HAMILTONIAN SYSTEMS IN
FLUID-MECHANICS

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1. INTRODUCTION / PREFACE

My goal - which I did not fully achieve:

When I got an overview over this project, my goal was to do the three cases in chapters 2, 3 and 4. I wanted to understand (to some extent at least) the setup of the problems, and then derive the Euler equations from the Poisson-bracketed and Hamiltonian. And then if time permitted perhaps "compare and contrast" the three cases.

What I have done:

I feel I do understand the setup of the three cases. I also did some "compare and contrast". I did not however succeed in deriving all the Euler equations.
My problem concerns two integrals \( \int_0^1 \nabla \cdot (v \cdot \nabla) v \, dx \) and 
\( \int_0^1 (v \cdot \nabla) v \, dx \). I was not able to transform them in a suitable way. Because of this I was not able to complete the derivation of the Euler equation in either case. But I have done all the other details and I did derive the mass conservation equations on case 2 and 3, and the entropy condition on case 3. I also did most of the calculations for the Euler equations — even though I could not complete them. In the first case (the incompressible case) I proved that the Euler and continuity equations can be written as 
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \quad \text{(see chap. 2)}.
\]

And how does it feel?

This was a most interesting project, and it felt very rewarding when I was able to derive the continuity equations. It felt equally disappointing that I should not be able to derive the Euler equations. So there are still things about this project I do not understand — maybe if I had a couple of more days... But that is useless thinking — the deadline is NOW.

But, I did learn a lot — even some new mathematical techniques such as some new integration by parts formulas and the Helmholtz decomposition theorem (which I also was exposed to in my 219-lecture).

About the report:

This is written in much more haste than the 219-report. It not so concisiously built up, and not so consistent. And the handwriting is not so nice. Another thing I would like to mention is the appendix. They are not a part of the report, and you don't have to read them — but they contain some of my attempts of doing the above material. The calculations... there are wrong and I don't know why!
2. INCOMPRESSIBLE FLUID IN $\mathbb{R}^2$

Let $\Omega$ be a domain in $\mathbb{R}$ with smooth boundary $\partial \Omega$. Now we assume $\Omega$ is filled with an incompressible and homogeneous fluid. Let $u(x,t)$ denote the spatial velocity. The equations of motion are the Euler equations:

\[
\begin{aligned}
\frac{du}{dt} + (u \cdot \nabla) u &= -\nabla p \\
\nabla \cdot u &= 0
\end{aligned}
\]

We also assume that $u \cdot n |_{\partial \Omega} = 0$.

Let us define $P$ to be the space on which $u \in \ldots$. Then $P = \{ u \mid u \text{ smooth}, \text{ domain } = \Omega, \nabla u = 0, u |_{\partial \Omega} = 0 \}$ where smooth means in essence $u \in (H^2(\Omega))^2$ where $H^2(\Omega)$ is the class of second space of "smoothness". This is a minimum choice corresponding to [12], but it seems to me the $(H^1(\Omega))^2$ should be sufficient, since there are no second order partial derivative terms in (1).

Now I'm going to state the Hamiltonian and the Poisson bracket, and then try to derive (1). The Hamiltonian and the Poisson bracket are:

\[
\begin{aligned}
(2) \quad H(u) &= \frac{1}{2} \int_{\Omega} ||u||^2 d^2x \\
(3) \quad \{ F, G \}(u) &= -\int_{\Omega} \nabla u \cdot \left[ \frac{\partial F}{\partial u_x}, \frac{\partial F}{\partial u_y} \right] d^2x
\end{aligned}
\]

where $u, \frac{\partial F}{\partial u_x}, \frac{\partial F}{\partial u_y} \in P$ and $F, G : P \to \mathbb{R}$.

Let me explain what (3) means. First, $\frac{\partial F}{\partial u_x}$ is the Jacobian derivative of $F$ with respect to $u_x$.
\begin{equation}
(4) \quad DF(\mathbf{u}) \cdot \delta \mathbf{u} = \left. \frac{d}{d\epsilon} F(\mathbf{u} + \epsilon \delta \mathbf{u}) \right|_{\epsilon = 0} = \int_{\Omega} \frac{\delta F}{\delta \mathbf{u}} \cdot \delta \mathbf{u} \, d^3x
\end{equation}

where \( \int_{\Omega} \frac{\delta F}{\delta \mathbf{u}} \cdot \delta \mathbf{u} \, d^3x = \langle \delta F, \delta \mathbf{u} \rangle \) is the pairing between the space \( S = \{ u \in (L^2(\Omega))^d \mid \nabla \cdot u = 0, \ u \cdot n |_{\partial \Omega} = 0 \} \) and its dual \( S^* \). Note that \( P \) is a subspace of \( S \). Also note that \( S \) and \( S^* \) are isomorphic by Riesz representation theorem, so we can identify \( S \) and \( S^* \) using the above pairing (inner product in this case).

\begin{equation}
(5) \quad [u, v] = (u \cdot \nabla) v - (v \cdot \nabla) u \quad u, v \in P
\end{equation}

This is the Jacobi-Lie bracket. So \( (P, [\cdot, \cdot]) \) is a Lie algebra.

