R Keswell

A paper on :–

Stability of the Inverted Pendulum
— a Topological Explanation
by Mark Levi

1/4/88
Mark – I thought you might like to see the report on your paper written by a student in my mechanics course.
Jerry

Introduction

The stability of an inverted pendulum with vertically oscillating point of suspension is a popular subject — perhaps more popular than its importance merits. It is a well known result that stability only exists for certain parameter values. Previous discussions have usually relied on detailed and cumbersome calculations to mathematically explain the phenomenon which although correct provides no physical insight into the situation. Mark Levi, in his paper, turns to topology in order to shed new light on the subject and offers a physical interpretation of the phenomenon. His argument is very general (— the acceleration of the pendulum support is only assumed periodic) and hence is largely qualitative. He reformulates the problem in terms of a Poincare map and its Floquet matrix and then proceeds to examine the stability of this matrix. The culmination of the paper is a theorem and corollary which attempt to link the physics of the system loosely with the appearances of stability. This is a welcome development and contrasts directly with the usual stability analysis of the Mathieu equation offered previously.

PAST DISCUSSIONS

The following two discussion summaries represent traditional thinking on the subject. Phelps and Hunter [1] derive the equation of motion of the inverted pendulum assuming a sinusoidal driving frequency and using Lagrange's techniques. They obtain :—

$$l\ddot{\phi} - (\alpha\cos\Omega t + g)\sin\phi = 0$$

with ϕ the angular displacement from the vertical and taking the rod of the pendulum to be light and acceleration of the suspension point equal to $a(t) = \alpha\cos\Omega t$. Linearizing around the upward vertical gives the Mathieu equation

$$l\ddot{\phi} - (\alpha\cos\Omega t + g)\phi \simeq 0 \quad \text{or more standardly}$$

McLachlan p40 Stability diagram for Mathieu eqn $\ddot{\phi} + (a - 2q\cos 2t)\phi = 0$

3.25. Continuity of the characteristic numbers as functions of q. By considering successive convergents, it may be demonstrated

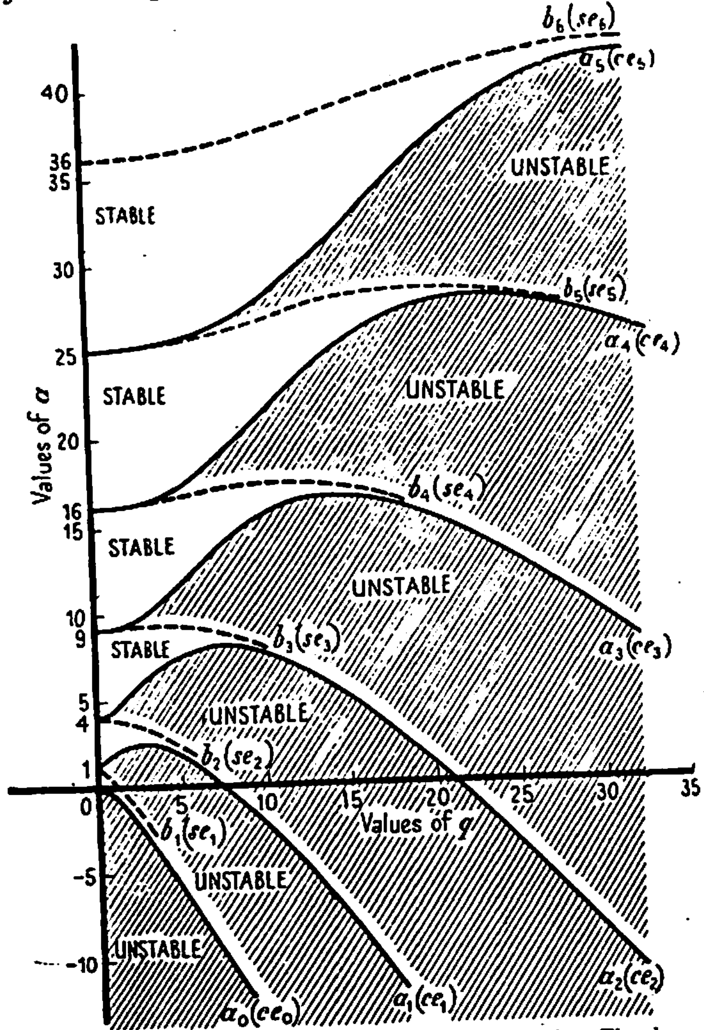


FIG. 8 (A). Stability chart for Mathieu functions of integral order. The characteristic curves a_0, b_1, a_1, b_2,... divide the plane into regions of stability and instability. The even-order curves are symmetrical, but the odd-order curves are asymmetrical about the a-axis. Nevertheless the *diagram* is symmetrical about the a-axis.

that the continued fractions for computing the a (see §§ 3.11, 3.14, 3.15) may be expressed as the quotient of two integral functions. The denominator of this quotient cannot vanish if the value of a is a characteristic number for a periodic Mathieu function of integral

order. Hence the continued
values (a, q) which satisfy
cendental equation for a.
characteristic numbers for c
Since the transcendental eq

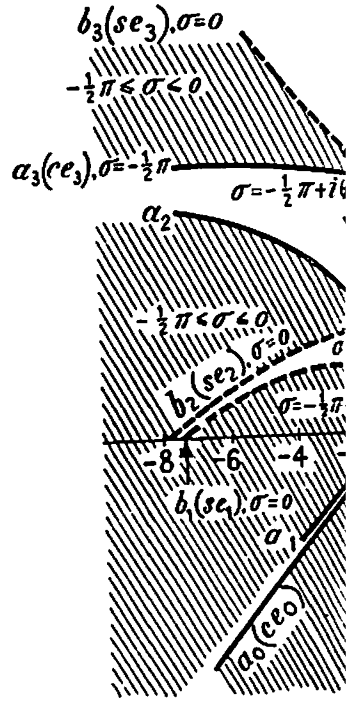


Fig. 8 (B). Illustrating variatio
regions $q > 0$ and $q < 0$. a_m, b_m
a_m, a_{m+1}; b_m, b_{m+1} are mutually

from the recurrence relatio
functions of q (see Fig. 9).

