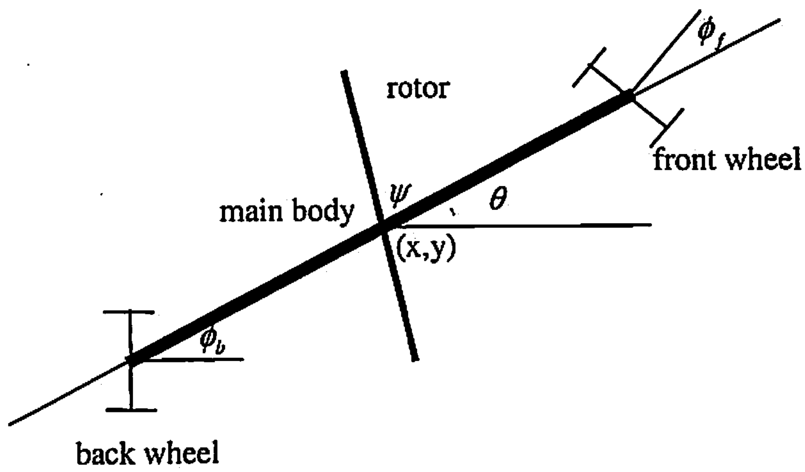


**Figure 5 A simplified model of snakeboard**

Obviously, $T_qQ = H_qQ \oplus V_qQ$ and $H_qQ$ depends smoothly on $q$. And it can be verified that $H_{\Phi_g(q)}Q = T_q\Phi_g(H_qQ)$. Thus there exists a connection one form as mentioned in the last section. Let

$$A(q)\cdot\dot{q} = \begin{pmatrix} \dot{x} - \frac{\rho}{2}\cos\theta(\dot{\psi}_1 + \dot{\psi}_2) + y(\dot{\theta} - \frac{\rho}{2w}(\dot{\psi}_1 - \dot{\psi}_2)) \\ \dot{y} - \frac{\rho}{2}\sin\theta(\dot{\psi}_1 + \dot{\psi}_2) - x(\dot{\theta} - \frac{\rho}{2w}(\dot{\psi}_1 - \dot{\psi}_2)) \\ \dot{\theta} - \frac{\rho}{2w}(\dot{\psi}_1 - \dot{\psi}_2) \end{pmatrix}$$

It can be checked that $A(q)\cdot\xi_Q = \xi$ and $A(\Phi_g(q))\cdot D_q\Phi_g(\dot{q}) = Ad_gA(q)\cdot\dot{q}$. Thus it's a well defined connection on $Q$. It's also easily checked that $H_qQ = ker(A(q))$. Rewrite the connection one form, we get:

$$A(q)\cdot\dot{q} = \begin{bmatrix} \cos\theta & \sin\theta & y \\ -\sin\theta & \cos\theta & -x \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{pmatrix} \dot{x}\cos\theta - \dot{y}\sin\theta \\ \dot{x}\sin\theta + \dot{y}\cos\theta \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} -\rho & 0 \\ 0 & 0 \\ 0 & -\frac{\rho}{w} \end{bmatrix}\begin{pmatrix} \dot{\psi}_1 + \dot{\psi}_2 \\ \dot{\psi}_1 - \dot{\psi}_2 \end{pmatrix}\right),$$

which is of the form

$$A(q)\cdot\dot{q} = Ad_g(\xi + \mathbb{A}(\psi)\cdot\dot{\psi}),$$

where

$$\mathbb{A}(\psi) = \begin{bmatrix} -\rho & -\rho \\ 0 & 0 \\ -\frac{\rho}{w} & \frac{\rho}{w} \end{bmatrix}$$

, and

$$\dot{\psi} = \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix}.$$

It describes how the changes in the shape variable result in the net changes in the group variables, i.e.

$$\xi = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = -\mathbb{A}(\psi)\cdot\dot{\psi} = \begin{pmatrix} \frac{\rho}{2}(\dot{\psi}_1 + \dot{\psi}_2) \\ 0 \\ \frac{\rho}{2w}(\dot{\psi}_1 - \dot{\psi}_2) \end{pmatrix}.$$

Note the above equation viewed in the body fixed frame. To view this relationship in the fixed reference frame, we just need to make a coordiante transformation by multipying the above equation with $Ad_g$.

In fact, when one derives the constrained equations of motion using the Lagrange-d'Alembert principle, the **curvature** of the connection appears in one side of the equation and the constrained Lagrangian on the other side. The effect of the constraints is taken into account through the curvature term, which gives a nice and intrinsic angle of view. In [1] there is a more detailed discussion about curvature.

