Math 189 Project
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HAMILTONIAN SYSTEMS IN FLUID-MECHANICS

CONTENTS

1. INTRODUCTION / PREFACE

My goal - which I did not fully achieve:

When I got an overview over this project, my goal was to do the three cases in chapters 2, 3 and 4. I wanted to understand (to some extent at least) the setup of the problems, and then derive the Euler equations from the Poisson-bracket and Hamiltonian. And then if time permitted perhaps "compare and contrast" the three cases.

What I have done:

I feel I do understand the setup of the three cases. I also did some "comparing and contrasting". I did not however succeed on deriving all the Euler equations...

My problem concerns two integrals $\int v \cdot (v \cdot \nabla) V \, dx$ and $\int v \cdot (V \cdot \nabla) v \, dx$. I was not able to transform them in a suitable way. Because of this I was not able to complete the derivation of the Euler equation in either case. But I have done all the other details and I did derive the mass-conservation equations on case 2 and 3, and the isentropy condition on case 3. I also did most of the calculations for the Euler equations - even though I could not complete them. In the first case (the incompressible case) I "proved" that the Euler and continuum equations can be written as
$\frac{\partial v}{\partial t} + \mathbb{P}\{(v \cdot \nabla) v\} = 0$ (see chap. 2).

<u>And how does it feel?</u>

This was a most interesting project, and it felt very rewarding when I was able to derive the continuums equations. It felt equally dissapointing that I should not be able to derive the Euler equations. So there are still things about this project I do not understand - maybe if I had a couple of more days ... But that is useless thinking - the deadline is NOW.

But, I did learn a lot - even some new "mathematical" technique such as some new integration by parts formulas and the Helmholtz decomposition theorem (which I also was exposed to in my 219-project).

<u>About the report:</u>

This is written in much more haste than the 219-report. It not so conciensciously built up, and not so consistent. And the handwriting is not so nice. Another thing I would like to mention is the appendix. They are not a part of the report, and you don't have to read them - but they contain some of my attempts of doing the above mentioned integr. The calculations there are WRONG And I don't know why!

2. Incompressible Fluid in $\mathbb{R}^2$

Let Ω be a domain in $\mathbb{R}$ with smooth boundary $\partial\Omega$. Now we assume Ω is filled with an incompressible and homogeneous fluid. Let $v(x,t)$ denote the spatial velocity. The equations of motion are the Euler-equations:

$$(1)\quad \begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p \\ \nabla\cdot u = 0 \end{cases}$$

We also assume that $u\cdot n|_{\partial\Omega} = 0$.

Let us define P to be the space in which v lives. Then $P = \{u \mid u \text{ smooth, domain}(u) = \Omega,\ \nabla\cdot u = 0,\ u\cdot n|_{\partial\Omega} = 0\}$ where smooth means for instance $u \in (H^2(\Omega))^2$ where $H^2(\Omega)$ is the classical Sobolev space of "degree 2". This is a minimum choice corresponding to [2], but it seems to me the $(H^1(\Omega))^2$ should be sufficient, since there are no second order partial derivative terms in (1).

Now I'm going to state the Hamiltonian and the Poisson-bracket, and then try to derive (1.). The Hamiltonian and the Poisson bracket are:

$$(2)\qquad H(v) = \frac{1}{2}\int_\Omega \|v\|^2\, d^2x$$

$$(3)\qquad \{F,G\}(v) = -\int_\Omega v\cdot\left[\frac{\delta F}{\delta v}, \frac{\delta G}{\delta v}\right] d^2x$$

where $v, \frac{\delta F}{\delta v}, \frac{\delta G}{\delta v} \in P$ and $F, G : P \to \mathbb{R}$.

Let me explain what (3) means. First, $\frac{\delta F}{\delta v}$ is the functional-derivative of F with respect to v, by definition:

(4) $$DF(v)\cdot\delta v = \frac{d}{d\varepsilon}F(v+\varepsilon\cdot\delta v)\Big|_{\varepsilon=0} = \int_\Omega \frac{\delta F}{\delta v}\cdot\delta v\, d^2x$$

where $\int_\Omega \frac{\delta F}{\delta v}\cdot\delta v\, d^2x = \langle \frac{\delta F}{\delta v}, \delta v\rangle$ is the pairing between the space $S = \{u \in (L^2(\Omega))^2 \mid \nabla\cdot u = 0,\ u\cdot n|_{\partial\Omega} = 0\}$ and its dual S^*. Note that P is a subspace of S. Also note that S and S^* are isomorphic by Riez representation theorem, so we can identify S and S^* using the above pairing (innerproduct in this case).

(5) $$[u,v] = (u\cdot\nabla)v - (v\cdot\nabla)u \qquad u,v\in P$$

This is the Jacobi-Lie bracket. So $(P,[\cdot,\cdot])$ is a Lie-algebra.

