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MATH 275

THE LAGRANGE D'ALEMBERT EQUATIONS AND CONSTRAINTS

Contents:

Sameer - Very good work. Your insight into the VF paper is very good. A

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SECTION ONE : $T(TQ)$ and $T^*(TQ)$

Let $\tau_Q : TQ \to Q$ be the projection
Thus $T\tau_Q : T(TQ) \to TQ$.
In local coordinates $T\tau_Q : (U \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow U \times \mathbb{R}^n$

$$((u,e),(e_1,e_2)) \longmapsto (u,e_1)$$

If $v \in TQ$, let $T^V_v(TQ)$ denote the tangent space to the fiber through v. $T^V_v(TQ)$ is called the <u>vertical</u> <u>subspace</u> of $T_v(TQ)$

$T^V_v(TQ) = \ker(T_v\tau_Q)$
In local coordinates, if $v = (u,e)$, then
$T^V_v(TQ) = \{((u,e),(0,e_2)) \mid e_2 \in \mathbb{R}^n\}$

<u>Vertical lift</u> : If $v, w \in T_qQ$, the <u>vertical</u> lift of w with respect to v, denoted $\mathrm{ver}(w,v)$ is the element of $T_v(TQ)$ represented by the curve $c(t) = v + tw$.
In local coordinates, if $v = (u,e)$, $w = (u,f)$, then
$\mathrm{ver}(w,v) = ((u,e),(0,f))$

We can define, for $v \in T_qQ$, a map

$$\gamma_v : T_qQ \to T_v(TQ)$$

$w \longmapsto \mathrm{ver}(w,v)$ or in local coordinates,

$$f \longmapsto (0,f).$$

Note that $\gamma_v : T_qQ \longrightarrow T^V_v(TQ)$ is an isomorphism

A vector field X on TQ is called <u>vertical</u> if $X(v)$ is vertical $\forall\, v \in TQ$, i.e., $T\tau_Q \cdot X = 0$.

External forces correspond to vertical vector fields :
Consider a 2nd order system $\ddot{q} = F(q,\dot{q})$
Writing it as a first order system :

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ F(q,v) \end{bmatrix}$$

The state space consists of positions and velocities. Thus we can naturally describe the system by a vector field on TQ, where Q is the configuration space.

Add external forces : $\ddot{q} = F(q,\dot{q}) + Y_2(q,\dot{q})$

$$\Rightarrow \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ F(q,\dot{q}) \end{bmatrix} + \begin{bmatrix} 0 \\ Y_2(q,\dot{q}) \end{bmatrix}$$

Note that $\begin{bmatrix} 0 \\ Y_2(q,\dot{q}) \end{bmatrix}$ can be regarded as a vertical tangent vector at $(q,\dot{q}) \in TQ$. Thus a vector field on TQ will be called a force vector field iff it is vertical.
The horizontal subspace $T_v^{*H}(TQ)$ of $T_v^*(TQ)$ is defined as $(T_v^V(TQ))^\perp$.

In local coordinates $T_v^{*H}(TQ) = \{((u,e),(\alpha_1,0)) \mid \alpha_1 \in \mathbb{R}^n\}$

A 1-form ω on TQ is called horizontal, if $\omega(v)$ is horizontal for each $v \in TQ$.

Consider the map $\tau : T(TQ) \to T(TQ)$ defined at each $v \in TQ$ by

$$\tau_v = \gamma_v \circ T\tau_Q$$

In local coordinates,

$$\tau_v(e_1, e_2) = (0, e_1).$$

Thus τ_v maps $T_v(TQ)$ onto $T_v^V(TQ)$

Note that τ is a smooth vector bundle map.

Let $\tau^* : T^*(TQ) \to T^*(TQ)$ be defined at each $v \in TQ$ by $(\tau^*)_v = (\tau_v)^*$

In local coordinates $\tau^*_v(\alpha_1, \alpha_2) = (\alpha_2, 0)$.

Thus τ^*_v maps $T^*_v(TQ)$ onto $T^{*H}_v(TQ)$.

τ^* is a smooth vector bundle map.

τ^* is called the <u>horizontalizer</u>.

SECTION 2: Writing the Lagrange d'Alembert equations when external forces are known

Let $L : TQ \to \mathbb{R}$ be a regular Lagrangian.
Let Z be the corresponding (2nd order) Lagrangian vector field.
Let Y be the vertical vector field representing the external forces.