We know that Hamilton equations are \( \dot{F} = \{F, H^g\} \). Let us use this and (2) and (3) to derive (1).

**Alternative form of (1):**

First we need to show that (1) can be put in another form, without the \( \nabla p \) term. Let \( P : (L^2)^2 \to S \) be the \( (L^2)^2 \) projection onto \( S \). \( S \) has an orthogonal supplement in \( (L^2)^2 \) call it \( O \), (where \( O = \{\nabla p \mid p \in H^1(\Omega)\} \)). Let us check that these two spaces are orthogonal:

\[ \langle u, \nabla p \rangle = \int_{\Omega} u \cdot \nabla p \, d^3x = \int_{\Omega} p \, u \cdot n \, dS - \int_{\partial \Omega} p \, (\nabla u) \cdot dS = 0 \]

So we get \( Pu = u \), \( P \nabla p = 0 \) and applying \( P \) to (1) gives:

\begin{equation}
(6) \quad \frac{\partial u}{\partial t} + P(u \cdot \nabla) u = 0
\end{equation}

This is an equation for \( u \), now we need an equation for \( p \). Assume \( u \) is a given field that solves (7), then we can find \( p \) by solving the following equation, applying \( (1 - P) \) to (1):

\begin{equation}
\frac{\partial p}{\partial t} + (1 - P)(u \cdot \nabla) p = 0
\end{equation}
\[(7) \quad (1-\mathbf{p}) \cdot (\nabla \cdot \nabla) \mathbf{u} = \nabla \mathbf{p} \]

since \((1-\mathbf{p}) \cdot \mathbf{u} = 0\), \((1-\mathbf{p}) \cdot \nabla \mathbf{p} = \nabla \mathbf{p}\). Now take the divergence of (8) and also multiply (8) with \(n\) (the outer normal-vector of \(\partial \Omega\)):

\[
\left\{\begin{align*}
\nabla^2 \mathbf{p} & = \nabla \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u}] \\
\frac{\partial \mathbf{p}}{\partial \mathbf{n}} \bigg|_{\partial \Omega} & = [(\mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \mathbf{n} \bigg|_{\partial \Omega}
\end{align*}\right.
\]

Since \(\nabla \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u}] = \nabla \cdot [(1-\mathbf{p}) \cdot (\mathbf{u} \cdot \nabla) \mathbf{u}] = \nabla \cdot [(1-\mathbf{p}) (\mathbf{u} \cdot \nabla) \mathbf{u}] = \nabla \cdot [(1-\mathbf{p}) \mathbf{u} \cdot \nabla \mathbf{u}] = (1-\mathbf{p}) \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{n} + (1-\mathbf{p}) \mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{n} = (1-\mathbf{p}) \mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{n}.
\]

\[\text{It can be shown that (7) has a unique solution, for } \mathbf{u} \in \mathbf{P}. \text{ This means that } \mathbf{p} \text{ can be uniquely determined } \text{ if the solution of (7), } \mathbf{p}, \text{ is known. In this way (6) is connected to (7). More details in [3].}
\]

We now want to write \((7) \text{ and } (8)\).
The last equality comes from [L], and I have not been able to redo it. Now we note that \( \frac{\delta F}{\delta u} \in \mathcal{P} \), and this means \( \nabla q \cdot \frac{\delta F}{\delta u} = 0 \) \( \forall q \in \Omega \). So we get:

\[
\{ F, H \}_{(u)} = - \int \frac{\delta F}{\delta u} (u \cdot \nabla) u \, d^2 x
\]

\[
= - \int \frac{\delta F}{\delta u} \cdot \mathcal{P} (u \cdot \nabla) u \, d^2 x
\]

This follows from the fact that \( \frac{\delta F}{\delta u} \cdot (1 - \mathcal{P}) (u \cdot \nabla) u \, d^2 x = 0 \).

Let \( F = \int u \cdot V \, d^2 x \), \( V \in \mathcal{P} \), \( \dot{u} = 0 \), \( V \) otherwise arbitrary. Then it is easy to see that \( \frac{\delta F}{\delta u} = V \).

\[
F = \frac{d}{dt} \int u \cdot V \, d^2 x = \int \frac{\partial u}{\partial t} \cdot V \, d^2 x
\]

\[
= \{ F, t \}_{(3)} = - \int V \cdot \mathcal{P} (u \cdot \nabla) u \, d^2 x
\]

so we have

\[
\int (\frac{\partial u}{\partial t} + \mathcal{P} (u \cdot \nabla) u) \cdot V \, d^2 x = 0,
\]

and since \( V \) is arbitrary:

\[
\frac{\partial u}{\partial t} + \mathcal{P} (u \cdot \nabla) u \, d^2 x = 0.
\]

Since this is an equivalent form of (7) we are now done.
3. **Erostronic Flow in $\mathbb{R}^2$**

Let $\Omega$ be a domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$. A flow is *erostrophic* if there exists a function $h$, called enthalpy, such that

$$\nabla h = \frac{1}{\delta} \nabla p$$

Note: In [3] this flow is called *isentropic*.