3.26. Additional com
able to comment upon th
cient of $\cos mz$ in $ce_m(z, q)$
all values of q. We sh
remaining coefficients are

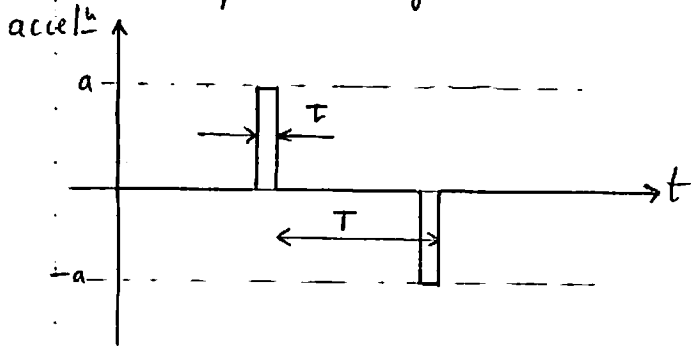
Consider $ce_2(z, q)$ whose
gives

$$\frac{d^2\phi}{d\xi^2} + (a - 2q\cos 2\xi)\phi \simeq 0$$

with $\xi = \frac{\Omega t}{2}$; $a = \frac{-4g}{l\Omega^2}$ & $q = \frac{2x}{l\Omega^2}$

They then proceed to examine the stability of this Mathieu equation which is long and detailed. McLachlan [2] gives the well known stability chart of a against q on p40 {a photocopy is included here.} Suffice to say that the stability – instability regions are complicated areas which have no obvious physical interpretation.

Henry Kalmus [3] provides an analysis of the operating conditions of an inverted pendulum assuming that instead of sinusoidal excitation a series of short impulses is applied to the point of suspension. This he believes leads to a more intuitive understanding of the phenomenon of stability. The rectangular acceleration function for the pendulum pivot consist of alternate positive and negative short-time pulses as follows:–



For his treatment he takes the pendulum to be located in gravity-free space. He proceeds to step by step analyse the motion for each α as follows.

by integrating $\ddot{\alpha} \simeq \frac{-a(t)}{l}\sin\alpha$ over an acceleration period obtain

$$[\dot{\alpha}]_0^\tau \simeq \frac{-a}{l}\sin\alpha\int_0^\tau dt \quad \text{assuming } \sin\alpha \text{ approximately constant over } [0,\tau]$$

$$\simeq \frac{-a\tau}{l}\sin\alpha$$

Hence each acceleration spike effectively kicks the pendulum by an angular velocity $\pm\frac{a\tau}{l}\sin\alpha$ (alternately +, –) every T secs. Thus it is clear that there will exist certain pendulum frequencies for which these angular velocity impulses will produce stability, and corresponding others which are unstable. To add relevancy to his argument, Kalmus shows also that his contrived rect–

angular acceleration function leads to a triangular displacement function which is not too different from the normal sinusoidal wave. Although Kalmus' argument is reasonable and seems to explain the general stability phenomenon, one must remember that gravity was ignored and a rather hypothetical forcing system was used.

Both these past discussions illustrate that the forced inverted pendulum is far from easily solved and understood. In both cases very specific forcing systems were taken to facilitate analysis. In the case of Phelps and Hunter the results obtained are difficult to understand whereas for Kalmus they are reasonable but only specific to a contrived forcing system with no gravity. Mark Levi presents a very different approach to traditional discussions illustrated above.

The Topological Explanation

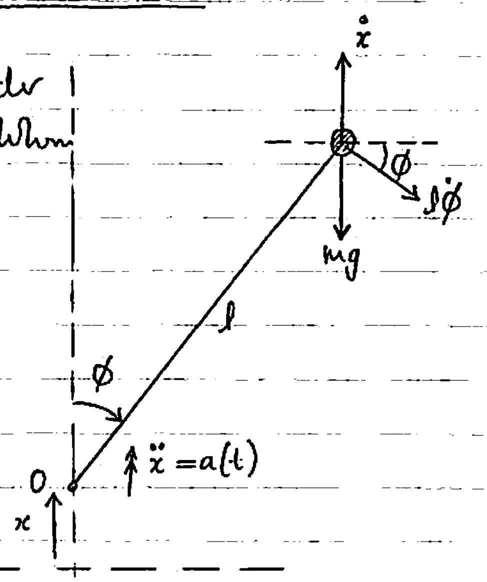
In his paper, Mark Levi offers a simple topological explanation for the stability of the inverted pendulum. He essentially rewrites the linearized equation of motion in matrix form and examines the stability of the associated Floquet matrix which he claims is necessary and sufficient for the pendulum to be stable. As we have effectively a one dimensional problem this matrix is 2X2 and as system is Hamiltonian the determinant is 1. Now the problem is reduced to analyzing how stable 2X2 matrices of det = 1 relate to all 2X2 matrices of det = 1. This is best achieved by identifying the set of 2X2 matrices of det 1 [Sp(2)] with an open solid torus. Then it is seen that the stable matrix subset [S] forms an obstruction of the torus — i.e. any non contractible loop in Sp(2) must intersect S (..in at least two points in fact). Crucially Mark Levi shows that a traversal of the torus (i.e a circuit around the hole) corresponds physically to an increase in the number of times the pendulum is vertical in one period of forcing. Then it follows, assuming that the Floquet matrix depends on a parameter μ in such a way that μ changing from μ_1 to μ_2 forces M to traverse the torus, that there exists a value(s) of μ in $[\mu_1, \mu_2]$ for which the pendulum is

Nice & Clear!

stable. This is the crux of his argument to explain why the inverted pendulum is stable for certain parameter values.

<u>In more detail</u>

Consider pendulum



pt of suspension oscillates up & down with accel^n $= a(t)$

$$V = mg(x + l\cos\phi).$$

$$T = \tfrac{1}{2}m\left\{(l\dot{\phi}\cos\phi)^2 + (\dot{x} - l\dot{\phi}\sin\phi)^2\right\}$$

$$= \tfrac{1}{2}m\left\{l^2\dot{\phi}^2 + \dot{x}^2 - 2\dot{x}l\dot{\phi}\sin\phi\right\}$$

$$\therefore L = \tfrac{1}{2}m\left\{l^2\dot{\phi}^2 + \dot{x}^2 - 2\dot{x}l\dot{\phi}\sin\phi\right\} - mgl(x/l + \cos\phi)$$

