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- Rigid Body with internal Torques

- A cotangent Bundle picture

- A ~~ ~~ connection on

$$\begin{array}{c} SO(3) \times T^3 \\ \downarrow \\ T^3 \end{array}$$

- Symplectic ~~ ~~ form on $P_0 = J^{-1}(0) / G$

Rozita - You have developed some excellent skills in this area. Keep it up!!

JM.

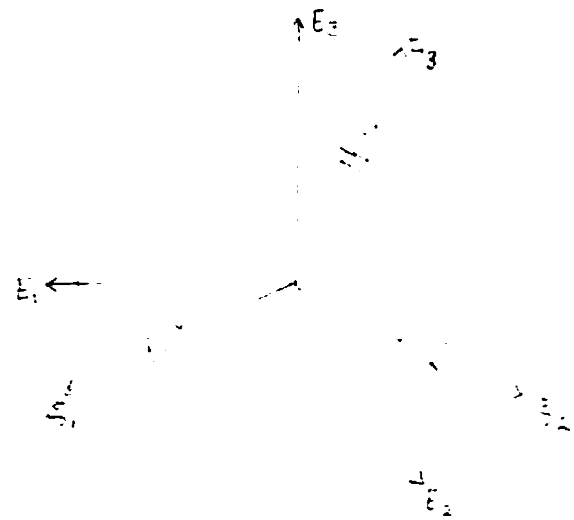
– Rigid Body with internal torques :

Consider a rigid body carrying three symmetric rotors attached along the body axes specified by the unit vectors (orthogonal) $\{\xi_1, \xi_2, \xi_3\}$.

Let $\{E_1, E_2, E_3\}$ be an orthonormal frame fixed in the inertial space.

I_ℓ : moment of inertia of the body with the rotors locked

$I_r = \mathrm{diag}\,(I_{1r}, I_{2r}, I_{3r})$, I_{ir} = moment of inertia of the ith rotor about its spin axis



Ω = body angular velocity

Ω_r = vector of spin velocity of the rotors

R = body attitude i.e. $\xi_i = RE_i$

m = total body angular momentum

π = spatial angular momentum

$$\left.\begin{array}{ll} \pi = Rm & \dot{\pi} = \dot{R}m + R\dot{m} \\ \text{no external torques} & \dot{\pi} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \dot{m} = -R^{-1}\dot{R}\,m \\ \boxed{= m \times \Omega} \end{array} \qquad \hat{\Omega} = R^{-1}\dot{R}$$

L = Lagrangian of the free system

$$= \frac{1}{2}\,\Omega \cdot (I_\ell - I_r)\,\Omega + \frac{1}{2}\left[(\Omega + \Omega_r) \cdot I_r(\Omega + \Omega_r)\right]$$

$\Rightarrow$ conjugate momenta $\begin{cases} \dfrac{\partial L}{\partial \Omega} = I_l \Omega + I_l \Omega_r = m \\ \dfrac{\partial L}{\partial \Omega_r} = I_r(\Omega + \Omega_r) = l \end{cases} \Rightarrow \Omega = (I_l - I_r)^{-1}(m-l)$

So the equations of the motion with internal torques [redacted] u in rotors are:

$$\begin{cases} \dot m = m \times (I_l - I_r)^{-1}(m-l) \\ \dot l = u \end{cases}$$

In [1] it has been shown for special type of feedback u (control)

the body equations in the above can be reduced as Hamiltonian eqns

on $so(3)^*$ with Lie-Poisson structure. For example for control:

$u = k\dot m$ where k is a constant real matrix

s.t. $I = (1-k)^{-1}(I_l - I_r)$ is symmetric

we get $km - l = p$ is conserved then

$$\begin{aligned} \dot m &= m \times (I_l - I_r)^{-1}(m - l) \\ &= m \times (I_l - I_r)^{-1}(m - km + p) \\ &= m \times (I_l - I_r)^{-1}(1-k)(m - \xi) \qquad \xi = (1-k)^{-1}p = (1-k)^{-1}(km - l) \end{aligned}$$

$\dot m = m \times I^{-1}(m - \xi)$ is Hamiltonian w.r.t usual Lie poisson structure on $so(3)^*$

$= m \times \nabla H$ for $H = \frac{1}{2}(m-\xi) \cdot I^{-1}(m-\xi)$

& $C = \frac{1}{2}\|m\|^2$ is a Casimir determining the momentum spheres.

