MATH 275

THE LAGRANGE D'ALEMBERT EQUATIONS
AND
CONSTRAINTS

Contents:
Section 1: T(TQ) and T*(TQ)
Section 2: The equations of motion with known external forces
Section 3: The Verhulst approach to deriving the Lagrange d'Alembert equations for systems with constraints
Section 4: The traditional constrained variational approach to the Lagrange d'Alembert equations
Section 5: Justifying the Lagrange d'Alembert equations
Section 6: Concluding remarks

Samer - Very good work. Your insight into this problem is very good. A
Section One: \( T^{\text{v}} \) and \( T^{\text{h}} \)

Let \( T_{\mathbf{q}} : T\mathbf{q} \to \mathbf{q} \) be the projection.

Thus, \( T_{\mathbf{q}} : T(T\mathbf{q}) \to T\mathbf{q} \).

In local coordinates:
\[
T_{\mathbf{q}} : (\mathbb{R}^n \times \mathbb{R}^m) / (\mathbb{R}^n \times \{0\}) \to \mathbb{R}^n
\]
\[
((u,e) \mapsto (u, e_1)
\]

If \( u \in T\mathbf{q} \), \( v \in T^v(T\mathbf{q}) \) denote the tangent space to the fiber

through \( u \). \( T^v(T\mathbf{q}) \) is called the \textit{vertical} subspace of \( T\mathbf{q}(T\mathbf{q}) \).

\[
T^v(T\mathbf{q}) : \mathbb{R} \times T_{\mathbf{q}}(T\mathbf{q})
\]

In local coordinates, if \( u = (u, e) \), then
\[
T^v(T\mathbf{q}) : \{((u, e), (u, e_2)) \mid e_2 \in \mathbb{R}^m\}
\]

The \textit{vertical lift} \( \nu \mathbf{v} \) of \( \nu \) is a vector field on \( T\mathbf{q} \) with initial point \( u \) such that \( \nu(w, v) \) is the element of \( T^v(T\mathbf{q}) \) represented by the curve \( c(t) = v + tw \).

In local coordinates, if \( v = (u, e) \), \( w = x(t) \), then
\[
\nu_c(w, v) = (u, e)(0, f)
\]

We can define, for \( u \in T\mathbf{q} \), \( v \in T\mathbf{q} \), \( w \in T\mathbf{q} \):

\[
y : T\mathbf{q} \to T^v(T\mathbf{q})
\]
\[
w \mapsto \nu(v, v) \text{ or in local coordinates,}
\]
\[
f \mapsto (0, f)
\]

Note that \( y : T\mathbf{q} \to T^v(T\mathbf{q}) \) is an isomorphism.

A vector field \( \mathbf{v} \) on \( T\mathbf{q} \) is called \textit{vertical} if \( X(0) \) is vertical

\( \forall u \in T\mathbf{q} \), i.e., \( T\mathbf{q} \cdot X = 0 \).
Every 2-form corresponding to vertical vector fields is a horizontal 2-form.

Consider a $2$-form on the system $\dot{q} = F(q, \dot{q})$ writing it as a first order system:

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} V \\ F(q, v) \end{bmatrix}$$

The state space consists of positions and velocities. Thus we can naturally describe the system by a vector field on $TQ$, where $Q$ is the configuration space.

Add external forces: $\ddot{q} = F(q, \dot{q}) + \gamma_{\dot{q}}(q, \dot{q})$

$$\Rightarrow \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} V \\ F(q, v) \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma_{\dot{q}}(q, \dot{q}) \end{bmatrix}$$

Note that $\begin{bmatrix} \gamma_{\dot{q}}(q, \dot{q}) \end{bmatrix}$ can be regarded as a vertical tangent vector at $(q, \dot{q}) \in TQ$. Thus a vector field on $TQ$ will be called a force vector field if it is vertical.

The horizontal subspace $T^h_v(TQ)$ of $T^v_v(TQ)$ is defined as $(T^v_v(TQ))^\perp$.

In local coordinates: $T^h_v(TQ) = \{ (u, e), (x_1, 0) \mid v \in R^n \}$

A 1-form $\omega$ on $TQ$ is called horizontal, if $\omega(v)$ is horizontal for each $v \in TQ$.

Consider the map $\tau: T(TQ) \rightarrow T(TQ)$ defined at each $v \in TQ$ by $\tau_v = v_0 \cdot T\tau q$.

In local coordinates,

$$\tau_v (\xi, \eta) = (0, \xi)$$

Thus $\tau$ maps $T^v_v(TQ)$ onto $T^v_v(TQ)$.