Let us briefly state the thermodynamic relations needed in this case. The following quantities are used:

- $p$: pressure
- $\rho$: density
- $T$: temperature
- $s$: entropy
- $h$: enthalpy (per unit mass)
- $\varepsilon = h - \frac{T}{\delta}$: internal energy (per unit mass)

The first law of thermodynamics:

$$dh = T ds + \frac{\rho}{\delta} dp$$

or equivalently:

$$d\varepsilon = T ds + \frac{\rho}{\delta} dp$$

We will assume that $p$ is a function of $s$ only. By (2)

d$h = \frac{1}{\delta} dp$, so (1) is satisfied.
The Euler equations for this case are:

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla h(\rho) \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0
\end{align*}
\]  

(4) 

where (5) are boundary conditions for (4). Let $P$ be the space of $\mathbf{u}$ and $\rho$ that are smooth enough, and tend to a fixed unknown field and density at infinity if $\Omega$ is unbounded. Furthermore $P$ is "nice" in all ways demanded in [2].

**Comment:** In [2] it says: "... space of $\mathbf{u}$ and $\rho$ that are $C^1$ (say $H^s$, $s > 2$) ...". This confuses me, to me it seems to choose $\mathbf{u} \in H^1(\Omega)^3$ and $\rho \in H^2(\Omega)$ -- and I really don't understand what the prerequisites, provisions to say. The other properties that makes $P$ "nice" seem to me to be related to the method discussed in [2], and may thus not be relevant to this "discussion".

The configuration space of compressible fluid motion is the group of diffeomorphisms on $\Omega$ whose Lie algebra consists of the space $\mathfrak{X}(\Omega)$ of all vectorfields on $\Omega$.

**Remark:** From [1] we have that for compressible flow, the configuration space is $\text{Diff}^\ast_0(\Omega)$. That is, the group of volume-preserving diffeomorphisms. The corresponding Lie algebra being that of divergence-free vectorfields on $\Omega$.

$\mathfrak{X}(\Omega)$ is represented on the vector space $\mathcal{F}(\Omega)$ of functions on $\Omega$ by the minus Lie-derivative:

\[ \mathcal{L}_X f = \mathcal{X}[f] = -df(X) \]
for $X \in \mathcal{X}(\Omega)$ and $f \in \mathcal{F}(\Omega)$. On the dual of the
second-order product $\mathcal{X}(\Omega) \otimes \mathcal{F}(\Omega)$ with variables $M = g$ and $g$, the
equations (4) are Hamiltonian with the following Hamiltonian and Lie-Poisson bracket:

\[
(6) \quad \mathcal{H}(M, g) = \int_\Omega \left( \frac{1}{2} \| M \|^2 + E(g) \right) \, d^3 x \\

(7) \quad \{ F, G \} = \int_\Omega M \cdot \left[ \sum \frac{\delta F}{\delta M} \right] \, d^3 x + \int_\Omega g \{ \frac{\delta F}{\delta g}, \frac{\delta G}{\delta g} \} \, d^3 x
\]

where $[ \cdot, \cdot ]$ is the Jacobi-Lie (skew) and $\frac{\delta F}{\delta M}, \frac{\delta G}{\delta g}$ the
functional derivatives - (both defined in chapter 2.1.3. $E(g)$ is internal energy per unit area, satisfying $E(g) \equiv 0$).

**Note:** $\frac{\delta F}{\delta M}$ is an element in the space of the $M$'s and $\frac{\delta F}{\delta g}$ an element in the space of the $g$'s, i.e. $\mathcal{X}(\Omega)$ and 
$\mathcal{F}(\Omega)$ respectively. I guess you need smoothness conditions for $M$ and $g$ to mean the space of $M$ and $g$.

Using (6) and (7) we let $\mathcal{H} = \mathcal{H}(M, g)$ and we use these equations (4).

**Functional derivatives:**

Let us now find $\frac{\delta \mathcal{H}}{\delta M}$ and $\frac{\delta \mathcal{H}}{\delta g}$.

\[
\frac{d}{dt} \mathcal{H}(M(s) + \delta M, g) = \int_\Omega \frac{1}{2} \delta M \cdot \left[ \frac{\delta \mathcal{H}}{\delta M} \right] \, d^3 x,
\]

\[
= \int_\Omega \frac{1}{2} g \cdot \delta M \, d^3 x = \int_\Omega \frac{\delta \mathcal{H}}{\delta M} \delta M \, d^3 x
\]

\[
\Rightarrow \frac{\delta \mathcal{H}}{\delta M} = \frac{1}{2} g = 0.
\]
\[
\frac{d}{ds} H(M, g + \varepsilon \delta g) \big|_{s=0} = \int \frac{\|HM\|^2}{\delta g^2 (g + \varepsilon \delta g)} \, d^3 x \bigg|_{s=0} + \Sigma (g + \varepsilon \delta g) \int d^3 x \bigg|_{s=0}
\]
\[
= \int \left( - \frac{\|HM\|^2}{2 \delta g^2} + \delta g \right) d^3 x = \int \left( - \frac{\|HM\|^2}{2 \delta g^2} + h(g) \right) \delta g d^3 x = \int \frac{\delta H}{\delta g} \, \delta g \, d^3 x
\]

so
\[
\frac{\delta H}{\delta g} = - \frac{\|HM\|^2}{2 \delta g^2} + h(g) = h(g) - \frac{1}{2} \| \mathbf{v} \| ^2.
\]