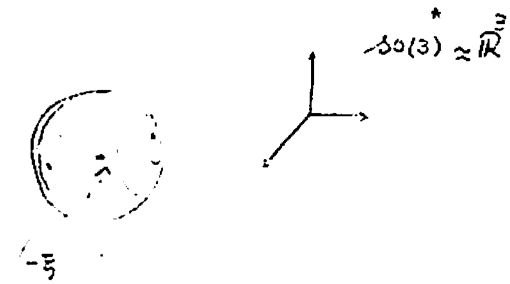
After a change of basis in $so(3)^*$ we get :

$$\dot{m}_b = (m_b + \xi) \times I^{-1} m_b$$

$$= \nabla C \times \nabla H$$

$$\begin{cases} m_b = m - \xi = m + (1-k)^{-1}(km - \ell) \\ \xi = (1-k)^{-1}(km - \ell) \end{cases}$$

$$\begin{cases} C = \frac{1}{2} \| m_b + \xi \|^2 \\ H = \frac{1}{2} m_b \cdot J^{-1} m_b \end{cases}$$



In fact $\begin{cases} \dot{m}_b = (m_b + \xi) \times I^{-1} m_b \\ \dot{\xi} = 0 \end{cases}$ is Lie-Poisson on $so(3)^* \oplus \mathbb{R}^3 = g^*$

where $G = SO(3) \times T^3$

Here ξ plays the role of parameter [illegible] of the

[illegible] $\xi = -(1-k)^{-1}(km - \ell)$.

– A Cotangent Bundle Picture :

$$Q = SO(3) \times T^3 \qquad T^*Q = T^*(SO(3)) \times T^*(T^3)$$

$$\quad R \ , \theta \qquad\qquad (R, P_R, \theta, P_\theta)$$

$$Q/G = T^3 . \qquad\qquad (R, m, \theta, \xi)$$

$$(R, m_b, \theta, \xi)$$

$$\begin{cases} \hat{m} = R^{-1} P_R \\ \xi = R^{-1} P_\theta \end{cases} \qquad m_b = m - \xi$$

$$T^*(SO(3) \times T^3) \xrightarrow{J} so(3)^* \approx \mathbb{R}^3$$

$$(R, m_b, \theta, \xi) \longmapsto R(m_b + \xi) = \mu$$

$$J^{-1}(\mu) \qquad J^{-1}(0) = \{(R, m_b, \theta, \xi) \ / \ \|m_b + \xi\|^2 = \|\mu\|^2\}$$

$$\downarrow \qquad\qquad \downarrow$$

$$P_\mu \approx J^{-1}(0)/G = \{(m_b, \theta, \xi) \ / \ \|m_b + \xi\|^2 = \|\mu\|^2\}$$

$$\downarrow$$

$$T^*(T^3)$$

$$(\theta, \xi)$$

For a fixed ξ_0 The motion is restricted to $so(3)^*$. Suppose the system for $\dot{m}_b = (m_b + \xi) \times I^{-1} m_b$ goes through a periodic solution in sphere $\|m_b + \xi\|^2 = \|\mu\|^2$. During this the body goes through a trajectory on $J^{-1}(\mu)$ but this trajectory is not a closed loop.