Note that $\tau$ is a smooth vector bundle map.
Let $\mathbf{T}^* : T^* (\mathbb{R}^k) \longrightarrow T^* (\mathbb{R}^k)$ be defined at each $v \in T^* \mathbb{R}^k$ by $(\mathbf{T}^* v)_j = (\mathbf{T}_j)^*$. In local coordinates, $\mathbf{T}^* (\alpha_1, \alpha_2) = (\alpha_2, 0)$. Thus $\mathbf{T}_j^*$ maps $T_j^* (\mathbb{R}^k)$ onto $T_j^* (\mathbb{R}^k)$. $\mathbf{T}^*$ is a smooth vector bundle map. $\mathbf{T}^*$ is called the horizontal projection.
Let \( \mathcal{L} : TQ \to \mathbb{R} \) be a regular Lagrangian.
Let \( \xi \) be the corresponding (3rd order) Lagrangian vector field.
Let \( Y \) be the vertical vector field representing the external forces.

\[ X = \xi + Y \] is the induced vector field.

For there is a bijection between the collection of 1-forms on \( TQ \) and the tensor bundle 1-forms on \( TQ \) given by
\[ Y \mapsto \gamma = -\gamma_{\xi} \partial_{\xi} \] where \( \gamma_{\xi} \partial_{\xi} \) is the pull back by \( \xi \) of the standard symplectic structure on \( T^*Q \).

To this end, we calculate:
\[ \gamma(Y, \xi) = \gamma_{\xi} \partial_{\xi} \] then
\[ \omega_Y (\xi, \xi) = (D_2)_{\gamma} (\xi, \xi) \partial_{\xi} \]

This we can also realize with horizontal 1-forms.

A physical interpretation for thinking of forces as horizontal 1-forms:
In Cartesian coordinates.

\[ \text{Let } E \text{ be the energy, } E = A - L, \text{ where } A \to T^*Q \text{ is defined by } A(\gamma) = \Pi_L(\gamma) \cdot \gamma \] Then
\[ \frac{dE}{dt} |_{t=t_0} = \omega_Y (\gamma(t), \xi(t)) \]

Proof:
\[ \frac{dE}{dt} |_{t=t_0} = \frac{dE(v(t))}{dt} \cdot \dot{v}(t_0) \]
\[ = \frac{dE(v(t_0))}{dt} \cdot X(v(t_0)) \]
\[ = \frac{dE(v(t_0))}{dt} \cdot \left[ Z(v(t_0)) + Y(v(t_0)) \right] \]
\[ = \frac{dE(v(t_0))}{dt} \cdot Y(v(t_0)) \quad \text{(since } E \text{ is conserved along the flow of } \xi \quad \text{and}\quad \gamma) \]
\[ = (\mathcal{L}_v \partial_{\xi}) \cdot Y(v(t_0)) \]
\[ = -(\mathcal{L}_v \gamma_{\xi}) \cdot \xi(v(t_0)) \]

Now
\[ -\mathcal{L}_v \cdot \xi = -\mathcal{L}_v \gamma_{\xi} \cdot (X - Y) = -\mathcal{L}_v \gamma_{\xi} \cdot X = \omega_Y \cdot X \]
\[ \implies \frac{dE}{dt} |_{t=t_0} = \omega_Y (v(t_0)) \cdot \dot{v}(t_0) \quad \text{QED} \]
The local Lagrange d’Alembert Principle:

\[ X = \mathbb{Z} + Y \]

\[ \Rightarrow \mathbb{Z} \mathcal{L} = \mathbb{Z} \mathcal{L}_L + \mathbb{Z} \mathcal{L}_L \]

\[ \Rightarrow -i \gamma \mathcal{S}_L = \mathcal{E} - i \chi \mathcal{Z}_L \]

\[ \Rightarrow \left[ \frac{i \gamma}{\mathcal{E}} = \mathcal{E} - i \chi \mathcal{Z}_L \right] \text{ This equation is called the local Lagrange d’Alembert principle.} \]

So we now assume:

\[ Y = \text{a constant vector field, and } Z = \text{a 3rd order tensor. Then } X \text{ is also 3rd order. Thus} \]

\[ \mathbf{v} \left( q, \mathbf{v} \right) = \begin{bmatrix} X_2(\mathbf{q}, \mathbf{v}) \end{bmatrix} \]

\[ \Rightarrow \mathbf{v} \left( q, \mathbf{v} \right) = \begin{bmatrix} \dot{q}_1 \end{bmatrix} \]