$X = Z + Y$ is the resultant vector field.

<u>Fact</u> There is a bijection between the collection of vertical vector fields on TQ and horizontal 1-forms on TQ given by $Y \mapsto \omega_Y = -i_Y \Omega_L$, where Ω_L is the pull back by $\mathbb{F}L$ of the standard symplectic structure on T^*Q.
This fact is readily verified by a coordinate calculation: if $Y(u,e) = (0, Y_2(u,e))$, then $\omega_Y(u,e) = (D_2 D_2 L(u,e) \cdot Y_2, 0)$

Thus we can also identify external forces with horizontal 1-forms.

A physical interpretation for thinking of forces as horizontal 1-forms:
<u>Proposition</u>: Let E be the energy. ($E = A - L$, where $A : TQ \to \mathbb{R}$ is defined by $A(v) = \mathbb{F}L(v) \cdot v$.) Then $\left.\frac{dE}{dt}\right|_{t=t_0} = \omega_Y(\dot{v}(t_0))$.

<u>Proof</u>: $\left.\frac{dE}{dt}\right|_{t=t_0} = dE(v(t_0)) \cdot \dot{v}(t_0)$

$$= dE(v(t_0)) \cdot X(v(t_0))$$

$$= dE(v(t_0)) \cdot [Z(v(t_0)) + Y(v(t_0))]$$

$$= dE(v(t_0)) \cdot Y(v(t_0)) \quad \text{(Since } E \text{ is conserved along the flow of } Z\text{)}$$

$$= [(i_Z \Omega_L) \cdot Y](v(t_0))$$

$$= -[(i_Y \Omega_L) \cdot Z](v(t_0))$$

Now $-i_Y \Omega \cdot Z = -i_Y \Omega_L \cdot (X - Y) = -i_Y \Omega \cdot X = \omega_Y \cdot X$

$$\Rightarrow \left.\frac{dE}{dt}\right|_{t=t_0} = \omega_Y(v(t_0)) \cdot \dot{v}(t_0) \qquad \underline{QED}.$$

The local Lagrange d'Alembert Principle:

$$X = Z + Y$$
$$\Rightarrow i_X\Omega_L = i_Z\Omega_L + i_Y\Omega_L$$
$$\Rightarrow -i_Y\Omega_L = dE - i_X\Omega_L$$
$$\Rightarrow \boxed{\omega_Y = dE - i_X\Omega_L}$$ This equation is called the local Lagrange d'Alembert principle.

In local coordinates:
Y is a vertical vector field, and Z is 2nd order. Then X is also 2nd order. Thus
let $X(q,v) = \begin{bmatrix} v \\ X_2(q,v) \end{bmatrix}$.

Let $Y(q,v) = \begin{bmatrix} 0 \\ Y_2(q,v) \end{bmatrix}$

Writing $\omega_Y = dE - i_X\Omega_L$ out in coordinates we get:

$$-D_1(D_2L)(q,v)\cdot v - D_2D_2L(q,v)\cdot X_2(q,v) + D_1L(q,v) + D_2D_2L(q,v)\cdot Y_2(q,v) = 0 \qquad -(2.1)$$

Now let $(q,\dot q)$ be an integral curve of X

$\Rightarrow \ddot q = X_2(q,\dot q)$
$\Rightarrow$ From (2.1),
$$-D_1(D_2L)(q,\dot q)\cdot\dot q - D_2D_2L(q,\dot q)\cdot\ddot q + D_1L(q,\dot q) + D_2D_2L(q,\dot q)\cdot Y_2(q,\dot q) = 0$$

$$\Rightarrow \frac{d}{dt}D_2L(q,\dot q) - D_1L(q,\dot q) = D_2D_2L(q,\dot q)\cdot Y_2(q,\dot q) \qquad -(2.2)$$

Conversely, let $(q,\dot q)$ satisfy (2.2)
Then from (2.1) we can conclude that $\ddot q = X_2(q,\dot q)$
Thus $(q,\dot q)$ is an integral curve of X.

Equations (2.2) are called the Lagrange d'Alembert equations.