## 4 Symmetry and connection

In the previous section, we analyzed a system with a sufficient number of kinematic constraints, i.e. the number of constraints is exactly the dimension of the position group $G$. The system's motion is fully determined by the constraints. We call the connection derived from such constraints a kinematic connection.

While there exist other cases in which there are no kinematic constraints. A falling cat is a typical example. So is a satellite in the outer space. In these cases, the connection arises from the conservation laws, which is called a mechanical connection.

Conservation laws arise when there are cyclic variables in the system. We assume that the variables $\theta^i$, $i = 1, \ldots, k$, are cyclic, that is, they don't appear explicitly in the Lagrangian, although their velocities do. Then the Lagrangian for the system has the form(in local coordiantes)

$$L(q, \dot{q}) = L(r, \dot{r}, \dot{\theta}) \qquad q = (r, \theta) \in \mathbb{R}^{n-k} \times \mathbb{R}^k,$$

The Euler-Lagrange equations of the system are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^i} - \frac{\partial L}{\partial r^i} = 0 \qquad i = 1, \ldots, n-k;$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}^j} = 0 \qquad j = 1, \ldots, k.$$

The second set of equations says that the corresponding conjugate momenta $p_i = \frac{\partial L}{\partial \dot{\theta}^i}$ are conserved quantities. In the case that the Lagrangian is given by the difference between the kinetic and potential energies of the system, those momenta are given by

$$p_i = g_{ia}\dot{r}^a + g_{ij}\dot{\theta}^j.$$

If the initial value of these momenta are zero, then they stay zero. We get

$$\dot{\theta}^j = -g^{ij}g_{ia}\dot{r}^a,$$

where $[g^{ij}]$ is the inverse of $[g_{ij}]$. This actually gives a connection

$$A(q)^j\dot{q} = \dot{\theta}^j + g^{ij}g_{ia}\dot{r}^a,$$

In this case, the shape space $M = \mathbb{R}^{n-k}$, and the bundle is given by $\pi : \mathbb{R}^n \to \mathbb{R}^{n-k}$, with $G = \mathbb{R}^k$ and addition as the group operation.

More generally speaking, conservation laws arise from the invariance of a Lagrangian under the action of a Lie group. We have the following **Neother's theorem:**

Let $L(q, \dot{q})$ be a Lagrangian which is invariant under the action of a Lie group, $G$. That is, $L(\Phi_g(q), D_q\Phi_g(\dot{q})) = L(q, \dot{q})$, for $g \in G$, $q \in Q$, and $\dot{q} \in T_qQ$. Then, for all curves $c(t) : [a, b] \to Q$ satisfying Lagrange's equations,

$$\frac{d}{dt}\langle\frac{\partial L}{\partial \dot{q}}(\dot{c}(t)), \xi_Q(c(t))\rangle = 0,$$

for all $\xi \in \mathfrak{g}$ Or, $\dot{p} = 0$, where $p = \langle\frac{\partial L}{\partial \dot{q}}, \xi_Q\rangle$ is the generalized momentum.

When $G$ is $SE(2)$ or $SE(3)$, Neother's theorem is just the conservation of linear and angular momentum.

One example is the planar space robot. A simplified model is shown in Figure 4. Let $M$ and $I$ represents the mass and inertia of the central body and let $m$ represent the mass of the arms, which is lumped at the tips. The arms are connected to the central body via revolute joints located at a distance $r$ from the center of mass of the body and have length $l$. Let $\theta$ be the angle of the central body w.r.t the horizontal, $\psi_1$ and $\psi_2$ the angles of the arms w.r.t the central body. Assume the body is free floating in the space and there is no friction. Let $p$ be the positon of a point on the central body. The Lagranqian of the systmes is given by

$$L = \frac{1}{2}(M+2m)\|\dot{p}\|^2 + \frac{1}{2}\begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\theta} \end{pmatrix}^T \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\theta} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} = a_{22} &= ml^2 \\ a_{12} &= 0 \\ a_{13} &= ml^2 + mr\cos\psi_1 \\ a_{23} &= ml^2 + mr\cos\psi_2 \\ a_{33} &= I + 2ml^2 + 2mr^2 + 2mrl\cos\psi_1 + 2mrl\cos\psi_2. \end{aligned}$$

Note $L$ does not depend on $\theta$. Therefore it follows that in the absence of external forces,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0.$$