Then, we get:

\[ -D_1(D_2(q, v) \cdot \mathbf{v}) - D_2(q, v) \cdot X_2(\mathbf{q}, \mathbf{v}) + D_1(\mathbf{q}, \mathbf{v}) = \dot{X}_2(\mathbf{q}, \mathbf{v}) = 0 \]

\[ -D_2(q, v) \cdot \mathbf{v} - \dot{\mathbf{v}} = 0 \]}

Now let \( (q, \mathbf{v}) \) \( \mathbf{v} \mathbf{v} = \) constant. Then \( \dot{X}_2 \)

\[ \Rightarrow \dot{q}_i = X_2(q, \dot{q}_i) \]

\[ \Rightarrow \text{ From (2.1)} \]

\[ -D_1(D_2(q, v) \cdot \mathbf{v}) = D_2(q, v) \cdot \mathbf{v} - D_2(q, \mathbf{v}) \cdot \dot{q}_2(\mathbf{q}, \mathbf{v}) + D_1(\mathbf{q}, \mathbf{v}) = 0 \]

\[ \Rightarrow \dot{q}_i = D_2(q, v) \cdot \mathbf{v} - D_1(q, \mathbf{v}) - \dot{X}_2(\mathbf{q}, \mathbf{v}) = 0 \]

\[ \Rightarrow \dot{q}_i = D_2(q, \mathbf{v}) \cdot \mathbf{v} - D_1(q, \mathbf{v}) - \dot{X}_2(q, \mathbf{v}) = 0 \]

Conversely, let \( (q, \dot{q}_i) \) \( \mathbf{v} \mathbf{v} = \) constant. Then from (2.1) we can conclude that \( \dot{q}_i = X_2(q, \dot{q}_i) \)

\[ \text{If we } (q, \dot{q}_i) \text{ is an integral curve, } X. \]
Equations (2.2) are called the Lagrange d'Alembert equations.

Thus we can state the following theorem:

**Theorem.** Let \( L \) be a singular Lagrangian and let \( Y \) be a virtual vector field. Let \( Y = Z + V \), where \( Z \) is the Lagrangian vector field. Then the following are equivalent:

1. \( (q, \dot{q}) \) is an integral curve \( X = Z + V \)
2. \( \frac{d}{dt} D_L(q, \dot{q}) - D_L(q, \dot{q}) = D_Z D_L(q, \dot{q}) \cdot Y \cdot (q, \dot{q}) \)

It will also be useful for us to note the following:

Suppose \( (q, \dot{q}) \) satisfies at time \( t \) the equation

\[
\frac{d}{dt} | D_L(q, \dot{q}, t) = D_L(q, \dot{q}, t) = D_Z D_L(q, \dot{q}, t) \cdot \dot{q}
\]

then

\[
\frac{d}{dt} \left[ \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \right] = Z(q(t), \dot{q}(t)) + \left[ \begin{bmatrix} 0 \\ \dot{q} \end{bmatrix} \right]
\]

Thus \( \dot{q} \) can be regarded as the vector column for \( q \).
The Vishik approach to deriving the Lagrange-d'Alembert equations in the presence of currents.

The contents of this section are taken from Vishik [2].

We shall assume that our system is constrained to move on a submanifold $S$ of $TQ$.

The constraint can thus be described by a codistribution $E$ defined in $\mathcal{C}$, $\tau = (T\psi S)^T \psi \nu \in S$

Thus $\psi (\tau) \in \mathcal{C}$ is annihilated by $\psi (\tau)$.

**Functional Constraints**: The submanifold $S$ is defined as the zero of the function

$$f = \left[ \begin{array}{c} f_1 \\ \vdots \\ f_k \end{array} \right] : TQ \to \mathbb{R}^k$$

$c \in \mathbb{R}^k$ is assumed to be a regular value of $f$.

If $c \in \mathbb{R}^k$, $(f_1, \ldots, f_k)$ is a linearly independent set by the nullity of $c \in \mathbb{R}^k$.

Thus $(f_1, \ldots, f_k)$ is a basis for $E(\nu)$.