**Derivations:**

Let us now define the following quantities:

\[
M = \int_{\Omega} M \cdot \mathbf{v} \, d^3 x \quad \text{and} \quad R = \int_{\Omega} g \cdot f \, d^3 x
\]

where \( \mathbf{V} \in X(\Omega) \) and \( f \in L^2(\Omega) \), \( \mathbf{V} = 0 = f \), \( \mathbf{V}, f \) otherwise arbitrary. Let us now calculate the functional derivatives of \( M \) and \( R \).

\[
\frac{d}{ds} M(M + \varepsilon \delta M) \bigg|_{s=0} = \int_{\Omega} \mathbf{v} \cdot \delta M \, d^3 x = \int_{\Omega} \frac{\delta M}{\delta \mathbf{M}} \, \delta M \, d^3 x
\]

so
\[
\frac{\delta M}{\delta \mathbf{M}} = \mathbf{V}.
\]

Likewise it is easy to see that:

\[
\frac{\delta M}{\delta g} = 0, \quad \frac{\delta R}{\delta g} = 0 \quad \text{and} \quad \frac{\delta R}{\delta\delta g} = f.
\]

Let us now calculate \( \dot{R} = \{ R, H \} = \)

\[
\{ R, H \} = 0 + \int_{\Omega} g \cdot \left( \frac{\delta H}{\delta \mathbf{M}} \cdot \nabla \frac{\delta R}{\delta g} - \frac{\delta R}{\delta \mathbf{M}} \cdot \nabla \frac{\delta H}{\delta g} \right) d^3 x
\]

\[
= \int_{\Omega} g \cdot \left( \nabla (g) \right) d^3 x = - \int \nabla \cdot (g \mathbf{v}) f \, d^3 x + \int_{\partial \Omega} f g \mathbf{u} \cdot n \, ds.
\]

\[
\dot{R} = \frac{d}{dt} \int_{\Omega} g \cdot f \, d^3 x = \int_{\Omega} \frac{\partial g}{\partial t} \cdot f \, d^3 x = \{ R, H \} = - \int_{\Omega} \nabla \cdot (g \mathbf{v}) f \, d^3 x.
\]

\[
\Rightarrow \int_{\Omega} \left( \frac{\partial g}{\partial t} + \nabla \cdot (g \mathbf{v}) \right) f \, d^3 x = 0.
\]
Since $f$ is arbitrary we have:

\begin{equation}
\frac{\partial x}{\partial t} + \nabla \cdot (gu) = 0
\end{equation}

Let us now calculate $\dot{\mathcal{M}} = \{\mathcal{M}, H\}$

\begin{equation}
\{\mathcal{M}, H\} = \int_\Omega \nabla \cdot \left[ \frac{EH}{\theta M} - \frac{SM}{\theta M} \right] \, d^2x + O - \int_\Omega g \left( \frac{EM}{\theta M} \cdot \nabla \frac{EH}{\theta M} \right) \, d^2x
\end{equation}

\begin{align*}
&= \int_\Omega \nabla \cdot \left( \frac{EH}{\theta M} \cdot \nabla \frac{EM}{\theta M} - \frac{EH}{\theta M} \cdot \nabla \frac{EH}{\theta M} \right) \, d^2x - \int_\Omega g \nabla \cdot \left( \frac{EH}{\theta M} \right) \, d^2x \\
&= \int_\Omega g u \cdot \left\{ (u \cdot \nabla) V - (V \cdot \nabla) u^2 \right\} \, d^2x \\
&\quad - \int_\Omega g \nabla \cdot \nabla \left( h - \frac{1}{2} \|u\|^2 \right) \, d^2x
\end{align*}

\begin{equation}
\dot{\mathcal{M}} = \frac{d}{dt} \int_\Omega \mathcal{M} \cdot V \, d^2x = \int_\Omega \frac{\partial EM}{\partial t} \cdot V \, d^2x = \int_\Omega \frac{\partial (gu)}{\partial t} \cdot V \, d^2x
\end{equation}

\begin{align*}
&= \int_\Omega \left( \frac{\partial EM}{\partial t} \cdot V + g \frac{\partial u}{\partial t} \cdot V \right) \, d^2x = \int_\Omega \left( \nabla \cdot (gu) \right) \cdot V + g \frac{\partial u}{\partial t} \cdot V \right) \, d^2x
\end{align*}