We know that Hamilton equations are $\dot{F} = \{F,H\}$. Let us use this and (2) and (3) to derive (1).

<u>Alternativ form of (1):</u>

First we need to show that (1) can be put in another form, without the ∇p-term. Let $\mathbb{P}: (L^2)^2 \to S$ be the (L^2) projection onto S. S has an orthogonal supplement in $(L^2)^2$ call it O, where $O = \{\nabla p \mid p\in H^1(\Omega)\}$. Let us check that these two spaces are orthogonal:

$$\langle u,\nabla p\rangle = \int_\Omega u\cdot\nabla p\, d^2x = \int_{\partial\Omega} p\, \underbrace{u\cdot n}_{0}\, dS - \int_\Omega p(\underbrace{\nabla\cdot u}_{0})\, d^2x = 0$$

So we get $\mathbb{P}u = u$, $\mathbb{P}\nabla p = 0$ and applying $\mathbb{P}$ to (1) gives:

(6) $$\frac{\partial u}{\partial t} + \mathbb{P}(u\cdot\nabla)u = 0$$

This is an equation for u, we now need an equation for p. Assume u is a given field that solves (7), then we can find p by solving the following equation, apply $(1-\mathbb{P})$ to (1):

$$(7) \qquad (1-\mathbb{P})(v\cdot\nabla)v = \nabla p$$

since $(1-\mathbb{P})v = 0$, $(1-\mathbb{P})\nabla p = \nabla p$. Now take the divergence of (8) and also multiply (8) with n (the outer normal-vector of $\partial\Omega$):

$$(8) \quad \begin{cases} \nabla^2 p = \nabla\cdot[(v\cdot\nabla)v] \\ \frac{\partial p}{\partial n}\Big|_{\partial\Omega} = [(v\cdot\nabla)v]\cdot n\,\Big|_{\partial\Omega} \end{cases}$$

since $\nabla\cdot((v\cdot\nabla)v) = \nabla\cdot(\mathbb{P}(v\cdot\nabla)v) + \nabla\cdot(1-\mathbb{P})(v\cdot\nabla)v = \nabla\cdot(1-\mathbb{P})(v\cdot\nabla)v$ and $(v\cdot\nabla)v\cdot n = (\mathbb{P}(v\cdot\nabla)v)\cdot n + (1-\mathbb{P})(v\cdot\nabla)v\cdot n = ((1-\mathbb{P})(v\cdot\nabla)v)\cdot n$. It can be shown that (8) has a unique solution for $v \in P$. This means that p can be uniquely determined when the solution of (7) is known. In this way (7) is equivalent to (6). More details in [3].

We now want to use (2) and (3) to [illegible]

<u>[illegible] of (6):</u>

Let us first calculate $\frac{\delta H}{\delta v}$:

$$\frac{d}{d\varepsilon} H(v+\varepsilon\delta v)\Big|_{\varepsilon=0} = \int v\,\delta v\, d^2x = \int \frac{\delta H}{\delta v}\,\delta v\, d^2x$$

so $\frac{\delta H}{\delta v} = v$. Now we calculate $\dot{F} = \{F, H\}$

$$\{F,H\}(v) = -\int v\,\left[\frac{\delta F}{\delta v}, \frac{\delta H}{\delta v}\right] d^2x = -\int v\cdot\left[\frac{\delta F}{\delta v}, v\right] d^2x$$

$$= -\int \left\{ v\cdot\left(\left(\frac{\delta F}{\delta v}\cdot\nabla\right)v\right) - v\cdot\left((v\cdot\nabla)\frac{\delta F}{\delta v}\right)\right\} d^2x$$

$$= -\int\left\{\frac{\delta F}{\delta v}\cdot\nabla\frac{\|v\|^2}{2} + \frac{\delta F}{\delta v}\cdot(v\cdot\nabla)v\right\} d^2x$$

The last equality comes from [2], and I have not been able to redo it. Now we note that $\frac{\delta F}{\delta v} \in P$, and this means $\nabla q \cdot \frac{\delta F}{\delta v} = 0 \quad \forall q \in H^1(\Omega)$. So we get:

$$\{F, H\}(v) = -\int \frac{\delta F}{\delta v} \cdot (v \cdot \nabla) v \, d^2x$$

$$= -\int \frac{\delta F}{\delta v} \cdot \mathbb{P}\{(v \cdot \nabla) v\} \, d^2x$$

This follows from the fact that $\frac{\delta F}{\delta v} \cdot (1-\mathbb{P})\{(v \cdot \nabla) v\} = 0$

Let $F = \int v \cdot V \, d^2x$, $V \in P$, $\dot{V} = 0$, V otherwise arbitrary. Then it is easy to see that $\frac{\delta F}{\delta v} = V$.

$$\dot{F} = \frac{d}{dt} \int v \cdot V \, d^2x = \int \frac{\partial v}{\partial t} \cdot V \, d^2x$$

$$= \{F, H\} = -\int V \cdot \mathbb{P}\{(v \cdot \nabla) v\} \, d^2x$$

So we have $\int \left(\frac{\partial v}{\partial t} + \mathbb{P}\{(v \cdot \nabla) v\} \right) \cdot V \, d^2x = 0$, and since V is arbitrary:

$$\frac{\partial v}{\partial t} + \mathbb{P}\{(v \cdot \nabla) v\} = 0.$$

Since this is an equivalent form of (1) we are now done.