In T there will be an attitude drift of:

$$\Delta\theta = \frac{2ET}{\|\mu\|} + \frac{T}{\|\mu\|}(\xi \cdot \Omega_{av}) \pm \Phi_{solid}$$

where $E = H$ (on the trajectory)

T: time period

Ω_{av}: average value of angular velocity over the trajectory

Φ: angle of the solid enclosed by the trajectory

This is found in [1] by integrating the pullback of the connection 1-form [illegible] closed curve $C_1 + C_2$ [illegible] where $\pi_\mu(C_1)$ = periodic trajectory ξ. [illegible] plane

Then by Stokes theorem

$$\int_{C_1+C_2} p\,dq = \int d(p\,dq) = \int_{\pi_\mu(\Sigma)} dA$$

and $\int_{C_2} p\,dq = \|\mu\|\Delta\theta$

Here fixing ξ we can restrict ourselves to $SO(3)$ & $so^*(3)$ where $i^*(p\,dq)$ is a connection 1-form on $S^1 = G_\mu$ principle Bundle: identifying $\mathfrak{g}_\mu$ with $\mathbb{R}$.

$$\begin{array}{ccc} G_\mu & \longrightarrow & J^{-1}(\mu) \\ & & \downarrow \pi_\mu \\ & & \|m_1+\xi\| = \mu \end{array}$$

Then its curvature will be ω_μ = reduced symplectic form dA

This fact is used in Reconstruction of the solutions [2]

A Connection on $SO(3) \times T^3 \to T^3$

Let $Q = SO(3) \times T^3$ $\quad G = SO(3)$

$TQ : (R\Omega, R\Omega_r)$ $\quad$ Ω: body angular velocity

$R\Omega = \Omega_Q(R)$ $\quad$ Ω_r: rotors velocity vectory in body frame.

Define $\mathcal{A} : TQ \longrightarrow \mathfrak{g}$

$$(R\Omega, R\Omega_r) \mapsto I_l^{-1}(I_l\Omega - I_r\Omega_r) = \Omega - I_l^{-1} I_r \Omega_r$$

$$\begin{cases} \mathcal{A}(R\Omega, 0) = \Omega \\ \mathcal{A} \text{ is independent of } R \text{ as it is } Ad^* \text{ invariant} \end{cases}$$

Infact $\mathcal{A}$ is the mechanical connection

$hor(TQ) = \{ (R(-I_l^{-1} I_r \Omega_r), R\Omega_r) \}$

$ver(TQ) = \{ (R\Omega, 0) \}$

for $v = (R\Omega, R\Omega_r)$ $\quad \begin{cases} ver\, v = (R(\Omega + I^{-1} I_r \Omega_r), 0) \\ hor\, v = (-R(I^{-1} I_r \Omega_r), R\Omega_r) \end{cases}$

$$d\mathcal{A}(hor\, v_1, hor\, v_2) = \mathcal{A}([hor\, v_1, hor\, v_2])$$
$$= \mathcal{A}\left(-R(-I_l^{-1} I_r \Omega_r^1 \times -I_l^{-1} I_r \Omega_r^2), -R(\Omega_r^1 \times \Omega_r^2)\right)$$
$$= -\left(I_l^{-1} I_r \Omega_r^1 \times I_l^{-1} I_r \Omega_r^2\right) + I_l^{-1} I_r (\Omega_r^1 \times \Omega_r^2)$$

$$\mathcal{B}_\mu(\Omega_r^1, \Omega_r^2) = -\langle \mu, (I_l^{-1} I_r \Omega_r^1 \times I_l^{-1} I_r \Omega_r^2) - I_l^{-1} I_r (\Omega_r^1 \times \Omega_r^2) \rangle \; ??$$

– Symplectic form on P_0

Let $\dim Q = n \quad \dim G = k$

$$\begin{tikzcd} T^*Q \arrow[d, leftrightarrow] \\ J^{-1}(0) \arrow[d] \\ P_0 = J^{-1}(0)/G \;,\; \Omega_0 \arrow[d] \\ J^{-1}(0)/G \approx T^*(Q/G) \;,\; \omega_0 - B \end{tikzcd}$$

<u>Sketch</u> :