The last equality follows from (8). Let $f = u \cdot V$

\begin{equation}
\int_\Omega \nabla \cdot (gu) f \, d^2x = - \int_\Omega \nabla f \cdot gu \, d^2x + \int_\Omega f g u \cdot ndS
\end{equation}

\begin{align*}
&= - \int_\Omega g u \cdot \nabla (u \cdot V) \, d^2x \\
&\quad - \int_\Omega g u \left\{ (u \cdot \nabla) V + (V \cdot \nabla) u + u \times (\nabla \times V) + V \times (\nabla \times u) \right\} \, d^2x
\end{align*}

The last identity follows from the standard identity:

\begin{equation}
\nabla \cdot (a \cdot b) = a \cdot \nabla b + b \cdot \nabla a + a \cdot (V \cdot b) - b \cdot (V \cdot a)
\end{equation}

Hence:

\begin{equation}
\nabla \cdot u \cdot u \times (\nabla \times V) = 0
\end{equation}
From (9), (10) and (11) we get

\[ \mathcal{M} = \{ \mathcal{M}, \mathcal{H} \} \]

\[ \mathcal{M} \]

(12) \[ \int_{\Omega} g \frac{\partial u}{\partial t} \cdot V \, d^2 x = \int_{\Omega} g \{ 2 \cdot (\mathbf{u} \cdot \nabla) V + V \times (\nabla \times \mathbf{u}) \} \, d^2 x - \int_{\Omega} g V \cdot (\nabla h - \nabla \frac{1}{2} \| V \|^2) \, d^2 x \]

By (12) \[ \nabla \frac{1}{2} \| V \|^2 = (\mathbf{u} \cdot \nabla) \mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}). \]
Also we have that \[ V \cdot (\mathbf{u} \times (\nabla \times \mathbf{u})) = - \mathbf{u} \cdot (\mathbf{V} \times (\nabla \times \mathbf{u})) \]
by the properties of the triple product.

(13) Then becomes:

\[ \int_{\Omega} g \frac{\partial u}{\partial t} \cdot V \, d^2 x = \int_{\Omega} g \{ 2 \cdot (\mathbf{u} \cdot \nabla) V + V \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} - V \cdot \nabla h \} \, d^2 x \]

Remark: I am running out of time, and thus seem to be wrong - unless:

\[ \int_{\Omega} 2 g \mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \mathbf{V} \, d^2 x = -2 \int_{\Omega} g V \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} \]

which seems highly unlikely to me. Although this seems to correspond with the too naive integration by parts. (Let \( D = \mathbf{u} \cdot \nabla \))

\[ \int_{\Omega} g \mathbf{u} \cdot D \mathbf{V} \, d^2 x = -\int_{\Omega} V \cdot D(g \mathbf{u}) \, d^2 x + \text{Surface term} \]

But I have not been able to find an integration by parts formula for the convection operator \( D \). This is the same problem that I had in chapter 2.
4. ADIABATIC FLOW IN $\Omega$.

Let $\Omega$ be a region in $\mathbb{R}^3$ with smooth boundary $\partial \Omega$. For adiabatic flow in $\Omega$, we have the following equations:

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p(\rho, s) \]

(1) \[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad \text{(conservation of mass)} \]

(2) \[ \frac{\partial s}{\partial t} + u \cdot \nabla s = 0 \quad \text{(conservation of entropy)} \]

(3) \[ u \cdot n \bigg|_{\partial \Omega} = 0 \]

where (1) are the adiabatic fluid equations, and (2) bounding conditions for (1). The follow two quantities are used:

\begin{align*}
\rho(x, t) & : \text{density} \\
\rho s(x, t) & : \text{mass density} \\
s(x, t) & : \text{specific entropy} \\
p(x, t) & : \text{pressure} \\
h(x, t) & : \text{entropy} \\
T(x, t) & : \text{temperature} \\
\varepsilon(x, t) & : \text{internal energy density}
\end{align*}

Actually, $p$ and $\varepsilon$ are functions of $\rho$ and $s$ only. Given $\varepsilon$, we now define $T$, $h$ and $p$ by:

(3) \[ s T = \frac{\partial s}{\partial s} \quad \text{and} \quad h = \frac{\partial s}{\partial s} \quad \text{and} \quad p = \rho \frac{\partial s}{\partial s}(\frac{\partial s}{\partial s}) \]

And we have the following identities:

(4) \[ d\varepsilon = g T ds + h ds \]

(5) \[ gh = \varepsilon + p \]
\[ (6) \quad dh = T \, ds + \frac{1}{2} \, dp = \left[ T + p \frac{\partial F}{\partial s} \right] \, ds + \frac{1}{2} \, c^2 \, ds \]

where \( c \) is the adiabatic sound speed, defined by

\[ (7) \quad c^2 = \frac{\partial p}{\partial s} = \frac{\partial h}{\partial \dot{s}} \]

The equations (6) are Hamiltonian in the variable \((M, g, s)\), where \( M, g, s \). It is not said in [2] what space \( M, \dot{g}, s \) lies in, but I guess this case is similar to that in chapter 3, so \((M, g, s)\) lies in the dual space of the semi-direct product \( X(\Omega) \otimes F(\Omega) \otimes F(\Omega) \). \( X(\Omega) \) and \( F(\Omega) \) defined in chapter 3.