Apply Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

gives:-

$$\frac{d}{dt}\left(ml^2\dot{\phi} - m\dot{x}l\sin\phi\right) = -m\dot{x}l\dot{\phi}\cos\phi + mgl\sin\phi$$

$$ml^2\ddot{\phi} - m\ddot{x}l\sin\phi - \cancel{m\dot{x}l\cos\phi\dot{\phi}} = -\cancel{m\dot{x}l\dot{\phi}\cos\phi} + mgl\sin\phi$$

$$\Rightarrow \quad l\ddot{\phi} - a\sin\phi = g\sin\phi$$

$$\Rightarrow \quad l\ddot{\phi} - (a(t) + g)\sin\phi = 0$$

Mark Levi uses the linearized equation @ $\phi = 0$ ie

$$l\ddot{\phi} - (a(t) + g)\phi \simeq 0$$

Let Period of forcing $= T$

now define $\psi = l\dot{\phi}$ and then above becomes

$$\psi = l\dot{\phi}$$

$$\& \quad \dot{\psi} = (g + a)\phi$$

$$\text{or} \quad \frac{d}{dt}\begin{pmatrix}\phi \\ \psi\end{pmatrix} = \begin{pmatrix}0 & 1/l \\ g+a & 0\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix}$$

then if $\underline{X}(t) = \begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix}$

have $\dot{\underline{x}}(t) = A(t)\underline{x}(t)$ with $A(t) = \begin{pmatrix} 0 & 1/l \\ g+a(t) & 0 \end{pmatrix}$ ----- (I)

[to check if Hamiltonian, associate momentum with ψ and assume this is canonical $\Rightarrow \Omega \equiv \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and require $A^T\mathbb{J} = -\mathbb{J}A$

$$LHS = \begin{pmatrix} 0 & g+a \\ 1/l & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -g-a & 0 \\ 0 & 1/l \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/l \\ g+a & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/l \\ g+a & 0 \end{pmatrix}$$

$$= -\mathbb{J}A = RHS \text{ as required}$$

$\therefore$ system is Hamiltonian]

Consider the Floquet matrix of (I), which exists by linearity of the system. The Poincaré map is defined as the map which assigns to each vector $\underline{x}_0 \in \mathbb{R}^2$, the vector $\underline{x}$ obtained by applying (I) for one period T. The Floquet matrix is the matrix for the Poincaré map or "mapping at a Period" according to Arnold [4] — see later. If (I) has the solution $\underline{x}(t) = X(t,\mu)\underline{x}(0)$ (with μ a parameter of $a(t)$ and hence of the equation of motion) then it follows that M (the Floquet matrix) $= M(\mu) = X(T,\mu)$. [the construction of such a matrix can be achieved by following the evolution of the base vectors over a forcing period T. This is however unnecessary for what follows — all that is required is existence of the matrix] Now crucially the following result must be proved: — The pendulum equation (I) is stable iff its Floquet matrix M is stable. Stability of (I) is defined as existing iff every sol$^{\underline{n}}$ of (I) stays bounded for all t. A matrix is stable iff all of its integer powers are bounded. Now the proof: —

this is linear stability

trivially if eqn (I) is stable

$$|\underline{x}(t)| < C \quad \forall t$$

$$\Rightarrow \quad |X(t,\mu)| < \frac{C}{|\underline{x}(0)|} \quad \forall t$$

and in particular

$$|X(nT,\mu)| < \frac{C}{|\underline{x}(0)|} \quad \forall n \in \mathbb{N}$$

as $X(nT,\mu) = X(T,\mu)X(T,\mu) \ldots \underset{n^{th}}{X(T,\mu)}$

$$= M^n$$

$$\therefore \text{ have } \quad |M^n| < \frac{C}{|\underline{x}(0)|}$$

$\therefore$ floquet matrix is stable

Now consider Floquet matrix as stable

then $\forall n \in \mathbb{N} \quad |M^n| < C$

ie $\quad |X(nT,\mu)| < C$

now let $0 < \tau < T$ and consider $t = nT + \tau$

$$|X(t)| = |X(nT+\tau)| = |X(nT)X(\tau)|$$

$$= |X(nT)||X(\tau)|$$

$$< C|X(\tau)|$$

now $|X(\tau)|$ for $\tau \in (0,T)$ is obviously bounded by D say

$$\therefore \quad |X(t)| < CD \quad \forall t$$

$$\Rightarrow \quad |X(t)\underline{x}(0)| < CD|\underline{x}(0)|$$

$$\Rightarrow \quad |\underline{x}(t)| < CD|\underline{x}(0)|$$

$\Rightarrow$ stability of equation (I)

Hence have result

As previously proved, the system is Hamiltonian hence must have X as symplectic ie $X^t \mathbb{J} X = \mathbb{J}$ or equivalently $\det X = 1$
thus $M \in Sp(2)$ [$Sp(n)$ is the set of symplectic $n \times n$ matrices]
We must now examine the topology of $Sp(2)$.

Any matrix M with $\det M = 1$ admits a unique polar factorization $M = PQ$ where $P^T = P$ is +ve definite & $\det P = 1$ & $Q^T = Q^{-1}$ is a rotation matrix. This is proved by setting $P = (MM^T)^{1/2}$ where the +ve definite square root is taken.

i) check this is symmetric

$$P^T = [(M^T)^T M^T]^{1/2} = [MM^T]^{1/2} = P$$

ii) $(\det P)^2 = \det MM^T = \det M \det M^T = 1$

also P is +ve def. $\therefore \det P > 0 \Rightarrow \det P = 1$

$\therefore$ P is +ve def. symmetric & $\det = 1$

Now set $Q = P^{-1}M = (MM^T)^{-1/2}M$

$$Q^T = M^T(MM^T)^{-1/2}$$

look at
$$Q^TQ = M^T(MM^T)^{-1/2}(MM^T)^{-1/2}M$$
$$= M^T(MM^T)^{-1}\underbrace{MM^T(M^T)^{-1}}_{I}$$
$$= M^T(M^T)^{-1}$$
$$= I$$
$$\Rightarrow Q^T = Q^{-1}$$

$\det Q = 1$ trivially from $Q = P^{-1}M$ $\therefore$ Q is a rotation

hence have required factorization

Now Q has form $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ $\alpha \in \mathbb{R}$ thus set of all Q topologically is a circle. Consider P as having form $\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ with

i) $P^T = P$ gives $P_{12} = P_{21}$ ii) $\det P = 1$ gives $P_{11}P_{22} - P_{12}^2 = 1$ hence have two degrees of freedom. Without loss of generality can parametrise P by