Bundle:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & T^*Q \arrow[r, "J"] \arrow[d, "\pi_Q"] & \mathfrak{g}^* \\ & Q & \end{tikzcd}$$

subbundle of the above :

$$\begin{tikzcd} \mathbb{R}^{n-k} \arrow[r] & J^{-1}(0) \arrow[d, "\pi_0"] \\ & Q \end{tikzcd}$$

affine subbundle :

$$\begin{tikzcd} \mathbb{R}^{n-k} \arrow[r] & J^{-1}(\mu) \arrow[d, "\pi_\mu"] \\ & Q \end{tikzcd}$$

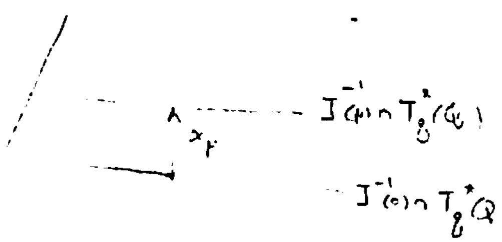
Then $J^{-1}(0) = \bigcup_{\mu \in 0}$ affine subbundles $\subset T^*Q$

– For $q_0 \in U$ open $\subset Q$ & $V = \left(\pi_Q\big|_{J^{-1}(0)}\right)^{-1}(U)$

Let $\varphi : V \longrightarrow Q \times \mathbb{R}^{n-k}$ be a trivialization of $J^{-1}(0)$

– Define the projection : $\quad \mathrm{hor} : T^*Q \longrightarrow T^*Q$

$$\alpha_q \longmapsto \alpha_q - \alpha_{J(\alpha_q)}(q)$$



(where $\alpha_\mu : TQ \to \mathbb{R}$ is a connection 1-form)

Claim : 1 - $\mathrm{hor}^* \Omega(\alpha_q) = \Omega(\alpha_q) + \pi_Q^* d\alpha_{J(\alpha_q)}$

$$\begin{array}{ccc} T^*Q & & 0 \to J^{-1}(0) \\ \downarrow \mathrm{hor} & & \downarrow \mathrm{hor} \\ J^{-1}(0) & & J^{-1}(0) \end{array}$$

2 - $\mathrm{hor}^{-1}(\beta_q) = \{ \beta_q + \alpha_\mu(q) \mid \mu \in \mathfrak{g}^* \}$

Proof : ——

Let $W = \mathrm{hor}^{-1}(V)$

Define : $\Phi : W \longrightarrow V \times \mathcal{O}$

$$\alpha_q \longmapsto (\alpha_q - \alpha_{J(\alpha_q)}(q) \, , J(\alpha_q))$$

$$(\mathrm{hor}(\alpha_q) \, , J(\alpha_q))$$

$V \downarrow$ $V \times \mathcal{O}$

claim : Φ is a trivialization of $J^{-1}(0) \to J^{-1}(0)$

Φ is G - ~~invariant~~ equivariant (because J is equivariant)

$\Phi(g \cdot W) = g \cdot \Phi(W)$

Finally : $\Rightarrow$ $\Phi_G : [W]_G \longrightarrow [V]_G \times \mathcal{O}$

is a trivialization of

$$\begin{array}{c} J^{-1}(0)/G \\ \downarrow \\ J^{-1}(0)/G \approx T^*(Q/G) \end{array}$$

& $\Phi_G^* (\underbrace{\omega_0 - B_{J(\alpha_q)}}_{\text{magnetic term}} \, , \omega_{\mathcal{O}}^+) = \Omega_0$

Refrences :

[1] . Bloch , Krishnaprasad , Marden . Sanchez de Alvarez (1992)
" Stabilization of Rigid Body Dynamics "

[2] Marsden , Montgomery . Ratiu (1990)
" Reduction , Symmetry and phases in mechanics "

[3] Krishnaprasad (1985)
" ~~Th~~ Lie-poisson Structures , dual spin spacecraft and asymptotic stability "