Thus $\mu = \frac{\partial L}{\partial \dot{\theta}} = a_{13}\dot{\psi}_1 + a_{23}\dot{\psi}_2 + a_{33}\dot{\theta}$ is the conserved quantity, which is exactly the angular momentum of the system. If the initial angular momentum is zero, then the conservation of angular momentum gives the following constraints:

$$a_{13}\dot{\psi}_1 + a_{23}\dot{\psi}_2 + a_{33}\dot{\theta} = 0.$$

Let $(\psi_1, \psi_2) \in M = S^1 \times S^1$ be the shape variables and $\theta \in G = S^1$ be the group variable, then the connection on this principal bundle is given by

$$A(q)\cdot\dot{q} = \dot{\theta} + \frac{a_{13}}{a_{33}}\dot{\psi}_1 + \frac{a_{23}}{a_{33}}\dot{\psi}_2,$$

and the horizontal lift of changes in shape varibles is given by

$$\dot{\theta} = -\frac{a_{13}}{a_{33}}\dot{\psi}_1 - \frac{a_{23}}{a_{33}}\dot{\psi}_2.$$

This enables us to compute the geometric phase.

## 5 Mixed kinematic and dynamic case

In this section, we turn to the case in which there are kinematic constraints but the number of the constraints is not sufficient to fully determine the motion of the system. At the same time, conservation laws may not be preserved because of the constraints.

Fortunately, there is still the quantity called generalized momentum evolving under the governing of the generalized momentum equation. It will give us a connection together with the constraints.

Below is an example called the snakeboard showing how to construct the connection in this case.

The simplified model of a snake board is shown in Figure 5. Let $(x, y, \theta) \in G = SE(2)$ denote the position of the main body, $(\psi, \phi_b, \phi_f) \in M = S^1 \times S^1 \times S^1$ denote the orientation angles of the rotor, back wheels, and front wheels respectively. So the configuration space is $Q = G \times M$. The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_r(\dot{\psi} + \dot{\theta})^2 + \frac{1}{2}J_w((\dot{\phi}_b + \dot{\theta})^2 + (\dot{\phi}_f + \dot{\theta})^2),$$

where $J$ is the intertia of the main body, $J_r$ the inertia of the rotor, and $J_w$ the inertia of the wheels about the vertical axis. $m$ is the total mass of the snake board.

The non-slip assumption at the wheels determines two constraints:

$$-\sin(\phi_f + \theta)\dot{x} + \cos(\phi_f + \theta)\dot{y} + l\cos(\phi_f)\dot{\theta} = 0,$$

$$-\sin(\phi_b + \theta)\dot{x} + \cos(\phi_b + \theta)\dot{y} - l\cos(\phi_b)\dot{\theta} = 0.$$

There are only two constraints but the dimension of $G$ is 3, i.e. the constraints are not enough to determine the motion of the motion of the snakeboard.

Now let the constraint distribution $D_q$ be the set of all velocities satisfying the above constraints. It turns out that

$$D_q = span\left[a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \phi_b}, \frac{\partial}{\partial \phi_b}\right],$$

where

$$a = -l[\cos\phi_b \cos(\phi_f + \theta) + \cos\phi_f \cos(\phi_b + \theta)],$$

$$b = -l[\cos\phi_b \sin(\phi_f + \theta) + \cos\phi_f \sin(\phi_b + \theta)],$$

$$c = \sin(\phi_b - \phi_f).$$

The vertial subspace, i.e. the tangent space of the group fiber, is given by

$$V_qQ = span\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\right].$$

We see $D_q$ and $V_qQ$ have a nonempty intersection. Recall that in the pure kinematic case, $T_qQ = D_q \oplus V_qQ$ and hence we can choose $D_q$ as the horizontal