3. BAROSTROPIC FLOW IN $\mathbb{R}^2$

Let Ω be a domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. A flow in Ω is barostropic if there exists a function h, called enthalpy, such that

$$(1) \qquad \nabla h = \frac{1}{\rho}\nabla p$$

Note: In [3] this flow is called isentropic.

Let us briefly state the thermodynamics needed in this case. The following quantities appear as functions of x and t:

p : pressure
ρ : density
T : temperature
s : entropy
h : enthalpy (per unit mass)
$\varepsilon = h - \frac{p}{\rho}$: internal energy (per unit mass)

The first law of thermodynamics:

$$(2) \qquad dh = T\,ds + \frac{1}{\rho}\,dp$$

or equivalently:

$$(3) \qquad d\varepsilon = T\,ds + \frac{p}{\rho^2}\,d\rho$$

We will assume that p is a function of ρ only. By (2) $dh = \frac{1}{\rho}\,dp$, so (1) is satisfied.

The Euler-equations for this case are:

$$(4) \quad \begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla h(\rho) \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \end{cases}$$

$$(5) \qquad v \cdot n \big|_{\partial\Omega} = 0$$

where (5) are boundary conditions for (4). Let P be the space of v and ρ that are smooth enough, and tend to a fixed vectorfield and density at infinity if Ω is unbounded. Furthermore P is "nice" in all ways demanded in [2].

<u>Comment:</u> In [2] it says: "... space of v and ρ that are C^1 (say H^s, $s > 2$) and ...". This confuses me, to me it's OK to choose $v \in (H^1(\Omega))^2$ and $\rho \in H^2(\Omega)$ - and I really don't understand what the parentheses is meant to say. The other properties that makes P "nice" seem to me to be related to the method discussed in [2], and may thus not be relevant to this "discussion".

The configuration space of compressible fluid motion is the group of diffeomorphisms of Ω whose Lie algebra consists of the space $\mathfrak{X}(\Omega)$ of all vectorfields on Ω

<u>Remark:</u> From [1] we have that for compressible flow, the configuration space is $\mathrm{Diff}_{vol}(\Omega)$. That is, the group of volumepreserving diffeomorphisms. The corresponding Lie algebra being that of divergencefree vectorfields on Ω.

$\mathfrak{X}(\Omega)$ is represented on the vector space $\mathcal{F}(\Omega)$ of functions on Ω by the minus Lie-derivative:

$$\mathcal{L}_X f = X[f] = -df(X)$$

for $X \in \mathfrak{X}(D)$ and $f \in \mathcal{F}(\Omega)$. On the dual of the semidirect product $\mathfrak{X}(\Omega) \circledS \mathcal{F}(\Omega)$ with variables $M = \rho v$ and ρ, the equations (4) are Hamiltonian with the following Hamiltonian and Lie-Poisson bracket:

$$(6) \quad H(M, \rho) = \int_\Omega \left(\frac{1}{2\rho} \|M\|^2 + \mathcal{E}(\rho) \right) d^2x$$

$$(7) \quad \{F, G\} = \int_\Omega M \cdot \left[\frac{\delta G}{\delta M}, \frac{\delta F}{\delta M} \right] d^2x + \int_\Omega \rho \left\{ \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} \right\} d^2x$$

where $[\, , \,]$ is the Jacobi-Lie bracket, and $\frac{\delta G}{\delta M}, \frac{\delta G}{\delta \rho}$ the functional derivatives - both defined in chapter 2. Here $\mathcal{E}(\rho)$ is internal energy per unit area, satisfying $\mathcal{E}'(\rho) = w(\rho)$.

<u>Note:</u> $\frac{\delta G}{\delta M}$ is an element in the space of the M's and $\frac{\delta G}{\delta \rho}$ is an element in the space of the ρ's, i.e. $\mathfrak{X}(\Omega)$ and $\mathcal{F}(\Omega)$ respectively. I guess you need smoothness conditions on $\mathfrak{X}$ and $\mathcal{F}$ to make the spaces of M and ρ.

Using (6) and [illegible] $\dot{F} = \{F, \cdot\}$ let us try to derive equations (4).