Thus we can state the following theorem:

Theorem. Let L be a regular Lagrangian and let Y be a vertical vector field. Let $X = Z + Y$, where Z is the Lagrangian vector field. Then the following are equivalent:

(i) $(q, \dot{q})$ is an integral curve of $X = Z + Y$

(ii) $\frac{d}{dt} D_2 L(q, \dot{q}) - D_1 L(q, \dot{q}) = D_2 D_2 L(q, \dot{q}) \cdot Y_2(q, \dot{q})$

It will also be of use to us to note the following:

Suppose $(q, \dot{q})$ satisfies at time t_0 the equation

$$\frac{d}{dt}\Big|_{t=t_0} D_2 L(q(t), \dot{q}(t)) - D_1 L(q(t_0), \dot{q}(t_0)) = D_2 D_2 L(q(t_0), \dot{q}(t_0)) \cdot x ,$$

then $\frac{d}{dt}\Big|_{t=t_0} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = Z(q(t_0), \dot{q}(t_0)) + \begin{bmatrix} 0 \\ x \end{bmatrix}$

Thus x can be regarded as the vector of external forces

SECTION 3 : The Vershik approach to deriving the Lagrange-d'Alembert equations in the presence of constraints.

The contents of this section are taken from Vershik [2].

We shall assume that our system is constrained to move on a submanifold S of TQ.

The constraints can then be described by a codistribution $\mathcal{E}$ defined on $S \subset TQ$: $\quad \mathcal{E}(v) = (T_v S)^{\perp} \; \forall \, v \in S$

Thus $v(t) \in S \iff \mathcal{E}$ annihilates $\dot{v}(t)$.

Functional Constraints :

The submanifold S is defined as the zeros of the function

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} : TQ \longrightarrow \mathbb{R}^k .$$

$0 \in \mathbb{R}^k$ is assumed to be a regular value of f.

If $v \in S$, $\{ df_1(v), \ldots, df_k(v) \}$ is a linearly independent set by regularity of $0 \in \mathbb{R}^k$.

Note that $T_v S = \ker(Tf)$.

Thus $\{ df_1(v), \ldots, df_k(v) \}$ is a basis for $\mathcal{E}(v)$.

Examples of Functional Constraints :

1) Linear Constraints. S is a distribution on Q.

$f_i(q,v) = \sum_j b_{ij}(q) v^j$, $i = 1, \ldots, k$ are linearly independent constraints.

Thus $\mathcal{E}_i := df_i(q,v) = \left[\frac{\partial f_i}{\partial q^1}(q,v) \cdots \frac{\partial f_i}{\partial q^n}(q,v) \;\; b_{i1}(q) \cdots b_{in}(q) \right]$

Note that $0 \in \mathbb{R}^k$ is indeed a regular value of $f = \begin{bmatrix} f_1 \\ f_k \end{bmatrix}$

2) Affine constraints : $S \cap T_qQ$ is an affine subspace of T_qQ for each $q \in Q$.

Thus $f_i(q,v) = \sum_j b_{ij}(q)(v^j - X^j(q))$, $i = 1, \ldots, k$.

where X is some vector field on Q.

$\theta_i := df_i = \left[\frac{\partial f_i}{\partial q^1} \ldots \frac{\partial f_i}{\partial q^n} \; b_{i1}(q) \cdots b_{in}(q) \right]$

Note that $0 \in \mathbb{R}^k$ is again a regular value of $f = \begin{bmatrix} f_1 \\ f_k \end{bmatrix}$.

Admissibility of constraints :

A constraint distribution θ on $S \subset TQ$ is said to be admissible if $\dim(\tau^*\theta) = \dim \theta$

Note that $\tau^*(\alpha_1, \alpha_2) = (\alpha_2, v)$

$\tau^* : \theta \to \tau^*\theta$ is linear and onto

Thus $\dim \theta = \dim \tau^*\theta$ iff τ^* is 1-1 on θ, i.e., $\theta \to \tau^*\theta$, i.e., iff $\theta \cap \ker \tau^* = \{0\}$, i.e., θ does not contain any horizontal covectors.

Note that if $\omega \in \theta$ is horizontal at v, and $\omega(v) \cdot Z(v) \neq 0$, then $\omega(v) \cdot (Z(v) + Y(v)) = \omega(v) Z(v)$ is nonzero for any external force field Y.

Thus there will be no force vector field which will ensure that the resultant satisfies the constraints. Thus it is reasonable to require that our constraints be admissible.

Ideal Constraints :

A constraint codistribution θ is said to be ideal if it annihilates the fundamental vector field on TQ, which is defined by $\tilde{X}(v) = vl(v,v)$.