**Examples of Functional Constraints**: 1) Linear Constraints. $S$ is a distribution on $G$.

$$(f_1, \ldots, f_k) = \sum_j b_j \psi (\nu) \partial_j, \quad i = 1, \ldots, k \text{ are linearly independent sections.}$$

Thus $E_i := d_f (q, \nu) = \left[ \frac{\partial f_i}{\partial q^j} \psi (q, \nu) \quad \frac{\partial f_i}{\partial \nu^j} \psi (q, \nu) \quad b_1 (q) \ldots b_k (q) \right]$

Note that $0 \in \mathbb{R}^k$ is indeed a regular value of $f = \left[ f_1 \right]$. 
1) All the vectors \( \mathbf{v} \in \mathbb{T}_Q^G \) are in affine subspaces \( \mathbb{T}_Q \mathbf{g} \), for each \( \mathbf{g} \in G \).

Thus \( f_i(i, \mathbf{v}) = \sum b_{ij}(\mathbf{q}) (v_j - X^{ij}(\mathbf{q})) \), \( i = 1, \ldots, k \).

where \( X \) is some vector field on \( \mathbb{Q} \).

\[ \mathbf{v} = df_i = \left( \frac{\partial f_i}{\partial q^1}, \ldots, \frac{\partial f_i}{\partial q^N}, b_1(\mathbf{q}), \ldots, b_m(\mathbf{q}) \right) \]

Note that \( 0 \in \mathbb{R}^k \) is again a regular value of \( f \), \( f = \left[ f_i \right] \).

Almost by definition:

A continuous distribution \( \mathcal{E} \) on \( S \subset \mathbb{Q} \) is said to be admissible
if \( \dim(t^*\mathcal{E}) = \dim \mathcal{E} \).

Note that \( t^* : (\mathbb{Q}, \mathbb{R}) \rightarrow (\mathbb{R}^N, \mathbb{R}) \) is linear and onto.

Thus \( \dim \mathcal{E} = \dim t^* \mathcal{E} \) iff \( \mathcal{E}^* = \mathbb{R}^N \). i.e., \( \mathcal{E} \) is \( \mathbb{R}^N \)-linear.

\( \mathcal{E} \) does not contain any finite-rank linear subspace.

Note that if \( \mathcal{E} \) is horizontal at \( v \) and \( w \in \mathbb{R} \cdot \mathbb{R}v \), then \( w \in \mathcal{E} \), \( (\mathbb{R} \cdot \mathbb{R}v + v^*) = \mathbb{R}v^* \) is nonzero for any

extended vector field \( v \).

Thus there will be no force vector field which will ensure that
the resultant satisfies the constraints. Thus it is reasonable to
require that our constraints be admissible.

Ideal Constraints:

A continuous distribution \( \mathcal{E} \) is said to be ideal if it contains the
vertical direction \( v \), which is defined by

\[ \mathbf{X}(v) = v(\mathbf{0}) \, . \]

Thus \( \mathcal{E} \) is ideal iff \( \mathbf{X}(\mathbf{v}) = 0 \), \( \mathbf{v} \in \mathbb{R} \).
So by our claim, \( x(q, y) = (0, 0) \).

It is easy to check that these constraints are ideal, but affine constraints are not.

**Definition:** Horizontal 1-forms taking values in \( \tau^* \mathfrak{g} \) are said to be constraint reaction 1-forms.

**Theorem 3.4:** If a constraint is ideal, then constraint reaction 1-forms do not exist on curves on \( \mathcal{T}A \) which are lifts of curves on \( \mathfrak{g} \).

(i.e., curves \( \gamma \) in form \( \gamma^* \).

**Proof:**

Let \( \omega \in \tau^* \mathfrak{g} \) be \( \omega = \tau^* \mathfrak{g} \) and \( \gamma^* \).

\[
\omega \cdot (\gamma^*) = \tau^* \mathfrak{g} \cdot (\gamma^*) = \hat{\omega} [\tau(\gamma^*)] \cdot \hat{\omega} (0, \gamma^*) = 0,
\]

for \( \omega \in \tau \) and \( \epsilon \) is ideal. \( \text{QED} \).

Let \( \mathcal{T}A = \mathcal{L} \) be a connection on a Lie group \( \mathcal{L} \).