<u>Functional derivatives:</u>

Let us now find $\frac{\delta H}{\delta M}$ and $\frac{\delta H}{\delta \rho}$.

$$\frac{d}{d\varepsilon} H(M + \varepsilon \delta M, \rho) \Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_\Omega \left(\frac{1}{2\rho} \|M + \varepsilon \delta M\|^2 + \mathcal{E}(\rho) \right) d^2x \Big|_{\varepsilon=0}$$

$$= \int_\Omega \frac{1}{\rho} M \, \delta M \, d^2x = \int_\Omega \frac{\delta H}{\delta M} \delta M \, d^2x$$

So $\frac{\delta H}{\delta M} = \frac{1}{\rho} M = v$.

$$\frac{d}{d\varepsilon} H(M, \rho + \varepsilon\delta\rho)\Big|_{\varepsilon=0} = \int_\Omega \left[\frac{\|M\|^2}{2(\rho+\varepsilon\delta\rho)} + \mathcal{E}(\rho + \varepsilon\delta\rho)\right] d^2x \Big|_{\varepsilon=0}$$

$$= \int\left(-\frac{\|M\|^2}{2\rho^2} + \mathcal{E}'(\rho)\right)\delta\rho\, d^2x = \int\left(-\frac{\|M\|^2}{2\rho} + h(\rho)\right)\delta\rho\, d^2x = \int \frac{\delta H}{\delta\rho}\delta\rho\, d^2x$$

So $\frac{\delta H}{\delta\rho} = -\frac{\|M\|^2}{2\rho^2} + h(\rho) = h(\rho) - \frac{1}{2}\|v\|^2$.

<u>Derivations</u>:

Let us now define the following quantities:

$$\mathcal{M} = \int_\Omega M \cdot V\, d^3x \qquad \text{and} \qquad R = \int \rho \cdot f\, d^3x$$

where $V \in \mathfrak{X}(\Omega)$ and $f \in \mathcal{F}(\Omega)$, $\dot{V} = 0 = \dot{f}$, V, f otherwise arbitrary. Let us now calculate the functional derivatives of $\mathcal{M}$ and R.

$$\frac{d}{d\varepsilon}\mathcal{M}(M + \varepsilon\delta M)\Big|_{\varepsilon=0} = \int_\Omega V \cdot \delta M\, d^2x = \int_\Omega \frac{\delta\mathcal{M}}{\delta M}\delta M\, d^2x$$

so $\frac{\delta\mathcal{M}}{\delta M} = V$, Likewise it is easy to see that:

$$\frac{\delta\mathcal{M}}{\delta\rho} = 0, \quad \frac{\delta R}{\delta M} = 0 \quad \text{and} \quad \frac{\delta R}{\delta\rho} = f$$

Let us now calculate $\dot{R} = \{R, H\}$:

$$\{R, H\} = 0 + \int_\Omega \rho \cdot \left\{\frac{\delta H}{\delta M}\cdot\nabla\frac{\delta R}{\delta\rho} - \frac{\delta R}{\delta M}\cdot\nabla\frac{\delta H}{\delta\rho}\right\} d^2x$$

$$= \int_\Omega \rho\, v\cdot(\nabla f)\, d^2x = -\int_\Omega \nabla\cdot(\rho v) f\, d^2x + \int_{\partial\Omega} f\rho\, \underbrace{v\cdot n}_{0}\, dS.$$

$$= -\int_\Omega \nabla\cdot(\rho v) f\, d^2x$$

$$\dot{R} = \frac{d}{dt}\int_\Omega \rho f\, d^2x = \int_\Omega \frac{\partial\rho}{\partial t}\cdot f\, d^2x = \{R, H\} = -\int_\Omega \nabla\cdot(\rho v) f\, d^2x$$

$$\Rightarrow \quad \int_\Omega \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v)\right) f\, d^2x = 0$$

Since f is arbitrary we have:

$$(8) \qquad \underline{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0}$$

Let us now calculate $\dot{\mathcal{M}} = \{\mathcal{M}, H\}$

$$(9) \quad \{\mathcal{M}, H\} = \int_\Omega M\cdot\left[\frac{\delta H}{\delta M}, \frac{\delta \mathcal{M}}{\delta M}\right] d^2x + 0 - \int_\Omega \rho\, \frac{\delta \mathcal{M}}{\delta M}\cdot\nabla\frac{\delta H}{\delta M}\, d^2x$$

$$= \int_\Omega M\cdot\left(\left(\frac{\delta H}{\delta M}\cdot\nabla\right)\frac{\delta \mathcal{M}}{\delta M} - \left(\frac{\delta \mathcal{M}}{\delta M}\cdot\nabla\right)\frac{\delta H}{\delta M}\right) d^2x - \int_\Omega \rho\, V\cdot\nabla\frac{\delta H}{\delta M}$$

$$= \int_\Omega \rho v\cdot\{ (v\cdot\nabla)V - (V\cdot\nabla)v\}\, d^2x$$

$$- \int_\Omega \rho\, V\cdot\nabla\left(h - \tfrac{1}{2}\|v\|^2\right) d^2x$$

$$(10) \quad \dot{\mathcal{M}} = \frac{d}{dt}\int_\Omega M\cdot V\, d^2x = \int_\Omega \frac{\partial M}{\partial t}\cdot V\, d^2x = \int_\Omega \frac{\partial}{\partial t}(\rho v)\cdot V\, d^2x$$

$$= \int_\Omega \left(\frac{\partial \rho}{\partial t} v\cdot V + \rho\frac{\partial v}{\partial t}\cdot V\right) d^2x = \int_\Omega \left(\nabla\cdot(\rho v)\, v\cdot V + \rho\frac{\partial v}{\partial t}\cdot V\right) d^2x$$