Thus θ is ideal if $\theta(v) \cdot \tilde{X}(v) = 0 \quad \forall v \in S$.

In this case we write, $X(q,v) = (v,\dot v)$
It is easy to check that linear constraints are ideal, but affine constraints are not.

Definition: Horizontal 1-forms taking values in $\tau^*\Theta$ are said to be constraint reaction 1-forms

Theorem 3.1: If a constraint is ideal, then constraint reaction 1-forms do no work on curves in TQ which are lifts of curves on Q. (i.e., curves of the form $(q,\dot q)$.)

Proof
Let $w \in \tau^*\Theta$. Then $\exists\ \hat\omega \in \Theta$ s.t. $\tau^*\hat\omega = \omega$

$\frac{d}{dt}(q,\dot q) = (\dot q, \ddot q)$

$$\omega\cdot(\dot q, \ddot q) = \tau^*\hat\omega\,(\dot q,\ddot q) = \hat\omega\left[\tau(\dot q,\ddot q)\right] = \hat\omega\,(0,\dot q) = 0,$$

Since $\hat\omega \in \Theta$ and Θ is ideal. QED.

Let $L: TQ \to \mathbb{R}$ be a regular Lagrangian, Z the Lagrangian v.f., $S \subset TQ$ the constraint submanifold, Θ (defined on S) the constraint codistribution. We want a constraint reaction force $\omega \in \tau^*\Theta$ such that if Y is the corresponding vector field and $X = Z + Y$, then $\Theta(X) = 0$. In this situation we have the following theorem:

Theorem 3.2: Let $D_2D_2L(q,v)$ be positive definite $\forall\ (q,v) \in TQ$ and let Θ be an admissible constraint codistribution. Then $\exists$ a unique constraint reaction force $\omega \in \tau^*\Theta$ on S and a 2nd order vector field X on S s.t.

$$X = Z - \Omega_L^{\#}(\omega), \quad \Theta(X) = 0.$$

(Note $-\Omega_L^{\#}(\omega_Y) = Y$)

<u>Proof</u> We want to find $\omega \in \tau^* \mathcal{G}$ s.t
$$\theta(Z - \Omega_L^{\#}(\omega)) = 0$$
i.e, we want a $f \in \mathcal{G}$ s.t $\theta(Z - \Omega_L^{\#} \tau^* f) = 0$.
Let $\mathcal{G} = \text{span}\{\theta_1, \ldots, \theta_k\}$

Let $\theta^i = [a_{i1} \ldots a_{in} \; b_{i1} \ldots b_{in}]$ in local coordinates.
Let $f = \sum_i x_i \theta^i = x^T \begin{bmatrix} \theta^1 \\ \vdots \\ \theta^k \end{bmatrix}$

$$\Rightarrow \tau^* f = \left[x^T \begin{bmatrix} b^1 \\ b^k \end{bmatrix} \quad 0 \right] = [x^T B \quad 0]$$
where $B = \begin{bmatrix} b^1 \\ b^k \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ b_{k1} & \cdots & b_{kn} \end{bmatrix}$

Thus $-\Omega_L^{\#}(\tau^* f) = \begin{bmatrix} 0 \\ (D_2 D_2 L)^{-1} B^T x \end{bmatrix}$.

We need to solve the equation:

$$\theta(-\Omega_L^{\#} \tau^* f) = -\theta(Z)$$
$$\text{i.e.,} \quad \theta (D_2 D_2 L)^{-1} B^T x = -\theta(Z) \qquad - (3\cdot1)$$
Now $D_2 D_2 L$ is positive definite. Thus $(D_2 D_2 L)^{-1}$ is also positive definite.
Further, the admissibility of the constraints implies that B has linearly independent rows.
Thus $B (D_2 D_2 L)^{-1} B^T$ is nonsingular.
Hence there exists a unique solution to the equation (3·1).
Hence $\exists !$ $\omega \in \tau^* \mathcal{G}$ s.t. $\theta(Z - \Omega_L^{\#}(\omega)) = 0$

<u>QED</u>.

<u>Remark</u>: This theorem gives us a way to uniquely break up the vector field Z into an external force vector field Y and a resultant vector field X which satisfies the constraints.
We have to assume that Y is such that $\omega_Y \in \tau^* \mathcal{G}$.
In the linear case this assumption guarantees that the external forces will do no work.