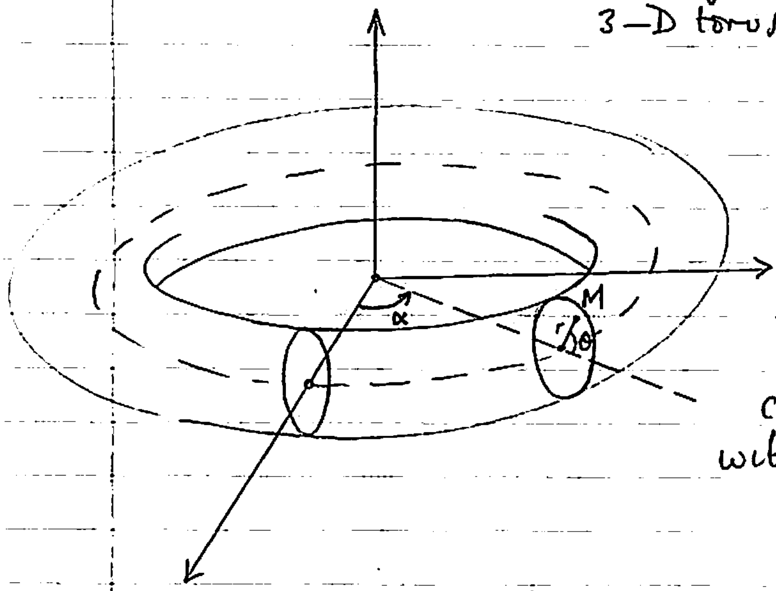
$$P = \begin{pmatrix} \cosh\gamma + \sinh\gamma\cos\beta & \sinh\gamma\sin\beta \\ \sinh\gamma\sin\beta & \cosh\gamma - \sinh\gamma\cos\beta \end{pmatrix} \quad \text{with } \gamma > 0$$

then if $r = \tanh^2\gamma$

and $\theta = \beta$ we have a natural isomorphism of the set of all P to the open unit disc. Now have

$$M = \begin{pmatrix} \cosh\gamma + \sinh\gamma\cos\beta & \sinh\gamma\sin\beta \\ \sinh\gamma\sin\beta & \cosh\gamma - \sinh\gamma\cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Thus $\exists$ a natural isomorphism of the set of all M, $Sp(2)$, to the open solid 3-D torus as shown



the matrix M as defined above is identified with the point (α, r, θ) in toroidal coords inside the torus.

Identifying $Sp(2)$ with the torus, one can think of $X(t,\mu)$ $0 \le t \le T$ as a curve within the torus.

Now consider how the set of stable 2x2 matrices appears within this torus. Mark Levi's argument hinges on proving that this set obstructs the torus — ie. $\exists$ an open unit disc (corresponding to only stable matrices) which cuts the torus in such a way that no traversal of the torus can avoid crossing the disc. To achieve this it is sufficient to look at the set of stable matrices $\Sigma' = \{PR,\ P^T = P > 0, \det P = 1 : R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ — a $\pi/2$ rotation$\}$. R is a Q matrix with $\alpha = \pi/2$. P can range over all possible matrices satisfying the appropriate conditions and hence Σ' is represented by the entire open disc at $\alpha = \pi/2$ which clearly obstructs the torus. Must now show that Σ' is a set of stable matrices:—

P is symmetric & $\therefore$ diagonalizable
let λ_1, λ_2 & $\underline{v_1}$ & $\underline{v_2}$ be eigenvalues & corresponding normalized eigenvectors. without loss of generality can assume $\underline{v_1}$ is $\perp$'r to $\underline{v_2}$. Therefore can take $R\underline{v_1} = \underline{v_2}$ & $R\underline{v_2} = -\underline{v_1}$

Now consider
$$
\begin{aligned}
(PR)^2 \underline{v_1} &= PRP\underline{v_2} \\
&= PR\lambda_2 \underline{v_2} \\
&= -\lambda_2 P \underline{v_1} \\
&= -\lambda_2 \lambda_1 \underline{v_1} \\
&= -(\det P)\underline{v_1} \\
&= -\underline{v_1}
\end{aligned}
$$

similarly $(PR)^2 \underline{v_2} = PRP\,{-\underline{v_1}}$

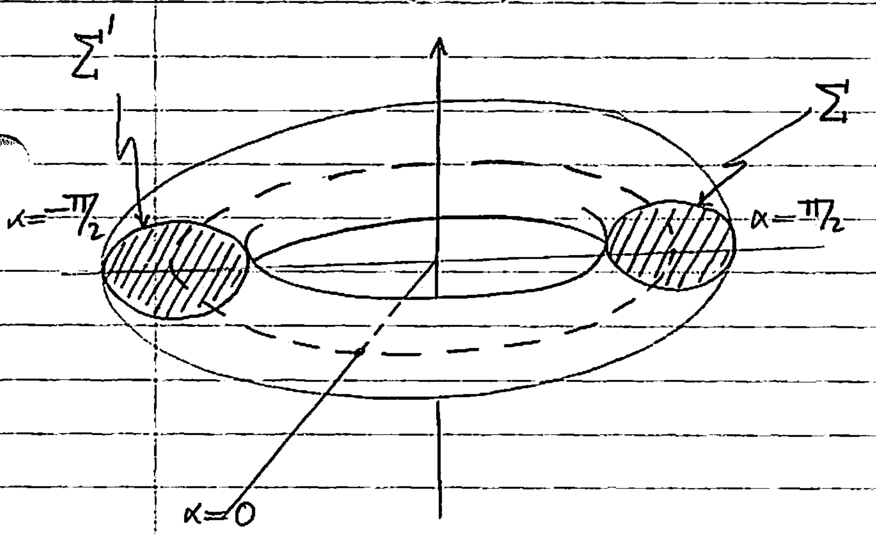
$$= PR(-\lambda_1 \underline{v_1})$$
$$= -\lambda_1 P(\underline{v_2})$$
$$= -\lambda_1\lambda_2 v_2$$
$$= -(\det P)\underline{v_2}$$
$$= -V_2$$

thus in $(\underline{v_1}, v_2)$ basis $(PR)^2 = -I$

<u>hence trivially stable as all powers of PR are bounded.</u>

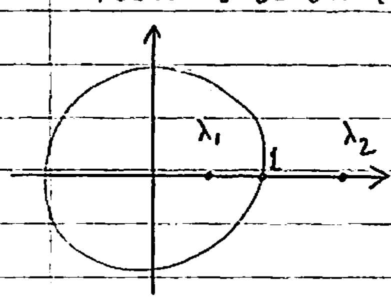
Notice there is also a similar open disc at $\alpha = -\pi/2$ representing

$$\Sigma_1^{'} = \left\{ PR', P^T = P > 0 \;\; \det P = 1 : R' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

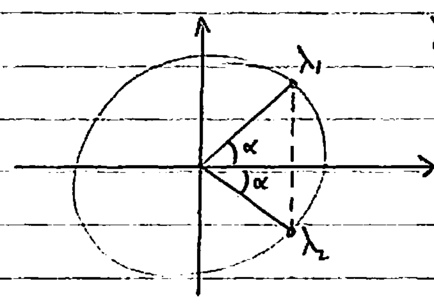


these two areas obviously obstruct the torus

Gelfand and Lidskii in their article on the stability of linear Hamiltonian systems [5] actually find the entire region of stable matrices in the torus. Stability requires both eigenvalues of M to have modulus = 1 which restricts the characteristic polynomial to have non-real roots since being complex conjugate and reciprocal in value necessitates both roots to be on the unit circle.



