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REPORT ON MY WORK RELATED TO MATH 275

In the annual evaluation last week, I had already presented an overall view of my work in the last 2 semesters. Here, I will only talk about my work that is more directly related to Math 275.

In the last several months, I have studied the first 5 chpaters of the Lectures on Mechanics closedly, and supplemented it with the related materials in Chapters 19 & 20 of the Mechanics & Symmetry as well as the paper by Simo, Lewis and Marsden on the energy-momentum method.

I went through almost all the proofs carefully. The materials that gave me the most trouble was the section on block diagonalization and there are still certain areas in this particular section that I am not clear about and would like to get some help from you (especially on how to visualize the various subspaces of S).

[As an aside: In writing this report, an idea hits me and I feel there may be some values in working out a proof of the block diagonalization theorem in the abelian case. As you have said repeatedly in your lectures, in order to understand the contagent bundle reduction well, it is important to master first the 2 extreme cases (Lie Poisson case and the abelian case). Since the block diagonalization theorem and its proof is much more complicated than the contangent bundle reduction theorem, a simpler proof of its special (abelian) case can be very instructive for the beginners like us and can be used as a steppingstone for understanding the general case. Besides, there are some really interesting concrete models, including the double spherical pendula, that is abelian. I would like to know whether such a proof exists and is written down somewhere (in Chapter 21?). If not, maybe I should spend sometime to work out a proof of it. In any case, is it possible for you to give a lecture(s) on the proof of the energy-momentum method sometime in the near future?]

Also, I have worked through all the examples in the first 5 chapters of the Lectures on Mechanics, like rigid body, simple spherical pendulum, double spherical pendula and classical water molecule. I did this because I deeply believe what you have said repeatedly in your lectures that it is important to have concrete examples, especially good examples, in mind in working on geometric mechanics. In my opinion, the ultimate test of the value of the geometric mechanics is whether it can provide the sharp mathematical tools in solving practical problems in sciences. For us, the student of geometric mechanics, working on examples will teach us the skills needed in future practical applications, deepen our understanding of the existing theory and make us aware of the inter-connection between the application and the theory. Hence, it is essential to the training of a good future geometric mechanist who can forge ever sharper tools for the sciences. This is the reason why I approached Aaron to organize together the study group this summer so that we as well as other students in Math 275 can learn from each other by going over all the examples in the whole books. However, it will be really important if you can give several timely lectures on some of the most difficult part of the theory, like the reduced energy-momentum method and its proof, in the (Wed.) informal seminar, because those are the materials no beginners like us would understand well enough to give a talk on.

Included is the calculation I have done in preparation for Aaron's talk (5/17) on the classical water molecule. Due to shortness of time, he was only able to cover section 3.2 on last Tues. According to him, he had already turned in the corrections for that particular section. As for section 3.5, he showed me the corrections that he had gotten earlier from you. It turns out that those corrections in Section 3.5 is slightly different from mine which prompt me to submit my calculation here for you to check. As for the corrections for other examples, I have already turned in last week and would not be included here.

Section 3.5 Water Molecule

1. To compute the locked inertia tensor $I(r,s)$, we need
   - (i) the infinitesimal generators $\eta_a(q)$, $\tilde{\eta}_a(q)$ and
   - (ii) the kinetic energy metric $\langle\langle\ \rangle\rangle$ (from Lagrangian)

no mistake here: Both of them have been done in Section 3.2. And it is straight forward to get

$$I(r,s)\eta = \left[\frac{m}{2}\|r\|^2 + 2M\bar{m}\|s\|^2\right]\eta - \left[\frac{m}{2}r\cdot(r\cdot\eta) + 2M\bar{m}\,s\,(s\cdot\eta)\right]$$

2. To form the mechanical connection we need to invert $\mathbb{I}(r,s)$ (assume r and s are orthogonal)

∴ to solve $\mathbb{I}(r,s)\eta = u$ where $u \in so^*(3) \cong \mathbb{R}^3$ (not orthonormal)

correction: Notice that r, s, $r\times s$ form an **orthogonal** basis and we can write $\eta = ar + bs + c(r\times s)$ (3.5.19)

To solve $\mathbb{I}(r,s)\eta = u$ is to find a, b, c in terms of u. Taking the dot product of (3.5.19) with r, s, $r\times s$ respectively.