Now we shall obtain coordinate expressions for the vector field X. Some care will be necessary while doing this, since X is defined only on the submanifold $S \subset TQ$.

We shall use the following generalization of the theorem in section 2:

Let Y be a vertical vector field on S and let $X = Z + Y$ be tangent to S. Then the following are equivalent

(i) $(q, \dot{q})$ is an integral curve of X (on S.)

(ii) $\frac{d}{dt} D_2 L(q, \dot{q}) - D_1 L(q, \dot{q}) = D_2 D_2 L(q, \dot{q}) \cdot Y_2(q, \dot{q})$
and $(q, \dot{q}) \in S$.

We shall now prove the following theorem:

Theorem 3.3: Consider the vector field X on S as obtained by Theorem 3.2. Then the following are equivalent:

(i) $(q, \dot{q})$ is an integral curve of X on S.

(ii) $(q, \dot{q}) \in S$ and $\exists$ a curve $x(t) \in \mathbb{R}^n$ s.t.

$$\frac{d}{dt} D_2 L(q, \dot{q}) - D_1 L(q, \dot{q}) = B^T(q, \dot{q})\, x(t).$$

Proof

(i) => (ii)

We have an external force v.f. Y s.t. $-i_Y \Omega_L = \omega_Y \in \tau^* \Theta$ s.t.
$X = Z + Y$

Thus we have

$$\frac{d}{dt} D_2 L(q, \dot{q}) - D_1 L(q, \dot{q}) = D_2 D_2 L(q, \dot{q}) \cdot Y_2(q, \dot{q})$$

Now $\omega_Y = (D_2 D_2 L(q, \dot{q}) \cdot Y_2(q, \dot{q}), 0) \in \tau^* \Theta$.

Thus $\exists$ a curve $x(t)$ s.t

$$\frac{d}{dt} D_2 L(q, \dot{q}) - D_1 L(q, \dot{q}) = B^T(q, \dot{q})\, x(t).$$

Conversely, suppose $(q,\dot q) \in S$ and

$$\frac{d}{dt} D_2L(q,\dot q) - D_1L(q,\dot q) = B^T(q,\dot q)\,x(t) \quad \text{for some}$$

curve $x(t)$.

Choose an instant t_0

Then $$\left.\frac{d}{dt}\right|_{t=t_0} D_2L(q,\dot q) - D_1L(q(t_0),\dot q(t_0)) = B^T(q(t_0),\dot q(t_0))\,x(t_0) \quad -(3\cdot 2)$$

Define Y_2 by $$D_2D_2L(q(t_0),\dot q(t_0))\cdot Y_2 = B^T(q(t_0),\dot q(t_0))\,x(t_0)$$

Let $Y = (0, Y_2)$ be a vector at $(q(t_0),\dot q(t_0))$

Let $\tilde X(q(t_0),\dot q(t_0)) = Z(q(t_0),\dot q(t_0)) + Y = (\dot q(t_0), \tilde X_2(q(t_0),\dot q(t_0)))$

From this equation we get:

$$-D_1(D_2L)(q(t_0),\dot q(t_0))\cdot \dot q(t_0) - D_2D_2L(q(t_0),\dot q(t_0))\cdot \tilde X_2(q(t_0),\dot q(t_0))$$
$$- D_1L(q(t_0),\dot q(t_0)) + D_2D_2L(q(t_0),\dot q(t_0))\cdot Y_2 = 0$$

By expanding equation $(3\cdot 2)$ and comparing with the above we get:

$$\ddot q(t_0) = \tilde X_2(q(t_0),\dot q(t_0))$$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=t_0}(q(t_0),\dot q(t_0)) = \tilde X(q(t_0),\dot q(t_0))$$

But $(q,\dot q)\in S \Rightarrow \tilde X$ is tangent to S.

But Y satisfies $-i_Y\Omega_L = D_2D_2L(q(t_0),\dot q(t_0))\cdot Y_2 \in \tau^*\Theta$

Hence by the uniqueness in theorem 3.2, $\tilde X(q(t_0),\dot q(t_0)) = X(q(t_0),\dot q(t_0))$.

Now t_0 was arbitrary

$\Rightarrow (q,\dot q)$ is an integral curve of X.

QED.