λ_1, λ_2 real
M corresponds to a hyperbolic rotation
$\therefore$ unstable



λ_1, λ_2 complex conjugates
M is a rotation through α
$\therefore$ stable

The characteristic polynomial is

$$X^2 - (\mathrm{tr}M)X + \underset{=1}{\det M} = 0$$

Clearly for non-real roots $(\mathrm{tr}M)^2 < 4$

ie $|\mathrm{tr}M| < 2$

$$M = \begin{pmatrix} \cosh\gamma + \sinh\gamma\cos\beta & \sinh\gamma\sin\beta \\ \sinh\gamma\sin\beta & \cosh\gamma - \sinh\gamma\cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\mathrm{tr}M = (\cosh\gamma + \cancel{\sinh\gamma\cos\beta})\cos\alpha + \sinh\gamma\,\cancel{\sin\beta}\sin\alpha$$
$$- \sinh\gamma\,\cancel{\sin\beta}\sin\alpha + (\cosh\gamma - \cancel{\sinh\gamma\cos\beta})\cos\alpha$$

$$= 2\cosh\gamma\cos\alpha.$$

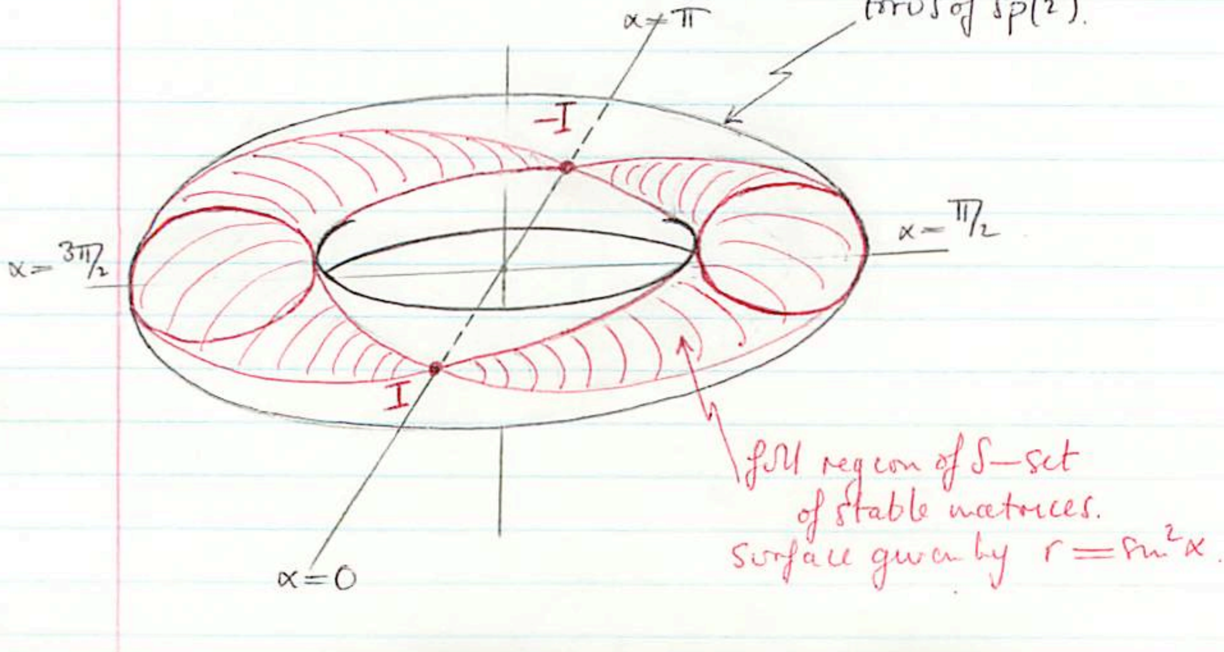
hence require $|\cos\alpha|\cosh\gamma < 1$

boundary given by $|\cos\alpha|\cosh\gamma = 1$

$$|\cos\alpha|^2 = \mathrm{sech}^2\gamma = 1 - r \qquad (r = \tanh^2\gamma)$$

$$\Rightarrow \underline{r = \sin^2\alpha}. \qquad (0 \leqslant \alpha \leqslant 2\pi).$$

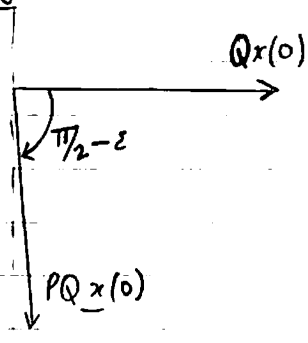
hence have situation as follows.