(i) $\|r\|^2 a\left[\frac{m}{2}\|r\|^2 + 2M\bar{m}\|s\|^2\right] - \frac{m}{2}\|r\|^2 a\|r\|^2 = u\cdot r \quad \therefore\ a = \dfrac{u\cdot r}{2M\bar{m}\|r\|^2\|s\|^2}$

(ii) $\|s\|^2 b\left[\frac{m}{2}\|r\|^2 + 2M\bar{m}\|s\|^2\right] - 2M\bar{m}\|s\|^2 b\|s\|^2 = u\cdot s \quad \therefore\ b = \dfrac{2u\cdot s}{m\|r\|^2\|s\|^2}$

correction: (iii) $\|r\|^2\|s\|^2 c\,\gamma = u\cdot(r\times s) \quad \therefore\ c = \dfrac{u\cdot(r\times s)}{\gamma\|r\|^2\|s\|^2}$

Footnote: Note: Instead of going directly to get the formula $\mathbb{I}(r,s)^{-1}$ out of a, b, c

i.e. $\eta = \dfrac{r\cdot u}{2M\bar{m}\|r\|^2\|s\|^2}r + \dfrac{2s\cdot u}{m\|r\|^2\|s\|^2}s + \dfrac{u\cdot(r\times s)}{\gamma\|r\|^2\|s\|^2}(r\times s)$

(*) or $I(r,s)^{-1} = \dfrac{1}{2M\bar{m}\|r\|^2\|s\|^2}r\otimes r + \dfrac{2}{m\|r\|^2\|s\|^2}s\otimes s + \dfrac{1}{\gamma\|r\|^2\|s\|^2}(r\times s)\otimes(r\times s)$

The book tries to get a simpler formula by substituting η into (3.5.19)

which is easier to use in calculating the connection.

$$\gamma\eta - \left[\frac{m}{2} r\,(a\|r\|^2) + 2M\bar m\, s\,(b\|s\|^2)\right] = u$$

$$\therefore\quad \eta = \frac{1}{\gamma}u + \frac{m}{2\gamma}\,\frac{u\cdot r}{2M\bar m\cancel{\|r\|^2}\|s\|^2}\cancel{\|r\|^2} r + \frac{2M\bar m}{\gamma}\,\frac{2u\cdot s}{m\|r\|^2\|s\|^2}\|s\|^2 s$$

$$\text{so}\quad \eta = \frac{1}{\gamma}u + \frac{m\, r\cdot u}{4M\bar m\|s\|^2\gamma}\, r + \frac{4M\bar m\, s\cdot u}{m\|r\|^2\gamma}\, s$$

correction

no $\|r\|^2$ in the denominator of the 2nd term.

no $\|s\|^2$ in the denominator of the 3rd term

$$\therefore\quad \mathbb{I}(r,s)^{-1} = \frac{1}{\gamma}\,\mathrm{Id} + \frac{m}{4M\bar m\|s\|^2\gamma}\, r\otimes r + \frac{4M\bar m}{m\|r\|^2\gamma}\, s\otimes s \qquad [35.22]$$

(when r and s are orthogonal)

5 To compute the mechanical connection α, we need to solve

using α here is o.k. but $\alpha(\dot r,\dot s)$ made the student think that α is defined on TQ

$$\left[\frac{m}{2}\|r\|^2 + 2M\bar m\|s\|^2\right]\alpha(\dot r,\dot s) - \left[\frac{m}{2} r\,(r\cdot\alpha(\dot r,\dot s)) + 2M\bar m\, s\,(s\cdot\alpha(\dot r,\dot s))\right] = r\times\pi + s\times\sigma$$

From (A), we immediately get

$$\alpha = \frac{r\cdot(r\times\pi + s\times\sigma)}{2M\bar m\|r\|^2\|s\|^2}\, r + \frac{2s\cdot(r\times\pi + s\times\sigma)}{m\|r\|^2\|s\|^2}\, s + \frac{(r\times s)\cdot(r\times\pi + s\times\sigma)}{\gamma\|r\|^2\|s\|^2}\,(r\times s)$$