The last equality follows from (8). Let $f = v\cdot V$

$$(11) \quad \int_\Omega \nabla\cdot(\rho v) f\, d^2x = -\int_\Omega \nabla f\cdot\rho v\, d^2x + \int_{\partial\Omega} f\rho\, \underset{\sim}{v}\cdot n\, dS$$

$$= -\int_\Omega \rho v\cdot\nabla(v\cdot V)\, d^2x$$

$$= -\int_\Omega \rho v\{(v\cdot\nabla)V + (V\cdot\nabla)v + v\times(\nabla\times V) + V\times(\nabla\times v)\}\, d^2x$$

This last equality follows from the standard identity

$$(12) \qquad \nabla(a\cdot b) = a\cdot\nabla b + b\cdot\nabla a + a\times(\nabla\times b) + b\times(\nabla\times a)$$

Note that $v\cdot v\times(\nabla\times V) = 0$

From (9), (10) and (11) we get

$$\dot{\mathcal{M}} = \{\mathcal{M}, H\}$$

$$\Updownarrow$$

$$(12) \quad \int_\Omega \rho \frac{\partial v}{\partial t} \cdot V \, d^2x = \int_\Omega \rho v \{ 2(v\cdot\nabla)V + V \times (\nabla \times v) \} \, d^2x - \int_\Omega \rho V \cdot (\nabla h - \nabla \tfrac{1}{2}\|v\|^2) \, d^2x$$

By (12) $\nabla \frac{1}{2}\|v\|^2 = (v\cdot\nabla)v + v \times (\nabla \times v)$. Also we have that $V \cdot v \times (\nabla \times v) = - v \cdot V \times (\nabla \times v)$ by the properties of the tripple product. (12) then becomes:

$$\int_\Omega \rho \frac{\partial v}{\partial t} \cdot V \, d^2x = \int_\Omega \rho \{ 2\, v \cdot (v\cdot\nabla)V + V\cdot(v\cdot\nabla)v - V\cdot\nabla h \} \, d^2x$$

<u>Remark</u>: I am running out of time, and this seems to be wrong – unless:

$$\int 2 \rho v \cdot (v\cdot\nabla)V \, d^2x = -2\int \rho \, V\cdot(v\cdot\nabla)v$$

which seems highly unlikely to me. Although this seem to correspond with the too naive integration by parts. (Let $D = v\cdot\nabla$)

$$\int \rho v \cdot DV \, d^2x = -\int V \cdot D(\rho v) \, d^2x + \text{Surface term}$$

But I have not been able to find an integration by parts formula for the convection operator D. This is the same problem that I had in chapter 2.

4. ADIABATIC FLOW IN $\mathbb{R}^3$

Let Ω be a region in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For adiabatic flow in Ω, we have the following equations:

$$(1)\quad \begin{cases} \frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{1}{\rho}\nabla p(\rho, s) & \\ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0 & \text{(conservation of mass)} \\ \frac{\partial s}{\partial t} + v\cdot\nabla s = 0 & \text{([illegible])} \end{cases}$$

$$(2)\qquad v\cdot n\big|_{\partial\Omega} = 0$$

where (1) are the adiabatic fluid equations, and (2) boundary conditions for (1). The following quantities will be used:

- $v(x,t)$: spatial velocity
- $\rho(x,t)$: mass density
- $s(x,t)$: specific entropy
- $p(x,t)$: pressure
- $h(x,t)$: enthalpy,
- $T(x,t)$: temperature
- $\varepsilon(x,t)$: internal energy density

Actually p and ε are functions of ρ and s only. Given ε we now define T, h and p by:

$$(3)\qquad \rho T = \frac{\partial\varepsilon}{\partial s} \quad\text{and}\quad h = \frac{\partial\varepsilon}{\partial\rho} \quad\text{and}\quad p = \rho^2\frac{\partial}{\partial\rho}\left(\frac{\varepsilon}{\rho}\right)$$

And we have the following identities:

$$(4)\qquad d\varepsilon = \rho T\,ds + h\,d\rho$$

$$(5)\qquad \rho h = \varepsilon + p$$

$$(6) \quad dh = T\,ds + \frac{1}{\rho}\,dp = \left[T + p\,\frac{\partial T}{\partial \rho}\Big|_s\right] ds + \frac{1}{\rho}\,c^2\,d\rho$$

where c is the adiabatic sound speed, defined by

$$(7) \quad c^2 = \frac{\partial p(\rho,s)}{\partial \rho} = \rho\,\frac{\partial h(\rho,s)}{\partial \rho}$$