Now we have the following Hamiltonian and Lie-Poisson bracket:

\[ (8) \quad H(M, g, s) = \int_{\Omega} \left[ \frac{1}{2} \frac{\dot{M}}{\dot{g}}^2 + \dot{g}(g, s) \right] \, d^3x \]

\[ (9) \quad \{F, G\} = \int_{\Omega} M \cdot \left[ \frac{\dot{G}}{\dot{M}}, \frac{\dot{F}}{\dot{M}} \right] \, d^3x \]

\[ + \int_{\Omega} \left( \frac{\dot{G}}{\dot{M}} \cdot \nabla \right) \frac{\dot{F}}{\dot{M}} \, d^3x \]

\[ + \int_{\Omega} \nabla \cdot \left( \frac{\dot{G}}{\dot{M}} \frac{\dot{F}}{\dot{M}} - \frac{\dot{F}}{\dot{M}} \frac{\dot{G}}{\dot{M}} \right) \, d^3x \]

where the \( \text{Jaco} \)bi-Lie bracket and \( \text{v} \)functional derivative are defined in chapter 2. We now want to use (6), (9) and \( F = \{F, H\} \) to derive (9).

**Functional derivatives:**

Let us now calculate \( \frac{\delta H}{\delta M}, \frac{\delta H}{\delta g}, \frac{\delta H}{\delta s} \). From chapter 3, we have \( \frac{\delta H}{\delta \frac{\delta M}{\delta \dot{g}}} = \frac{\delta H}{\delta \frac{\delta H}{\delta \dot{g}}} = 0 \). This is true here too since the kinetic-energy term in \( H \) is the same.
\[
\frac{1}{\mu} \mathcal{H}(M, g + \mu \delta g, s) = \int_{\Omega} \left( \frac{1}{2} \left( \frac{\delta(g + \mu \delta g)}{\delta s} \right)^2 + \varepsilon (g + \mu \delta g, s) \right) d^3 x
\]
\[
= \int_{\Omega} \left( -\frac{\|v\|^2}{2} + \frac{\partial v}{\partial s} \right) \delta g d^3 x = \int_{\Omega} \left( -\frac{\|v\|^2}{2} + h \right) \delta g d^3 x
\]
\[
= \int_{\Omega} \frac{\delta H}{\delta s} \delta g d^3 x
\]

So \( \frac{\delta H}{\delta s} = -\frac{1}{2} \|v\|^2 + h \) - note that this is the same as in chapter 2.

\[
\frac{1}{\mu} \mathcal{H}(M, s, s + \mu \delta s) = \int_{\Omega} \left( \frac{1}{2} \|v\|^2 - \varepsilon (s, s + \mu \delta s) \right) d^3 x
\]
\[
= \int_{\Omega} \frac{\partial \varepsilon}{\partial s} \delta s d^3 x = \int_{\Omega} \frac{\delta H}{\delta s} \delta s d^3 x
\]
\[
\therefore \frac{\delta \varepsilon}{\delta s} = \frac{\partial \varepsilon}{\partial s} = \varepsilon_{\parallel}
\]

**Illustration:**

Let us define \( R, M, \) and \( S \) similarly as those we did in chapter 3.

\[
R = \int_{\Omega} g f d^3 x, \quad M_v = \int_{\Omega} M \cdot V d^3 x, \quad S = \int_{\Omega} f d^3 x
\]

Let us calculate their functional derivatives:

\[
\frac{\delta R}{\delta M} = 0, \quad \frac{\delta R}{\delta S} = f
\]

\[
\frac{\delta M}{\delta M} = V, \quad \frac{\delta M}{\delta S} = 0
\]

Thus we have from chapter 3. And it is easy to see that the following holds:

\[
\frac{\delta R}{\delta S} = 0 = \frac{\delta M}{\delta S}, \quad \frac{\delta S}{\delta M} = 0 = \frac{\delta S}{\delta S}, \quad \frac{\delta S}{\delta S} = f
\]
Now \[ \dot{M} = 3M, \quad H \dot{3} = \int M \cdot \left[ \frac{\xi E M}{\xi} \frac{\xi H}{\xi M} \right] d^3x + u - \frac{g}{\xi} (\frac{\xi E M}{\xi} \frac{\xi H}{\xi M} \dot{H}) d^3x. \]

As in chapter 3 this equation only depends on the first term in (9).

\[ \dot{R} = \frac{\xi R}{\xi} \dot{H} = 0 + \int_\Omega (\frac{\xi H}{\xi M} \frac{\xi R}{\xi} - (\frac{\xi H}{\xi M} \frac{\xi R}{\xi} \frac{\xi H}{\xi M} \frac{\xi H}{\xi M} \frac{\xi M}{\xi}) d^3x + 0. \]

As in chapter 3 this only depends on the first term in (9).