We consider the constant reaction field \( \mathcal{E} \) (defined on \( \mathcal{L} \)) as the constraint codistribution. We want a constraint reaction field on \( \mathcal{T} \) such that if \( Y \) is the corresponding vector field and \( \tau = \mathcal{E} + Y \), then \( \mathcal{E}(x) = 0 \). In this situation, we have the following theorem:

**Theorem 3.2:** Let \( D_2 \mathcal{D}_2 Y(y) \) be positive definite \( \forall \) \((y, x) \in \mathcal{T} \mathcal{A} \) and let \( \mathcal{E} \) be an admissible constraint codistribution. Then there is a unique constraint reaction field \( \mathcal{E} \) on \( \mathcal{T} \) and a \( 2^\text{nd} \) order vector field \( X \in \mathcal{L} \) such that

\[
X \cdot \tau = \mathcal{S}_L(\omega) \quad \text{and} \quad \mathcal{E}(x) = 0.
\]

(\( \mathcal{N} \cdot \mathcal{E} - \mathcal{S}_L(\omega) \cdot Y \))
Proof: We want to find \( \mathbf{v} \in T^* \mathbf{z} : \)

\[ \mathbf{c}(\mathbf{z} - \mathbf{z}_L^*(\mathbf{w})) = \mathbf{0} \]

i.e., we want a \( \mathbf{v} \in \mathcal{E} : \mathbf{c}(\mathbf{z} - \mathbf{z}_L^*(\mathbf{T}^* \mathbf{f})) = \mathbf{0} \).

Let \( \mathcal{E} = \text{span}\{ \mathbf{e}_1, \ldots, \mathbf{e}_k \} \)

Let \( \mathbf{e}^i = [a_{il} \ldots a_{in} b_{il} \ldots b_{in}] \) in local coordinates.

Let \( \mathbf{f} = \sum_i \mathbf{e}^i = \mathbf{x}^T \begin{bmatrix} \mathbf{e}^1 \\ \vdots \\ \mathbf{e}^k \end{bmatrix} \)

\[ T^* \mathbf{f} = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{b}^1 \\ \vdots \\ \mathbf{b}^k \end{bmatrix} \quad \mathbf{c} \mathbf{j} = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \]

where \( \mathbf{j} = \begin{bmatrix} \mathbf{b}^1 \\ \mathbf{b}^k \\ \vdots \\ \mathbf{b}^k \end{bmatrix} = \begin{bmatrix} b_{11} \ldots b_{1n} \\ \vdots \\ b_{k1} \ldots b_{kn} \end{bmatrix} \)

Thus \( -\mathbf{c}^* (T^* \mathbf{f}) = \begin{bmatrix} 0 \\ (D_x D_x L)^{-1} \mathbf{x}^T \end{bmatrix} \).

We need to solve the equation:

\[ \mathbf{c}(\mathbf{z}_L^*(T^* \mathbf{f})) = -\mathbf{c}(\mathbf{z}_L) \]

i.e., \( \mathbf{c} (D_x D_x L)^{-1} \mathbf{x}^T = -\mathbf{c}(\mathbf{z}_L) \) \[-\mathbf{c}(\mathbf{z}_L) \] \((2.1)\)

Now \( D_x D_x L \) is positive definite. Thus \( (D_x D_x L)^{-1} \) is also positive definite.

Further, the admissibility of the constraints implies that \( \mathbf{c} \) has linearly independent rows.

Thus \( (D_x D_x L)^{-1} \mathbf{j} \) is nonsingular.

Hence there exists a unique solution to the equation \((2.1)\).

Hence \( \exists ! \ \mathbf{v} \in T^* \mathbf{z} \) s.t. \( \mathbf{c}(\mathbf{z} - \mathbf{z}_L^*(\mathbf{w})) = \mathbf{0} \). \( \boxed{\text{QED}} \)

Remark: This theorem gives us a way to uniquely break up the vector field \( \mathbf{z} \) into an external force field \( \mathbf{y} \) and a resultant vector field \( \mathbf{z} \) which satisfies the constraints. We have to assume that \( \mathbf{y} \) is such that \( \mathbf{y} \in T^* \mathbf{E} \).

In the linear case, this assumption guarantees that the external forces will do no work.
Now we shall obtain coordinate expressions for the vector field $X$.
Some care will be necessary while doing this, since $X$ is defined only on the submanifold $SCTB$.

We shall use the following generalization of the theorem in Section 2:
Let $Y$ be a vector field on $S$ and let $X = Z + Y$ be tangent to $S$. Then the following are equivalent:

(i) $(q, q)$ is an integral curve of $X$ (or $S$).
(ii) $dD_2L(q, q) - D_1L(q, q) = D_2D_2L(q, q) \cdot Y(q, q)$

and $(q, q) \in S$.