The equations (1) are Hamiltonian in the variable (M,ρ,s) where $M = \rho v$. It is not said in [2] what space (M,ρ,s) live in, but I guess this case is similar to that in chapter 3, so (M,ρ,s) lives in the dual space of the semidirect product $\mathfrak{X}(\Omega) \circledS \mathcal{F}(\Omega) \circledS \mathcal{F}(\Omega)$. $\mathfrak{X}(\Omega)$ and $\mathcal{F}(\Omega)$ defined in chapter 3. Now we have the following Hamiltonian and Lie-Poisson bracket:

$$(8) \quad H(M,\rho,s) = \int_\Omega \left[\frac{1}{2}\frac{\|M\|^2}{\rho} + \varepsilon(\rho,s)\right] d^3x$$

$$(9) \quad \{F,G\} = \int_\Omega M \cdot \left[\frac{\delta G}{\delta M}, \frac{\delta F}{\delta M}\right] d^3x$$
$$+ \int_\Omega \rho \left\{\left(\frac{\delta G}{\delta M}\cdot\nabla\right)\frac{\delta F}{\delta \rho} - \left(\frac{\delta F}{\delta M}\cdot\nabla\right)\frac{\delta G}{\delta \rho}\right\} d^3x$$
$$+ \int_\Omega s\,\nabla\cdot\left(\frac{\delta G}{\delta M}\cdot\frac{\delta F}{\delta s} - \frac{\delta F}{\delta M}\cdot\frac{\delta G}{\delta s}\right) d^3x$$

where the Jacobi-Lie bracket and the functional derivative are defined in chapter 2. We now want to use (8), (9) and $\dot{F} = \{F,H\}$ to derive (1).

Functional derivatives:

Let us now calculate $\frac{\delta H}{\delta M}$, $\frac{\delta H}{\delta \rho}$, $\frac{\delta H}{\delta s}$. From chapter 3 we have $\frac{\delta H}{\delta M} = \frac{M}{\rho} = v$. This is true here too since the kinetic-energy term in H is the same.

$$\frac{d}{d\mu} H(M, \rho + \mu\delta\rho, s) = \int_\Omega \frac{d}{d\mu}\left(\frac{1}{2}\frac{\|M\|^2}{\rho+\mu\delta\rho} + \varepsilon(\rho+\mu\delta\rho, s)\right) d^3x$$
$$\overset{\mu=0}{=} \int_\Omega \left(-\frac{\|M\|^2}{2\rho^2} + \frac{\partial\varepsilon}{\partial\rho}\right)\delta\rho \, d^3x = \int_\Omega \left(-\tfrac{1}{2}\|v\|^2 + h\right)\delta\rho \, d^3x$$
$$= \int_\Omega \frac{\delta H}{\delta\rho}\delta\rho \, d^3x$$

So $\frac{\delta H}{\delta\rho} = -\frac{1}{2}\|v\|^2 + h$ – note that this is the same as in chapter 3.

$$\frac{d}{d\mu} H(M, \rho, s+\mu\delta s) = \int_\Omega \frac{d}{d\mu}\left(\frac{1}{2\rho}\|M\|^2 + \varepsilon(\rho, s+\mu\delta s)\right) d^3x$$
$$\overset{\mu=0}{=} \int_\Omega \frac{\partial\varepsilon}{\partial s}\delta s \, d^3x = \int_\Omega \frac{\delta H}{\delta s}\delta s \, d^3x$$

So $\frac{\delta H}{\delta s} = \frac{\partial\varepsilon}{\partial s} = \rho T$

Invariations:

Let us define R, $\mathcal{M}$, and S similar to what we did in chapter 3.

$$R = \int_\Omega \rho f \, d^3x, \quad \mathcal{M}_i = \int_\Omega M\cdot V \, d^3x, \quad S = \int_\Omega s f \, d^3x$$

Let us calculate their functional derivatives:

$$\frac{\delta R}{\delta M} = 0, \quad \frac{\delta R}{\delta\rho} = f$$
$$\frac{\delta\mathcal{M}}{\delta M} = V, \quad \frac{\delta\mathcal{M}_i}{\delta\rho} = 0$$

This we have from chapter 3. And it is easy to see that the following holds:

$$\frac{\delta R}{\delta s} = 0 = \frac{\delta\mathcal{M}}{\delta s}, \quad \frac{\delta S}{\delta M} = 0 = \frac{\delta S}{\delta\rho}, \quad \frac{\delta S}{\delta s} = f$$

Now $\dot{M} = \{M, H\} = \int_\Omega M \cdot \left[\frac{\delta M}{\delta M}, \frac{\delta H}{\delta M}\right] d^3x + 0 - \int_\Omega \rho \left(\frac{\delta M}{\delta M} \cdot \nabla\right) \frac{\delta H}{\delta \rho} d^3x$
As in chapter 3 this equation only depends on the first $\underbrace{\text{term}}_{\text{and second}}$ in (9).