In these cases we have from chapter 3 that:

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p \quad \text{(in chapter 3 \( \nabla h = \frac{1}{\rho} \nabla p \))} \]

\[ \frac{\partial p}{\partial t} + \nabla \cdot (\rho u) = 0 \]

Let us now calculate \( \dot{S} \).

\[ \dot{S} = \int_\Omega (\frac{\xi E}{\xi} \frac{\xi H}{\xi M} - \frac{\xi E}{\xi} \frac{\xi H}{\xi M} \frac{\xi H}{\xi M} \frac{\xi M}{\xi}) d^3x \]

\[ = \int_\Omega \nabla \cdot (u \cdot f - \rho) d^3x = \int_\Omega \nabla \cdot (\rho \nabla f) d^3x \]

\[ = -\int_\Omega \nabla \cdot \rho \nabla f d^3x + \int_\Omega \rho \nabla \cdot \nabla f d^3x = -\int_\Omega \rho \nabla \cdot \nabla f d^3x. \]

So we have:

\[ \frac{d}{dt} \int_\Omega \frac{\xi H}{\xi M} d^3x = \int_\Omega \frac{\xi H}{\xi M} \frac{\partial H}{\partial t} d^3x = -\int_\Omega u \cdot \nabla \cdot \rho \nabla f d^3x. \]

\[ \Rightarrow \int_\Omega \frac{\partial H}{\partial t} + u \cdot \nabla \cdot \rho \nabla f d^3x = 0 \]

Since \( f \) was arbitrary in \( F^*(\Omega) \) we get:

\[ \frac{\partial H}{\partial t} + u \cdot \nabla \cdot \rho \nabla f = 0 \]

So we are done.
5. A FEW COMMENTS

Comparing the three cases:

These three cases are very similar, and in a sense correspond to starting with a basic model (actually the simplest I can imagine) and then adding new features to the basic model. The second and third cases are both models for compressible flow, but the third has a more general internal energy term.

Actually the first model only needed one variable to specify the system, namely \( V \). The second model needed two, \( V \) and \( h \), and finally the third needed three, \( V \), \( h \) and \( s \). The corresponding Hamiltonians have one, two, and two terms. The corresponding Poisson brackets have one term per state variable, and the more complicated Poisson brackets contains the two complicated brackets terms. The same goes for the Hamiltonian.

When I solved \( F = \{T, H\} \) to get the Euler equations for these systems, I proceeded in the same fashion in all cases. The first case was a little bit different because the space on which \( V \) lives in here is more restricted than in the two other cases (they have the same space for \( f \) and \( s \)). I had to take special care of \( \Delta q \)-terms. In the other two cases, the calculations were so similar that I could use the results from case 2 in case 3, so I only needed to find one new equation in this case.
REFERENCES


Appendix 1: Erroneous Calculations

1. $\int u \cdot (V \cdot \nabla) u \, dx \quad (*)$

\[
\int u \cdot (V \cdot \nabla) u \, dx = \int u \left( V_1 \frac{2}{\delta x_1} + V_2 \frac{2}{\delta x_2} + V_3 \frac{2}{\delta x_3} \right) u \, dx
\]

\[
= \int \left( \frac{2}{\delta x_1} u + \frac{2}{\delta x_2} u + \frac{2}{\delta x_3} u \right) dx
\]

\[
= \int \frac{1}{2} \left( V_1 \frac{2}{\delta x_1} u \|u\|^2 - V_2 \frac{2}{\delta x_2} u \|u\|^2 - V_3 \frac{2}{\delta x_3} u \|u\|^2 \right) dx
\]

\[
= \int \nabla \frac{1}{2} \|u\|^2 \, dx
\]

2. $\int g u \cdot (V \cdot \nabla) u \, dx$

\[
\int g u \cdot (V \cdot \nabla) u \, dx = \int g \left( V_1 \frac{2}{\delta x_1} + V_2 \frac{2}{\delta x_2} - V_3 \frac{2}{\delta x_3} \right) u \, dx
\]

\[
= \int g \nabla \frac{1}{2} \|u\|^2 \, dx
\]

Warning: These calculations are wrong! There is an identity which states: $\nabla \|u\|^2 = 2 (u \cdot \nabla) u + 2 u \times (\nabla \times u)$. If you multiply this by $V$, the last term does not disappear in general. I don't know why I don't get these terms, and I tried (friday morning) to find a derivation of this identity or some similar calculation (but was not found... (the library was closed - and my books didn't suffice).}

I tried to calculate the $\int u \cdot (V \cdot \nabla) V \, dx$ term, but got $-2 \int V \cdot (\nabla u) u \, dx$. So this is also wrong. I tried over and over again, but could not get correct answers. I am
probably doing a terrible stupid mistake, but as of now my eyes seem blind to it.
APPENDIX 2: Early version of erroneous calculations

1. \[ \int_{\Omega} \nu \cdot (V \cdot \nabla) \nu \, dx \quad (\times) \]

\[ \int_{\Omega} \nu \cdot (V \cdot \nabla) \nu \, dx = \int (u_1, u_2, u_3) (V_1 \frac{\partial \nu}{\partial x_1} + V_2 \frac{\partial \nu}{\partial x_2} + V_3 \frac{\partial \nu}{\partial x_3}) \, dx \]