We shall now prove the following theorem:

Theorem 3.3: Let $X$ be the vector field on $S$ as obtained by Theorem 3.2. Then the following are equivalent:

(i) $(q, q)$ is an integral curve of $X$ on $S$.
(ii) $(q, q) \in S$ and there is a curve $x(t) \in \mathbb{R}^n$ s.t.

$$dD_2L(q, q) - D_1L(q, q) = \beta^T(q, q) x(t).$$

Proof

(i) $\Rightarrow$ (ii)

We have an external form $v(t) \wedge \gamma \Omega v = \omega_Y \in T^* S$ s.t.

$$X = Z + Y.$$

Thus we have

$$d \frac{d}{dt} D_2L(q, q) - D_1L(q, q) = D_2D_2L(q, q) \cdot Y(q, q)$$

Now $\omega_Y = (D_2D_2L(q, q) \cdot Y(q, q), 0) \in T^* S$.

Thus there is a curve $x(t)$ s.t.

$$d \frac{d}{dt} D_2L(q, q) - D_1L(q, q) = \beta^T(q, q) x(t).$$
Conversely, suppose \((q, p) \in \mathcal{S}\) and \(\frac{d}{dt} \mathbf{D}_2 L(q, p) = \mathbf{D}_1 L(q, p) = \mathbf{B}^{\mathcal{S}}(q, p) \times(t)\) for some curve \(X(t)\).

Choose an instant \(t_0\).

Then \(\frac{d}{dt} \mathbf{D}_2 L(q, p) - \mathbf{D}_1 L(q, p), \dot{q}(t_0) = \mathbf{B}^{\mathcal{S}}(q(q_{t_0}), p(q_{t_0})) \times(t_0) = \mathbf{B}^{\mathcal{S}}(q_{t_0}, p_{t_0}) \times(t_0) - \mathbf{B}^{\mathcal{S}}(q_{t_0}, p_{t_0}) \times(t_0) = 0\)

Define \(Y_2\) by \(\mathbf{D}_2 \mathbf{D}_2 L(q_{t_0}, p_{t_0}) \cdot Y_2 = \mathbf{B}^{\mathcal{S}}(q_{t_0}, p_{t_0}) \times(t_0)\)

Let \(Y = (0, x_2)\) be a vector at \((q(t_0), p(t_0))\).

Let \(\mathbf{X}(q_{t_0}, p_{t_0}) = \mathbf{X}(q(t_0), p(t_0)) + Y = (\dot{q}(t_0), \dot{x}_2(q(t_0), p(t_0)))\)

From this, equation \(3.2\) gives:
\[-\mathbf{D}_1 \mathbf{D}_2 L(q_{t_0}, p_{t_0}) \cdot \dot{q}(t_0) = \mathbf{D}_2 \mathbf{D}_2 L(q_{t_0}, p_{t_0}) \cdot x_2(q(t_0), p(t_0))\]
\[\mathbf{D}_1 L(q_{t_0}, p_{t_0}) \cdot \dot{q}(t_0) + \mathbf{D}_2 \mathbf{D}_2 L(q_{t_0}, p_{t_0}) \cdot \dot{x}_2(q(t_0), p(t_0)) = 0\]

By equation equation \(3.2\) and comparing with the above we get:
\[\dot{q}(t_0) = \dot{x}_2(q_{t_0}, p_{t_0})\]
\[\Rightarrow \frac{d}{dt} \mathbf{X}(q_{t_0}, p_{t_0}) = \mathbf{X}(q_{t_0}, p_{t_0})\]

So \((q, p) \in \mathcal{S} \Rightarrow \mathbf{X}\) is tangent to \(\mathcal{S}\).

But \(Y\) satisfies \(\mathbf{D}_2 \mathbf{D}_2 L(q_{t_0}, p_{t_0}) \cdot Y_2 = 0\).

Hence by the uniqueness in theorem 3.2 \(\mathbf{x}(q(t_0), p(t_0)) = \mathbf{X}(q_{t_0}, p_{t_0})\).

Now to show arbitrary
\[\Rightarrow (q, p)\) is an integral curve \(q, X\).

\[\text{Q.E.D.}\]

Note: The equations \(q, p) \in S \) and \(\frac{d}{dt} \mathbf{D}_2 L(q, p) - \mathbf{D}_1 L(q, p) = \mathbf{B}^{\mathcal{S}}(q, p) \times(t)\)

are called the Lagrange d'Alambert equations for a system with constraints.
SECTION 4: The traditional constrained Variational approach to the Lagrange d’Alembert equations.

The traditional approach can handle the cases of linear and affine constraints.

Let \( \Theta \) be the constraint manifold.

Let \( \mathbb{Q} = \{ q_1, q_2, \ldots, q_n \} \) be the space of all curves \( \{ q_0, q_1, \ldots, q_n, \ldots \} \).