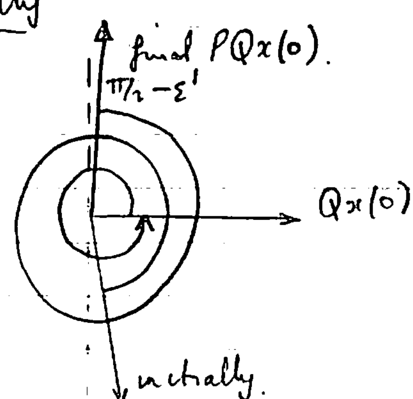
Now Mark Levi points out that if the system depends on some parameter μ in such a way that as μ changes from μ_1 to μ_2 the Floquet Matrix $M=M(\mu)$ traverses the torus, there exist at least two intermediary values of μ for which the system is stable (ie $M(\mu)$ lying in Σ' or Σ''). This follows immediately by realizing that $M(\mu)$ must cross Σ'_1 and Σ''_1. Obviously the next step is to physically interpret what a traversal of the torus by M actually represent. Mark Levi "roughly" states that one revolution of M corresponds to the change by one of the number of oscillations per forcing period. This, I believe, is not strictly true (—perhaps this is what "roughly" means...). To examine the situation it is sufficient to consider the parameter α of M only. This represent the angle of rotation affected by Q in the polar factorization and hence if M makes a revolution around the torus α changes by 2π. To ease the argument assume α increases by 2π without loss of generality. Recall that $x(T)=P(T)Q(T)x(0)$ hence if α increases by 2π then $\underline{x}(0)$ is rotated by a further 2π in the phase plane by $Q(T)$. Now remember P is +ve definite hence $PQ\underline{x}(0).Q\underline{x}(0)>0$ ie. angle between $PQx(0)$ & $Q\underline{x}(0)$ is always less than $\pi/2$. Thus if $Q(T)\underline{x}(0)$ rotates through an extra 2π, $PQ\underline{x}(0)$ will rotate through an extra γ with $\pi<\gamma<3\pi$. The two extreme cases:—

1) initially

PQx is initially $<\pi/2$ behind Qx and after rotation is $<\pi/2$ ahead.



finally

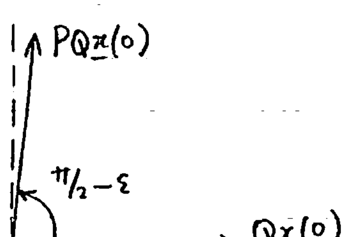


Here $\gamma = 3\pi - \varepsilon' - \varepsilon$

$\therefore \gamma < 3\pi$

11) initially

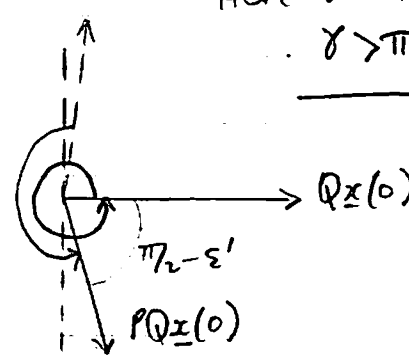
PQx is initially $<\pi/2$ ahead and finally $<\pi/2$ behind.



Here $\gamma = \pi + \varepsilon + \varepsilon'$

$\therefore \gamma > \pi$

finally

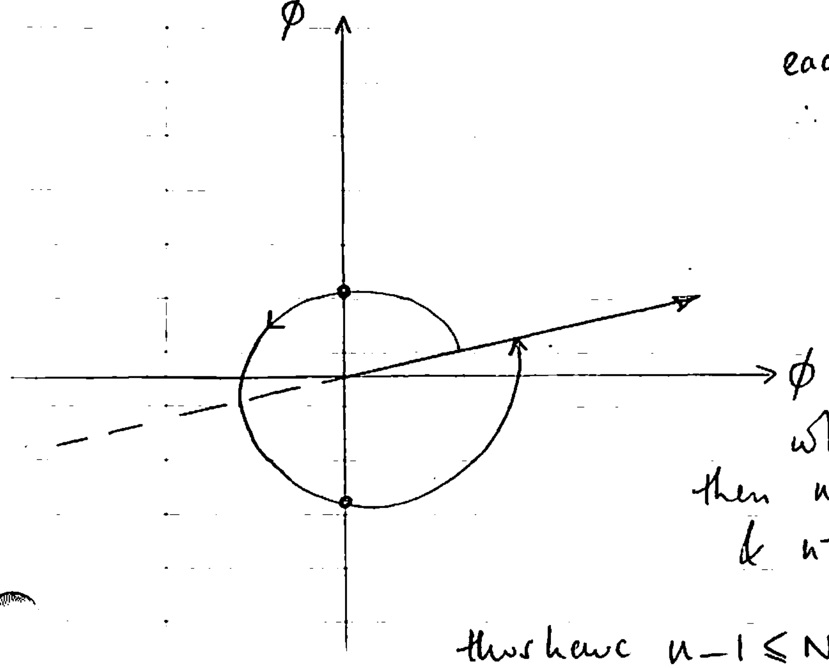


Certainly if $\underline{X}(T)$ rotates by at least 2π then we observe an additional oscillation, but $\underline{X}(T)$ need only rotate by an additional $\pi+\varepsilon$ with ε small — which is only just over a half an oscillation. Thus the number of oscillations increases by 0 or 1. Two revolutions about the torus will force $X(T)$ to rotate by at least 3π and at most 5π hence number of oscillations changes by either 1 or 2. Thus we can generalize that n revolutions will correspond to an increase in the number of oscillations by $n-1$ or n. However crucially can we reverse this relation? Does an increase by 1 in the number of oscillations necessarily imply that M has traversed the torus? we need $PQ\underline{x}(0)$ to have rotated by a further 2π for an extra oscillation but this only implies $Q\underline{x}(0)$ rotating by a further π to 3π. Hence a complete traversal of the torus is not guaranteed but we do have that $Q\underline{x}(0)$ rotates by at least π — i.e. α increases by at least π. This is sufficient to necessitate M passing through one of the stable obstructions and hence the existence of an intermediary stable value of μ. Although the answer to the question posed is no, we have perhaps the more relevant result that an increase by 1 in the number of oscillations (due to smooth parameter change) necessitates the existence of at least one intermediary stable value — this is Mark Levi's claim in his introduction.

The above gives a feeling for what a traversal of the torus by M represents, but what does the actual value of α — or angular position on the torus mean? It is now more convenient to consider the quantity N — the number of times the pendulum reaches the upward vertical (equilibrium position) in a forcing period. Mark Levi suggests that if $n = \left[\frac{|\alpha|}{\pi}\right]$ then $n-1 \leqslant N \leqslant n+3$ - - - - - - (*)

The number n represents the number of complete half turns $X(t)$ makes around the torus before reaching M. This corresponds to $Q(t)$ rotating $\underline{x}(0)$ by n complete half turns in the phase plane in one period. However we are interested in the action of $P(t)Q(t)$ on $\underline{x}(0)$. As before positive definiteness of P requires that $PQ\underline{x}(0)$ is always within $\pi/2$ of $Q(t)\underline{x}(0)$. If $Q(t)$ rotates $\underline{x}(0)$ by n complete turns we can have $Q(T)$ rotating $\underline{x}(0)$ by $n\pi+\theta$: $0 \leqslant \theta < \pi$ this allows $P(T)Q(T)$ to rotate $\underline{x}(0)$ by γ with $n\pi - \pi/2 < \gamma < n\pi + \pi + \pi/2$. Consider the phase

plane of $\underline{x}$:-
Note pendulum reaches vertical when $\phi = 0$
Consider effect of rotating an arbitary initial vector by γ