Let us look at: \[ \int \nu \cdot V \frac{\partial \nu}{\partial x_1} \, dx \]

\[ \int \nu \cdot V \frac{\partial \nu}{\partial x_1} \, dx = \int (u_1 \frac{\partial \nu}{\partial x_1} + u_2 \frac{\partial \nu}{\partial x_2} + u_3 \frac{\partial \nu}{\partial x_3}) \, dx = \int \frac{1}{2} V_1 \frac{\partial \nu}{\partial x_1} (\nu_1^2 + \nu_2^2 + \nu_3^2) \]

\[ = \int \frac{1}{2} V_1 \frac{\partial \nu}{\partial x_1} \| \nu \|^2 \, dx \]

So now (\times) becomes:

\[ \int \nu \cdot (V \cdot \nabla) \nu \, dx = \int \frac{1}{2} \left( \frac{\partial \nu}{\partial x_1} \| \nu \|^2 - \frac{\partial \nu}{\partial x_2} \| \nu \|^2 - \frac{\partial \nu}{\partial x_3} \| \nu \|^2 \right) \, dx \]

\[ = \int \frac{1}{2} V \cdot \nabla \| \nu \|^2 \, dx = \int V \cdot \nabla \frac{1}{2} \| \nu \|^2 \, dx \]

Remark: \( dV = d^3x \) and note that we would have got the same result had we been working in 2 dimensions instead of 3. This is obvious looking back at the calculations above.

2. \[ \int_{\Omega} \nu \cdot (u \cdot \nabla) V \, dx \quad (\ast \ast) \]

\[ \int_{\Omega} \nu \cdot (u \cdot \nabla) V \, dx = \int (u_1, u_2, u_3) \cdot \left( u_1 \frac{\partial V}{\partial x_1} + u_2 \frac{\partial V}{\partial x_2} + u_3 \frac{\partial V}{\partial x_3} \right) \, dx \]

Let us look at: \[ \int \nu \cdot \frac{\partial V}{\partial x_1} \, dx \]

\[ \int \nu \cdot \frac{\partial V}{\partial x_1} \, dx = - \int V \cdot \frac{\partial \nu}{\partial x_1} (u, u) \, dx + \int \frac{\partial \nu}{\partial x_1} (V \cdot u \cdot u) \, dx \]
\[ \int \sigma \cdot (\mathbf{u} \cdot \nabla) \mathbf{V} \, dx = - \int \mathbf{V} \cdot \left( \frac{2}{\delta x_1} (\mathbf{u}_1 \mathbf{u}) + \frac{2}{\delta x_2} (\mathbf{u}_2 \mathbf{u}) + \frac{2}{\delta x_3} (\mathbf{u}_3 \mathbf{u}) \right) \, dx \\
+ \int \nabla \cdot [(\mathbf{V} \cdot \mathbf{u}) \mathbf{u}] \, dx \]

By the divergence theorem, \[ \int_{\partial \Omega} \nabla \cdot [(\mathbf{V} \cdot \mathbf{u}) \mathbf{u}] \, d\mathbf{x} = \int_{\partial \Omega} (\mathbf{V} \cdot \mathbf{n}) \mathbf{u} \, d\mathbf{x} \]

since \( \mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0 \), this term disappears, so we get:

\[ \int \sigma \cdot (\mathbf{u} \cdot \nabla) \mathbf{V} \, dx = - \int \mathbf{V} \cdot \left( \frac{2}{\delta x_1} (\mathbf{u}_1 \mathbf{u}) + \frac{2}{\delta x_2} (\mathbf{u}_2 \mathbf{u}) + \frac{2}{\delta x_3} (\mathbf{u}_3 \mathbf{u}) \right) \, dx \]
\[ = - \int \left\{ \mathbf{V} \cdot (\nabla \cdot \mathbf{u}) \mathbf{u} + \mathbf{V} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} \right\} \, dx \]

From [3] we have the following identity:

\[ (\mathbf{H} \cdot \mathbf{F}) \cdot (\nabla \cdot \mathbf{G}) - \mathbf{F} \cdot (\mathbf{H} \cdot \nabla) \mathbf{G} = \mathbf{H} \cdot ((\mathbf{F} \times \nabla) \times \mathbf{G}) \]

Let \( \mathbf{H} = \mathbf{G} = \mathbf{u} \) and \( \mathbf{F} = \mathbf{V} \), then we get:

\[ (\mathbf{u} \cdot \mathbf{V}) \cdot (\nabla \cdot \mathbf{u}) = \mathbf{V} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{u} \cdot ((\mathbf{V} \times \nabla) \times \mathbf{u}) \]

The last term is 0 because \( \mathbf{u} \perp (\mathbf{V} \times \nabla) \times \mathbf{u} \). We now have:

\[ \int \sigma \cdot (\mathbf{u} \cdot \nabla) \mathbf{V} \, dx = -2 \int \mathbf{V} \cdot (\nabla \cdot \mathbf{u}) \mathbf{u} \, dx \]