We can define a map \( \mathfrak{E} : \mathbb{Q} \rightarrow \mathbb{R}^n \) by

\[
I(q) = \int L(q, \dot{q}) dt.
\]

\( \mathfrak{E} \) is a linear functional. Constraints are given by a distribution on \( \Theta \).

A curve \( q(t) \in \mathbb{Q} \) is a trajectory of the system if

\[
\frac{d}{dt}(q(t), \dot{q}(t)) = 0 \quad \text{and} \quad q_0 \text{ initial conditions}
\]

where \( \mathfrak{E} = 0 \) is the function defined above on the phase space \( \mathbb{Q} \), and \( \dot{q} \) is any variation which satisfies the constraints.

Thus, if the constraints are (locally) defined by the function

\[
f_i : \mathbb{Q} \rightarrow \mathbb{R}^1,
\]

then a curve \( q(\alpha) \) is a trajectory of the system if

\( q(\alpha) \) satisfies the constraints and

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \mathbf{b}(t) = 0 \quad \text{for some curve } x(t).
\]
Case 2. Affine constraints: The constraints are given by an affine distribution $\mathcal{Q}$.

A curve $\mathcal{q}(\tau)$ is a trajectory of the system $\mathcal{H}$ if

$$\dot{\mathcal{H}}(\mathcal{q}(\tau)) \cdot \mathcal{q} = 0 \quad \text{and} \quad \mathcal{q}(0) \text{ satisfies the constraints},$$

where $\delta \mathcal{q}$ is any variation $\delta \mathcal{q}$ that satisfies the linear constraints (i.e., $\delta \mathcal{q} \in T_{\mathcal{q}(0)} \mathcal{Q}$, which is the tangent space to the affine subspace specified by the constraints) and $\mathcal{I}$ is defined as in the previous case.

If the constraints are defined by $f_i(\mathcal{q}, \mathcal{v}) = \sum b_{ij} \mathcal{q}^j \mathcal{v}^j$, $i = 1, \ldots, k$, then $\mathcal{q}(\tau)$ is a trajectory of the system $\mathcal{H}$ if $\mathcal{q}(0)$ satisfies the constraints and

$$\frac{d}{d\tau} D_L(\mathcal{q}, \mathcal{v}) = D_L(\mathcal{q}, \mathcal{v}) = \mathcal{B}^T(\mathcal{q}) \mathcal{v}$$

for some time $\tau(\cdot)$.

It is easy to verify that these are exactly the same conditions as are obtained by Theorem 3.3.
SECTION 5: Justifying the Lagrange–d'Alembert approach.

We have seen in Section 3 that there is a unique way to decompose the Lagrangian vector field $\mathbf{L}$ into a resultent vector field $\mathbf{R}$ which satisfies the constraints and an external force vector field $\mathbf{F}$. To do this we have to assume that the forces do not work. Thus, it is natural to ask whether in the linear case, the conditions that the forces do not work and that the resultant vector field satisfies the constraints are sufficient to determine the forces of constraint. We shall show below that this is not the case.

We recall $\Omega(\mathbf{r} + \mathbf{y}) = 0$ (the constraint satisfaction) and $\mathbf{w}_y(q, \dot{q}) : \frac{d}{dt} \mathbf{Y} = 0$ (the non-work condition)

- the $\mathbf{B} : \text{span} \left\{ \mathbf{Y} \right\} = \text{span of the rows of } [A, B] \left( \text{see the example of linear constraints in } \S 5 \right)$

To be a constraint, $\mathbf{Y}(q, \dot{q}) = [\begin{bmatrix} \mathbf{y}_1(q, \dot{q}) \\ \mathbf{y}_2(q, \dot{q}) \end{bmatrix}]$

$V(q, \dot{q}) = [\begin{bmatrix} 0 \\ \mathbf{y}_2(q, \dot{q}) \end{bmatrix}]$. Also, $\mathbf{w}_y(q, \dot{q}) = [\begin{bmatrix} D_2 \mathbf{y}_2(q, \dot{q}) \\ \mathbf{y}_2(q, \dot{q}) \end{bmatrix}]$.