Note. The equations $(q,\dot q)\in S$ and

$$\frac{d}{dt} D_2L(q,\dot q) - D_1L(q,\dot q) = B^T(q,\dot q)\,x(t)$$

are called the Lagrange d'Alembert equations for a system with constraints.

SECTION 4. : The traditional constrained Variational approach to the Lagrange d'Alembert equations.

The traditional approach can handle the cases of linear and affine constraints.

Let Q be the configuration manifold

Let $\Omega(q_1, q_2, [a,b])$ be the space of all curves $q : [a,b] \to Q$, $q(a) = q_1$, $q(b) = q_2$.

We can define a map $I : \Omega \to \mathbb{R}$ by

$$I(q) = \int_a^b L(q, \dot{q})\,dt$$

Case 1 – linear constraints : Constraints are given by a distribution on Q.

A curve $q_0 : [a,b] \to Q$ is a trajectory of the system iff

$$dI(q_0)\cdot \delta q = 0 \quad \text{and } q_0(t) \text{ satisfies the constraints}$$

where $I : \Omega \to \mathbb{R}$ is the function defined above on the path space Ω containing q, and δq is <u>any</u> variation which satisfies the constraints.

Thus if the constraints are (locally) defined by the functions $f_i = \sum_j b_{ij}(q) v^j$, then a curve $q(t)$ is a trajectory of the system iff

$q(t)$ satisfies the constraints and

$\frac{d}{dt} D_2 L(q, \dot{q}) - D_1 L(q, \dot{q})$ is a linear combination of the rows of

$$B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ b_{k1} & \dots & b_{kn} \end{bmatrix}$$

i.e., iff

$q(t)$ satisfies the constraints and $\frac{d}{dt} D_2 L(q, \dot{q}) - D_1 L(q, \dot{q}) = B^T(q)\, x(t)$ for some curve $x(t)$.

Case 2 : Affine Constraints : The constraints are given by an affine distribution on Q.

A curve $q_0(t)$ is a trajectory of the system iff

$dI(q_0)\cdot\delta q = 0$ **and** q_0 satisfies the constraints,

where δq is any variation of q_0 that satisfies the linear constraints (i.e. $\delta q(t) \in$ the linear subspace of $T_{q(t)}Q$ which is the translation of the affine subspace specified by the constraints.) and I is defined as in the previous case.

If the constraints are defined by $f_i(q,v) = \sum b_{ij}(q)v^j$, $i = 1,\ldots,k$, then $q(t)$ is a trajectory of the system iff $q(t)$ satisfies the constraints and

$$\frac{d}{dt} D_2L(q,\dot q) - D_1L(q,\dot q) = B^T(q)\,x(t) \quad \text{for}$$

some curve $x(t)$.

It is easy to verify that these are exactly the same conditions as are obtained by Theorem 3.3.

SECTION 5: Justifying the Lagrange d'Alembert approach:

We have seen in Section 3 that there is a unique way to decompose the Lagrangian vector field Z into a resultant vector field X which satisfies the constraints and an external force vector field Y. To do this we have to assume ω_Y takes values in the codistribution $\tau^* E$. In the linear case this assures us that the forces of constraints do no work. Thus it is natural to ask whether in the linear case, the conditions that the forces of constraint do no work and that the resultant vector field satisfies the constraints are sufficient to determine the forces of constraint. We shall show below that this is not the case:

We need $\Theta(Z + Y) = 0$ (X satisfies the constraints)
and $\omega_Y(q,\dot q)\cdot \frac{d}{dt}\begin{bmatrix} q \\ \dot q\end{bmatrix} = 0$ (the no-work condition)

Let $\Theta = \operatorname{span}\begin{bmatrix}\theta_1 \\ \vdots \\ \theta_k\end{bmatrix}$ = span of the rows of $[A \quad B]$ (See the example of linear constraints in §3)

In local coordinates, $Z(q,\dot q) = \begin{bmatrix}\dot q \\ Z_2(q,\dot q)\end{bmatrix}$

$Y(q,\dot q) = \begin{bmatrix} 0 \\ Y_2(q,\dot q)\end{bmatrix}$. Also, $\omega_Y(q,\dot q) = [D_2D_2L(q,\dot q)\, Y_2, \ 0]$.