$\dot{R} = \{R, H\} = 0 + \int_\Omega \rho \left\{ \left(\frac{\delta H}{\delta M} \cdot \nabla\right) \frac{\delta R}{\delta \rho} - \left(\frac{\delta R}{\delta M} \cdot \nabla\right) \frac{\delta H}{\delta \rho} \right\} d^3x + 0.$
As in chapter 3 this only depends on the first term in (9.)

In these cases we have from chapter 3 that:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p \qquad \left(\text{in chapter 3 } \nabla h = \frac{1}{\rho} \nabla p\right)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

Let us now calculate $\dot{s}$.

$$\dot{S} = \{S, H\} = 0 + 0 + \int_\Omega s \nabla \cdot \left(\frac{\delta H}{\delta M} \cdot \frac{\delta S}{\delta s} - \frac{\delta S}{\delta M} \cdot \frac{\delta H}{\delta s} \right) d^3x$$

$$= \int_\Omega s \nabla \cdot (v \cdot f - 0) \, d^3x = \int_\Omega s \nabla \cdot (v f) \, d^3x$$

$$= -\int_\Omega \nabla s \cdot v f d^3x + \int_{\partial\Omega} s \underbrace{v \cdot n}_{0} f dS = -\int_\Omega v \cdot \nabla s \, f \, d^3x$$

So we have:

$$\frac{d}{dt} \int_\Omega f s \, d^3x = \int_\Omega f \cdot \frac{\partial s}{\partial t} d^3x = -\int_\Omega v \cdot \nabla s \cdot f d^3x$$

$$\Rightarrow \int_\Omega \left(\frac{\partial s}{\partial t} + v \cdot \nabla s \right) f d^3x = 0$$

Since f was arbitrary in $\mathcal{F}^*(\Omega)$ we get:

$$\frac{\partial s}{\partial t} + v \cdot \nabla s = 0$$

So we are done.

5. A FEW COMMENTS

Comparing the three cases:

These three cases are very similar, and in a sense corresponds to starting with a basic model (actually the simplest I can imagine) and then adding new features to the basic model. The second and third models are both models for compressible fluid, but the third has a more general internal energy term.

Actually the first model only needed one variable to specify the system, namely v. The second model needed two, v and h. And finally the third model needs three, v, h and s. The corresponding Hamiltonians have one, two and two terms. The corresponding Poissonbrackets have one term per state variable, and the more complicated brackets contains the less complicated brackets' terms. The same goes for the Hamiltonian.

When I solved $\dot{F} = \{F, H\}$ to get the Euler equations for these systems, I proceeded in the same fashion in all cases. The first case was a little bit different though, because the space on which v lives in here is more restricted than in the two other cases (they have the same space for the v's). I had to take spacial care of ∇q - terms. In the other two cases, the calculations were so similar that I could use the results from case 2 in case 3, so I only needed to find one new equation in this case.

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APPENDIX 1 : ERRONOUS CALCULATIONS

(1.) $\int v \cdot (V \cdot \nabla) v \, dx \qquad (*)$

$$\int v \cdot (V \cdot \nabla) v \, dx = \int v \left(V_1 \frac{\partial}{\partial x_1} + V_2 \frac{\partial}{\partial x_2} + V_3 \frac{\partial}{\partial x_3} \right) v \, dx$$

$$= \int \left(V_1 v \frac{\partial v}{\partial x_1} + V_2 v \frac{\partial v}{\partial x_2} + V_3 v \frac{\partial v}{\partial x_3} \right) dx$$

$$= \int \frac{1}{2} \left(V_1 \frac{\partial}{\partial x_1} \|v\|^2 - V_2 \frac{\partial}{\partial x_2} \|v\|^2 - V_3 \frac{\partial}{\partial x_3} \|v\|^2 \right) dx$$

$$= \int V \cdot \nabla \tfrac{1}{2} \|v\|^2 \, dx$$

(2.) $\int \rho v \cdot (V \cdot \nabla) v \, dx$

$$\int \rho v \cdot (V \cdot \nabla) v \, dx = \int \rho \left(V_1 v \frac{\partial v}{\partial x_1} - V_2 v \frac{\partial v}{\partial x_2} - V_3 v \frac{\partial v}{\partial x_3} \right) dx$$

$$= \int \rho \, V \cdot \nabla \tfrac{1}{2} \|v\|^2 \, dx$$

<u>Warning:</u> These calculations are wrong! There is an identity which states : $\nabla \|v\|^2 = 2(v \cdot \nabla) v + 2 v \times (\nabla \times v)$. If you multiply this by V, the last term does <u>not</u> disappear in general. I don't know why I dont get these terms, and I tried (friday evening) to find a derivation of the $\nabla \|v\|^2$ identity or some similar identity [illegible] (the library was closed - and my books didn't suffice).