We thus have the equations:

$$[A, B] \begin{bmatrix} \mathbf{y}_1(q, \dot{q}) \\ \mathbf{y}_2(q, \dot{q}) \end{bmatrix} = -[A, B] \begin{bmatrix} 0 \\ \mathbf{y}_2(q, \dot{q}) \end{bmatrix}$$

i.e., $B \mathbf{y}_2(q, \dot{q}) = -(A \dot{q} + B \mathbf{y}_2)$ $-(5.1)$

and $(D_2 \mathbf{y}_2(q, \dot{q}), \dot{q}) \mathbf{y}_2(q, \dot{q}) = 0$ $-(5.2)$

If we impose the additional condition that $\mathbf{w}_y(\mathbf{r} + \mathbf{y}) = 0$ in $\mathbb{T}^* \mathcal{E}$, i.e.,

$(D_2 D_2 \mathbf{y}_2)$ is of the form $B^T \mathbf{x}$, i.e., $\mathbf{y}_2 = \mathbf{y}$ in the form $(D_2 \mathbf{y}_2)^{-1} B^T \mathbf{x}$,

then we obtain $B (D_2 D_2)^{-1} B^T \mathbf{x} = -(A \dot{q} + B \mathbf{y}_2)$.
The rows of $Q$ are linearly independent. Assuming $(D_2D_L)$ is positive definite, it has a unique solution $X$ and hence a unique constraint reaction force $Y$, as was verified in §3.

Let us now analyze the situation without this additional assumption.

Rewrite equations (5.1) and (5.2) as

$$B Y_2 = 0 \quad (5.3)$$

and

$$a^T y_2 = 0 \quad (5.4).$$

Here the rows of $B$ are linearly independent.

Equation (5.3) defines an affine subspace $A \subset R^k$ of dimension $(n-k)$.

Equation (5.4) defines a subspace $V_1$ of dimension $(n-1)$.

Let $V_2$ = column space of $(D_2^2)^{-1} B^T$.

Then dim $V_2 = k$.

Claim: $V_2 \subset V_1$:

$$V_2 \subset V_1 : \quad (D_2^2)^{-1} B^T \subset (D_2^3 D_2^2)^{-1} B^T x$$

$$= (D_2^3 D_2^2)^{-1} (B^T x) \cdot q$$

$$= (B^T x) \cdot q$$

But each column $q$ of $B^T$ is orthogonal to $q$ since

$$B(q) \cdot q = 0 \quad \text{as } (q, q) \text{ satisfies the constraints.}$$

Thus $[(D_2^2 D_2^2)^{-1} B^T x] \cdot q = 0 \quad \text{GEO}$
We have seen in section 3.1.3 that \( V_2 \) intersects \( A \) in exactly one point - call it \( v_0 \).

We shall calculate the dimension of the affine space \( A \cap V_1 \) and \( V \) be the linear space obtained by translating \( A \) by \(-v_0\).

Note that \( V_1 \) and \( V_2 \) are invariant under translation by \(-v_0\).

Thus \( \dim (V_1 \cap V) = \dim (V \cap V_1) \).

Now \( \dim V_2 = k \), \( \dim V = (n-k) \), and \( V_2 \cap V = \xi_{0,2} \) (since \( V_2 \cap A = \xi_{0,2} \)).

\[ \Rightarrow \mathbb{R}^n = V_2 \oplus V \]

Now \( V_1 \supset V_2 \).

Thus \( V_1 + V = \mathbb{R}^n \)

\[ \dim (V_1 \cap V) = \dim V_1 + \dim V - \dim (V_1 + V) = (n-1) + (n-k) - n = n - (k + 1) \]

Thus the requirement that the forces of constraint do no work and that the resultant vector field satisfies the constraint determines the forces of constraint only if

\[ n - (k + 1) = 0 \]

i.e. \( k = (n-1) \)

i.e. the dimension of the constraint distribution is 1.

If the dimension of the constraint distribution is greater than 1, we cannot determine the forces of constraint in this way.
Section 6: Concluding Remarks

So far, we have described the Lagrange d'Alambert equations and attempted to justify them using the principle 'forces of constraint do no work'. We have seen that this principle is not sufficient to justify the Lagrange d'Alambert approach. There is also another plausible approach, called the 'Variational approach', which has been suggested for obtaining equations of motion of constrained systems. In this approach, it is proposed that the trajectory of the system is the curve $q(t)$ which minimizes

$$\int L(q, \dot{q}) \, dt$$

on the space curves which satisfy the constraints. It turns out that this gives different trajectories from the ones obtained by the Lagrange d'Alambert principle.

Thus we see that although this problem is an old and fundamental one, it is still not clearly understood.
REFERENCES:

1) A. Bloch, P.S. Krishnaprasad, J.E. Marsden and R. Murray, 1994, 'Nonholonomic Mechanical Systems with Symmetry'.