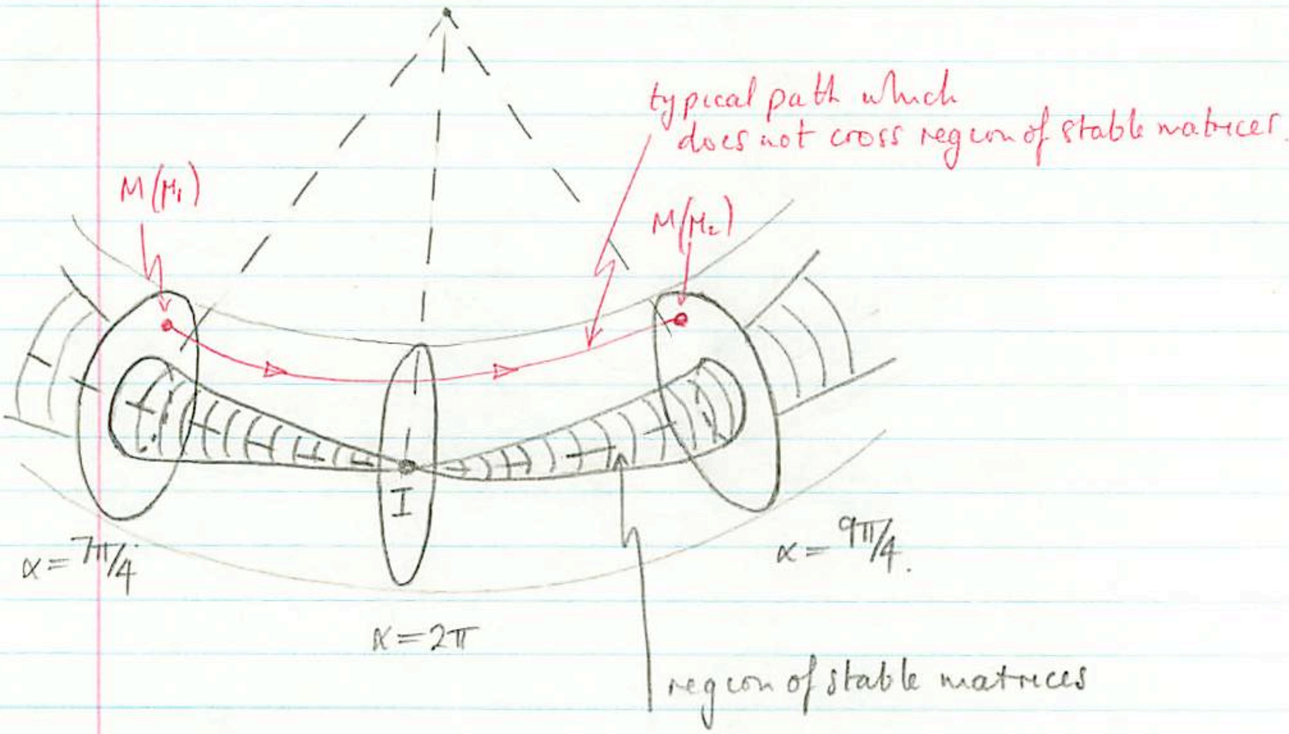


each half turn yields an equilibrium cross
∴ depending on starting vector
$n\pi - \pi/2$ turns can yield
$N = n-1$ or n
Similarly $(n+1)\pi + \pi/2$ can yield
$N = n+1$ or $n+2$
Note we also have the special case where the initial vector is such that $\phi = 0$
then $n\pi - \pi/2$ can yield $n+1 = N$
& $n\pi + 3\pi/2$ can yield $n+3 = N$

thus have $n-1 \leq N \leq n+3$

Thus we can loosely attach some physical meaning to the value of α.

The corollary of his theorem is perhaps the culmination of the paper :- "If for two distinct values of the parameter μ, the number of times the pendulum passes the top equilibrium during one period differ at least by 4, then for some intermediate μ the pendulum is stable". It is implicitly assumed that the system with $\mu = \mu_1$ & with $\mu = \mu_2$ do not necessarily have the same initial condition otherwise the corollary would reduce to the previous result. Mark Levi is suggesting that if $N_1(\mu_1)$ & $N_2(\mu_2)$ belong to different intervals (which they must if $|N_1 - N_2| > 4$) then as μ changes from μ_1 to μ_2 (or vice versa) M crosses Σ' or Σ'' at least once. Equivalently if n_1 & n_2 are different then α must pass through $\left(\frac{2m+1}{2}\right)\pi$ for some $m \in \mathbb{N}$ (i.e. the Σ' & Σ'' obstructions). However this does not necessarily follow. Consider μ_1 with $\alpha = 7\pi/4$, here $n_1 = [7/4] = 1$, and let μ_2 be such that $\alpha = 9\pi/4$ hence $[9/4] = 2 = n_2$. Thus we have intervals $0 \leq N_1 \leq 4$ & $1 \leq N_2 \leq 5$ and obviously can have $|N_1 - N_2| > 4$ but as μ changes from μ_1 to μ_2 α <u>need</u> only change smoothly and monotonically from $7\pi/4$ to $9\pi/4$ — hence not crossing an obstruction:—



The result will hold however if we require the N values for μ_1 & μ_2 to differ by at least 5 — this forces n_1 & n_2 to differ by at least 2 and thus requires α to change by at least π. Now M has to cross Σ_1 or Σ_1' and therefore a stable intermediary value of μ will exist.

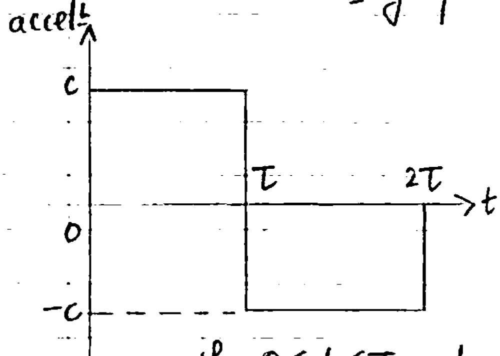
In the closing remark he attempts to qualify his treatment only of the linearized equation. Of course considering the linearized equation around $\phi = 0$ is fine only if small errors appearing due to this linearization do not build up over time. This condition essentially requires the point $\phi = 0$; $\dot{\phi} = 0$ to be a stable pt of the exact Poincaré map. It turns out in practice that KAM theory gives this stability except for, at most, a discrete set of μ. Hence it is general valid to consider only the linearized equations.