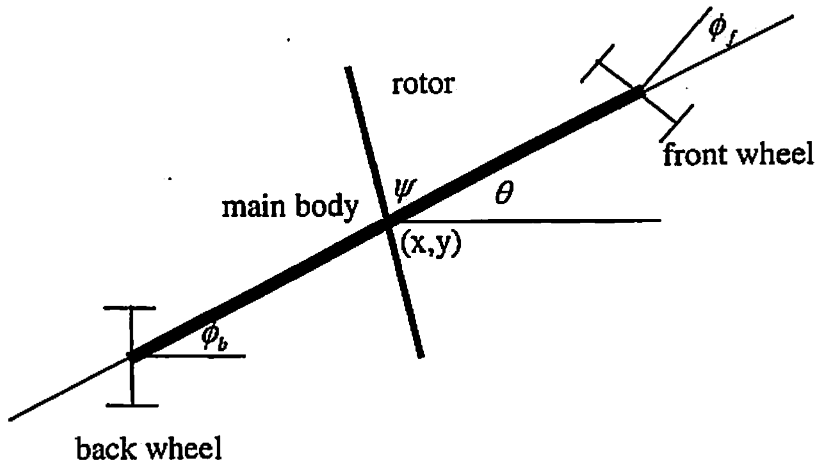


**Figure 5 A simplified model of snakeboard**

subspace and define a connection on the principal bindle. But things are different in this case. Let $S_q = D_q \cap V_q Q$ be the constrained fiber distribution. We have the following **proposition:**

If a constrained system on $Q = G \times M$ has a $G$-invariant Lagrangian $L$ and the constrained distribution $D$, and $c(t)$ is a curve satisfying the Lagrange's equations, then the following **generalized momnetum equation** holds for vector fields $\xi^c_Q \in S = D \cap VQ$:

$$\frac{d}{dt}p^c = \frac{\partial L}{\partial \dot{q}^i}\left(\frac{d}{dt}(\xi^c(c(t)))\right)^i_Q + \tau_i(\xi^c(c(t)))^i_Q$$

where $p^c = \frac{\partial L}{\partial \dot{q}^i}(\xi^c(c(t)))^i_Q$ is called the **constrained momentum.**

Here, for the snakeboard example, the constrained fiber distribution

$$S_q = span\left[a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta}\right].$$

The constrained momentum is

$$\begin{aligned} p^c &= \langle \frac{\partial L}{\partial \dot{q}^i}, \xi^c_Q \rangle \\ &= ma\dot{x} + mb\dot{y} + Jc\dot{\theta} + J_r c(\dot{\psi} + \dot{\theta}) + J_w c(\dot{\phi}_b + \dot{\theta}) + J_w c(\dot{\phi}_f + \dot{\theta}) \\ &= (mR^2 + \hat{J}c)\dot{\theta} + J_r c\dot{\psi} + J_w c(\dot{\phi}_b + \dot{\phi}_f) \end{aligned}$$

which is the angular momentum of the snakeboard. $\hat{J} = J + J_r + 2J_w$ is the sum of the inertia and $R$ is distance from the center of mass to the instantaneous center of rotation.

Combining the kinematic constraints and constrained momentum, a connection can be constructed. Write the three equations in the matrix form

$$\begin{bmatrix} -\sin(\phi_f + \theta) & \cos(\phi_f + \theta) & l\cos\phi_f \\ -\sin(\phi_b + \theta) & \cos(\phi_b + \theta) & -l\cos\phi_b \\ ma & mb & \hat{J}c \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ J_r c\dot{\psi} + J_w c(\dot{\phi}_b + \dot{\phi}_f) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p^c \end{pmatrix}$$

The first matrix can be writen as the product of two matrices

$$W(r)\cdot g = \begin{bmatrix} -\sin\phi_f & \cos\phi_f & l\cos\phi_f \\ -\sin\phi_b & \cos\phi_b & -l\cos\phi_b \\ -2ml\cos\phi_f\cos\phi_b & -ml\sin(\phi_b + \phi_f) & \hat{J}\sin(\phi_b - \phi_f) \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $r = (\psi, \phi_b, \phi_f)$ denotes the shape variable. Finally, we can get the connection as in the form

$$g^{-1}\dot{g} = A(r) \cdot \dot{r} + \gamma(r)p^c.$$

More discussion about the snakeboard can be found in [4], [5], [6] and [7].

There are other interesting examples such as the roller racer[9], kinematic snake [4].

## 6 Summuray

This review focused on the geometric insight of locomotion problems. We saw that a mechanical system has an inherent structure of a principal fiber bundle. A connection can be determined by the kinematic constraints or dynamic symmetries or both. The connection describes the essential mechanism of locomotion, coupling internal shape changes with external constraints which leads to a net change in the group variables, namely, geometric phase. Using these tools, we can parametrize the system with variables whose physical meanings can be easily seen, e.g. momenta, angular momenta, internal shape, etc. Besides, these geometric concepts are also very powerful in analysing the controllability of a locomotion system. Basicly, the possible control inputs are constrained in the tangent of the base space. By appropriately combination of inputs in different directions, one can get a output in a direction that is diffent to those of the inputs, e.g. the Lie bracket of two inputs. The controllability can be analyzed by investigating the controllability Lie algebra. Discussions about controllability can be found in most of the references.

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