Since $r\cdot(r\times\pi + s\times\sigma) = r\cdot(s\times 2M\bar m\dot s) = 2M\bar m\,\dot s\cdot(r\times s)$

$s\cdot(r\times\pi + s\times\sigma) = s\cdot(r\times\frac{m}{2}\dot r) = -m\,\dot r\cdot(r\times s)$

$(r\times s)\cdot(r\times\pi) = (r\times s)\cdot(r\times\frac{m}{2}\dot r) = \frac{m}{2}\dot r\cdot((r\times s)\times r) = \frac{m}{2}\dot r\cdot(\|r\|^2 s - \underbrace{(r\cdot s)}_{=0 \text{ because } r\perp s} r)$

$= \frac{m}{2}\|r\|^2\,\dot r\cdot s$

$(r\times s)\cdot(s\times\sigma) = (r\times s)\cdot(s\times 2M\bar m\dot s) = 2M\bar m\,\dot s\cdot((r\times s)\times s) = 2M\bar m\,\dot s\cdot((r\cdot s)s - \|s\|^2 r)$

$= -2M\bar m\|s\|^2\,\dot s\cdot r$

since $\frac{r}{\|r\|} = \hat{r}$, $\frac{s}{\|s\|} = \hat{s}$ $\frac{r\times s}{\|r\|\|s\|} = n$

$$\therefore\ \alpha^{(\dot r,\dot s)} = \frac{\dot s\cdot(r\times s)}{\|r\|^2\|s\|^2}\, r - \frac{\dot r\cdot(r\times s)}{\|r\|^2\|s\|^2}\, s + \frac{1}{\gamma}\left\{\frac{m}{2}\frac{\|r\|^2\,\dot r\cdot s}{\|r\|^2\|s\|^2} - \frac{2M\bar m\|s\|^2\,\dot s\cdot r}{\|r\|^2\|s\|^2}\right\}(r\times s)$$

correction

$$= \frac{\dot s\cdot n}{\|s\|}\,\hat r - \frac{\dot r\cdot n}{\|r\|}\,\hat s + \frac{1}{\gamma}\left\{\frac{m}{2}\|r\|^2 s\cdot\dot r - 2M\bar m\|s\|^2 r\cdot\dot s\right\}\frac{n}{\|r\|\|s\|}$$

④ To compute $\gamma_\mu(r,s)$

using γ_μ here ... that $\gamma_\mu(r,s)$ reminds the student that it is a 1 form on Q.

Note: Since $\gamma_\mu(r,s) \in T^*Q$, clearly its co-ordinate expression must have 2 components, and the formula in the book is wrong.

$$\langle \gamma_\mu(r,s), v_q\rangle = \langle \mu, \alpha(\dot r,\dot s)\rangle$$

$$= \frac{\dot s\cdot n}{\|s\|}\,\mu\cdot\hat r - \frac{\dot r\cdot n}{\|r\|}\,\mu\cdot\hat s + \frac{1}{\gamma}\left\{\frac{m}{2}\|r\|^2 s\cdot\dot r - 2M\bar m\|s\|^2 r\cdot\dot s\right\}\frac{\mu\cdot n}{\|r\|\|s\|}$$

$$= \left\langle \left(-\frac{\mu\cdot\hat s}{\|r\|}\,n + \frac{m}{2\gamma}(\mu\cdot n)\|r\|\,\hat s,\ \frac{\mu\cdot\hat r}{\|s\|}\,n - \frac{2M\bar m}{\gamma}\|s\|\,\hat r\right),\ (\dot r,\dot s)\right\rangle$$

correction

$$\therefore\ \gamma_\mu(r,s) = \left(-\frac{\mu\cdot\hat s}{\|r\|}\,n + \frac{m}{2\gamma}(\mu\cdot n)\|r\|\,\hat s,\ \frac{\mu\cdot\hat r}{\|s\|}\,n - \frac{2M\bar m}{\gamma}\|s\|\,\hat r\right)$$