We thus have the equations:

$$[A \quad B]\begin{bmatrix}\dot q \\ Z_2(q,\dot q)\end{bmatrix} = -[A \quad B]\begin{bmatrix}0 \\ Y_2(q,\dot q)\end{bmatrix}$$

i.e. $\quad B\, Y_2(q,\dot q) = -(A\dot q + B Z_2) \qquad -(5\text{-}1)$

and $\quad (D_2D_2L(q,\dot q)\cdot \dot q)\cdot Y_2(q,\dot q) = 0 \qquad -(5\text{-}2)$

If we impose the additional condition that $\omega_Y \in \tau^* E$, i.e.

$(D_2D_2L)\,Y_2$ is of the form $B^T x$, i.e., Y_2 is of the form $(D_2D_2L)^{-1}B^T x$, then we obtain $B(D_2D_2L)^{-1}B^T x = -(A\dot q + B Z_2)$

The rows of B are linearly independent. Assuming $(D_2 D_2 L)$ is positive definite, $\exists$ a unique solution x and hence a unique constraint reaction force Y, as was verified in §3.

Let us now analyze the situation without this additional assumption. Rewrite equations (5·1) and (5·2) as

$$B Y_2 = c \qquad -(5\cdot 3)$$

and

$$d^T Y_2 = 0 \qquad -(5\cdot 4).$$

Here the rows of B are linearly independent.

Equation (5·3) defines an affine subspace $\mathcal{A}$ of $\mathbb{R}^n$ of dimension $(n-k)$.
Equation (5·4) defines a subspace subspace V_1 of dimension $(n-1)$.

Let V_2 = column space of $(D_2 D_2 L)^{-1} B^T$.
Then $\dim V_2 = k$.

<u>Claim</u> : $V_2 \subset V_1$:
<u>Proof</u> $(D_2 D_2 L)\dot{q} \cdot ((D_2 D_2 L)^{-1} B^T x)$

$$= (D_2 D_2 L)[(D_2 D_2 L)^{-1} B^T x] \cdot \dot{q}$$

$$= (B^T x)\cdot(\dot{q})$$

But each column of B^T is orthogonal to $\dot{q}$ since $B(q)\cdot\dot{q} = 0$ as $(q,\dot{q})$ satisfies the constraints.

Thus $[(D_2 D_2 L)\dot{q}] \cdot [(D_2 D_2 L)^{-1} B^T x] = 0$ <u>QED</u>

We have seen in section three that V_2 intersects $\mathcal{A}$ in exactly one point – call it v.
We shall calculate the dimension of the affine space $\mathcal{A} \cap V_1$:
Let V be the linear space obtained by translating $\mathcal{A}$ by $(-v)$.
Note that V_1 and V_2 are invariant under translation by $(-v)$
Thus $\dim(\mathcal{A} \cap V_1) = \dim(V \cap V_1)$.

Now $\dim V_2 = k \quad \dim V = (n-k)$
and $V_2 \cap V = \{0\} \quad (\text{since } V_2 \cap \mathcal{A} = \{v\})$

$\Rightarrow \mathbb{R}^n = V_2 \oplus V$
Now $V_1 \supset V_2$
Thus $V_1 + V = \mathbb{R}^n$

$$
\begin{aligned}
\dim(V_1 \cap V) &= \dim V_1 + \dim V - \dim(V_1 + V) \\
&= (n-1) + (n-k) - n \\
&= n - (k+1)
\end{aligned}
$$

Thus the requirements that the forces of constraint do no work and that the resultant vector field satisfies the constraint determines the forces of constraint only if

$$n-(k+1) = 0.$$

i.e. $k = (n-1)$
i.e. the dimension of the constraint distribution is 1.

If the dimension of the constraint distribution is greater than 1, we cannot determine the forces of constraint in this way

SECTION 6 : Concluding Remarks

So far, we have described the Lagrange d'Alembert equations and attempted to justify them using the principle 'forces of constraint do no work'. We have seen that this principle is not sufficient to justify the Lagrange d'Alembert approach. There is also another plausible approach, called the 'Vakonomic approach' which has been suggested for obtaining equations of motion of constrained systems. In this approach it is proposed that the trajectory of the system is the curve $q(t)$ which minimize

$$\int L(q, \dot{q})\, dt$$

on the space of curves which satisfy the constraints. It turns out that this gives different trajectories from the ones obtained by the Lagrange d'Alembert principle.

Thus we see that although this problem is an old and fundamental one, it is still not clearly understood.

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