I tried to calculate the $\int v \cdot (v \cdot \nabla) V \, dx$ term too, and I got $-2 \cdot \int V \cdot (\nabla \cdot v) v \, dx$. So this is also wrong. I tried over and over again, but could not get correct answers. I am probably doing a terrible stupid mistake, but as of now my eyes seem blind to it.

APPENDIX 2: Early version of erroneous calculations

(1.) $$\int_{\Omega} v\cdot(V\cdot\nabla)v\,dx \qquad (*)$$

$$\int v\cdot(V\cdot\nabla)v\,dx = \int (v_1, v_2, v_3)\left(V_1\frac{\partial v}{\partial x_1} + V_2\frac{\partial v}{\partial x_2} + V_3\frac{\partial v}{\partial x_3}\right)dx$$

Let us look at: $\int v\cdot V_1\frac{\partial v}{\partial x_1}\,dx$

$$\int v\cdot V_1\frac{\partial v}{\partial x_1}\,dx = \int\left(v_1\frac{\partial v_1}{\partial x_1}V_1 + v_2\frac{\partial v_2}{\partial x_1}V_1 + v_3\frac{\partial v_3}{\partial x_1}V_1\right)dx = \int\frac{1}{2}V_1\frac{\partial}{\partial x_1}\left(v_1^2 + v_2^2 + v_3^2\right)$$

$$= \int\frac{1}{2}V_1\frac{\partial}{\partial x_1}\|v\|^2\,dx$$

So now $(*)$ becomes:

$$\int v\cdot(V\cdot\nabla)v\,dx = \int\frac{1}{2}\left(V_1\frac{\partial}{\partial x_1}\|v\|^2 - V_2\frac{\partial}{\partial x_2}\|v\|^2 - V_3\frac{\partial}{\partial x_3}\|v\|^2\right)dx$$

$$= \int\frac{1}{2}V\cdot\nabla\|v\|^2dx = \int V\cdot\nabla\frac{1}{2}\|v\|^2\,dx$$

<u>Remark</u>: $dx \equiv d^3x$ and note that we would have got the same result had we been working in 2 dimensions instead of 3. This is obvious looking ~~back~~ at the calculations above.

(2.) $$\int_{\Omega} v\cdot(v\cdot\nabla)V\,dx \qquad (**)$$

$$\int v\,(v\cdot\nabla)V\,dx = \int (v_1, v_2, v_3)\cdot\left(v_1\frac{\partial V}{\partial x_1} + v_2\frac{\partial V}{\partial x_2} + v_3\frac{\partial V}{\partial x_3}\right)dx$$

Let us look at: $\int v\cdot v_1\frac{\partial V}{\partial x_1}\,dx$

$$\int_{\Omega} v\cdot v_1\frac{\partial V}{\partial x_1}\,dx = -\int_{\Omega} V\cdot\frac{\partial}{\partial x_1}(v_1 v)\,dx + \int_{\Omega}\frac{\partial}{\partial x_1}(V\cdot v\, v_1)\,dx$$

$$\int v\cdot(v\cdot\nabla)V\,dx = -\int V\cdot\left(\frac{\partial}{\partial x_1}(v_1 v) + \frac{\partial}{\partial x_2}(v_2 v) + \frac{\partial}{\partial x_3}(v_3 v)\right) dx$$
$$+ \int \nabla\cdot[(V\cdot v)v]\,dx$$

By the divergens theorem $\int_\Omega \nabla\cdot[(V\cdot v)v]\,dx = \int_{\partial\Omega} (V\cdot v)\,v\cdot n\,dS$

Since $v\cdot n|_{\partial\Omega} = 0$ this term dissapears, so we get:

$$\int v\cdot(v\cdot\nabla)V\,dx = -\int V\cdot\left(\frac{\partial}{\partial x_1}(v_1 v) + \frac{\partial}{\partial x_2}(v_2 v) + \frac{\partial}{\partial x_3}(v_3 v)\right) dx$$
$$= -\int\{V\cdot(\nabla\cdot v)v + V\cdot(v\cdot\nabla)v\}\,dx$$

From [3] we have the following identity:

$$(H\cdot F)(\nabla\cdot G) = F\cdot(H\cdot\nabla)G - H\cdot((F\times\nabla)\times G)$$

let $H = G = v$ and $F = V$, then we get:

$$(v\cdot V)(\nabla\cdot v) = V\cdot(v\cdot\nabla)v - v\cdot((V\times\nabla)\times v)$$

The last term is 0 because $v \perp (V\times\nabla)\times v$. We now have:

$$\int v(v\cdot\nabla)V\,dx = -2\int V\cdot(\nabla\cdot v)v\,dx$$