Thus we see that Mark Levi introduces some interesting and very new results to the well published subject of the inverted pendulum.

Appendix

Obviously Mark Levi's topological argument is specific to the system studied however his method of reducing system to a Poincaré map or "Map at a Period" to examine stability is an important and far reaching technique. Arnold [4] p121 provides an explicit calculation using this technique applied to the pendulum assuming a step fn form for the forcing acceleration.

we have acceln graph:-



equation of motion is $l\ddot{\phi} = (g \pm c)\phi$

Calculation of the Floquet matrix is in two stages corresponding to $0<t<T$ & $T<t<2T$

we must follow the image of $\begin{pmatrix}1\\0\end{pmatrix}$ ie $\phi=1, \dot{\phi}=0$ & $\begin{pmatrix}0\\1\end{pmatrix}$ ie $\phi=0$ & $\dot{\phi}=1$

for $0<t<T$ have $l\ddot{\phi} = (g+c)\phi$

let $\alpha = \sqrt{g+c}$ then solns are $\phi = Ae^{\alpha t} + Be^{-\alpha t}$

if initially $\begin{pmatrix}1\\0\end{pmatrix}$ then $\phi = \cosh \alpha t$

$\Rightarrow \dot{\phi} = \alpha \sinh \alpha t$

if initially $\begin{pmatrix}0\\1\end{pmatrix}$ then $\phi = \frac{1}{\alpha}\sinh \alpha t$

$\Rightarrow \dot{\phi} = \cosh \alpha t$

hence matrix for $0<t<T$ is $\begin{pmatrix}\cosh \alpha T & 1/\alpha \sinh \alpha T \\ \alpha \sinh \alpha T & \cosh \alpha T\end{pmatrix}$

for $T<t<2T$ have $l\ddot{\phi} = (g-c)\phi = -(c-g)\phi$ assume c is large enough

let $\beta = \sqrt{\frac{c-g}{l}}$

then $\phi = A\cos\beta t + B\sin\beta t$

we have $\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}\cos\beta T\\ -\beta\sin\beta T\end{pmatrix}$ for $T<t<2T$

& $\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}1/\beta \sin\beta t\\ \cos\beta t\end{pmatrix}$

$\therefore$ Matrix for $T<t<2T$ is $\begin{pmatrix}\cos\beta T & 1/\beta\sin\beta T\\ -\beta\sin\beta T & \cos\beta T\end{pmatrix}$

This gives Floquet matrix as

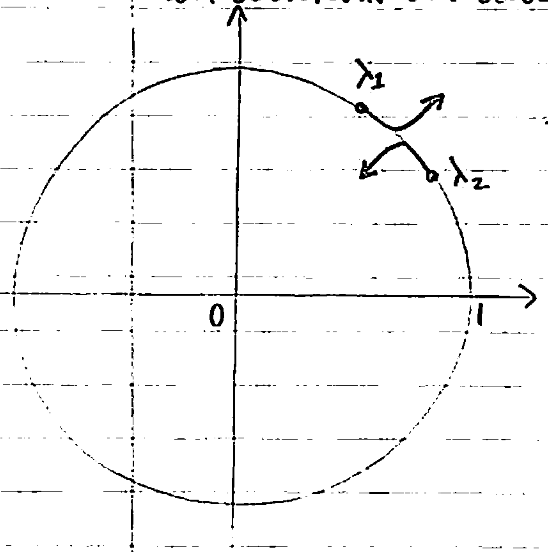
$$M = \begin{pmatrix} \cos\beta\tau & 1/\beta \sin\beta\tau \\ -\beta\sin\beta\tau & \cos\beta\tau \end{pmatrix} \begin{pmatrix} \cosh\alpha\tau & \frac{1}{\alpha}\sinh\alpha\tau \\ \alpha\sinh\alpha\tau & \cosh\alpha\tau \end{pmatrix}$$

we can now examine stability by looking at

$|\mathrm{tr} M| < 2$ condition

Arnold actually only proceeds to solve this for a special case where $c \gg g$ and amplitude of the forced oscillations $\ll l$ the length of the pendulum — see p121 for more. [He refers to the mapping represented by the Floquet Matrix as the mapping at a Period as opposed to Poincare map]

The inverted pendulum is a particular example of an oscillating system with periodically varying parameter (i.e. the pt of suspension is oscillating). As observed in the pendulum, such periodic parameter variation can cause previously unstable equilibrium points to become stable & vice versa. Arnold refers to the phenomenon as parametric resonance, and explains that it is dependent on the behaviour of the eigenvalues of the "mapping at a period" — or Poincare map. In the above pendulum example with one degree of freedom the analysis reduces to the behaviour of eigenvalues of symplectic transformations in the plane. However this analysis can be generalized to multi-dimensional systems with the crucial requirement that stability exists iff all eigenvalues of the Floquet matrix lie on the unit circle. To understand the result, perhaps the first thing to realize is that the mapping at a period obtained from a system of Hamilton's equations with periodic coefficients is symplectic. From this it follows that if λ is an eigenvalue of the Floquet matrix then so also is $1/\lambda$. Thus if one eigenvalue does not lie on the unit circle then $\exists$ at least one eigenvalue λ, $|\lambda| > 1$ this makes the matrix unstable and gives instability of the system. If we are considering a smooth parameter change in the system such that stability leads to instability, then it is seen that this transition is caused by the collision of eigenvalues on the unit circle. The situation is not simple however as not all collisions are dangerous. This is a subject for another paper...



a 'dangerous' collision of eigenvalues leading to instability

---

References

[1] Phelps F.M and J.H. Hunter — "An analytic solution of the inverted pendulum" Am J. Phys 33 (1965) p285

[2] N.W. McLachlen — "Theory & Applications of Mathieu functions"

[3] H. Kalmus — "The inverted Pendulum" Am J. Phys 38 (1970) p 874

[4] V. Arnold — "Mathematical Methods of classical Mechanics"

[5] Gelfand I.M. and Lidskii L.D. "On the structure of stability of linear Hamiltonian systems of differential equations with periodic coefficient" AMS Transl. (2) 8 (1958) pp143–181