How does a falling cat turn over?

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1 Introduction

It is a matter of common lore that a falling cat can right itself so as to land on its feet. However, how is it possible that the cat can turn over while maintaining zero total angular momentum? Angular momentum conservation still applies to cats. The answer lies in the cat's ability to change its shape. By changing its shape in a rotating manner, the cat forces its body as a whole to rotate the other way to compensate and maintain the condition of zero total angular momentum. Thus the cat can right itself while falling.

In the next section I will describe a dynamical model, due to Kane and Scher [1], for a cat which shares salient features with observations of real cats. Then I will introduce the geometric viewpoint for the problem, and find the natural mechanical connection, following the work of Montgomery [4]. This connection can then be used to derive the Kane-Scher dynamics. Finally, I will mention how the cat's problem is only one of a larger class of isoholonomic problems.

2 The Kane-Scher Cat

Kane and Scher in 1969 produced the first falling cat dynamical model that respects the observed dynamical features of falling cats. These features are [1]:

1. The cat bends, but does not twist, its spine.
2. The cat is held initially at rest upside down with its spine bent forward. After release, it bends first to one side, then backwards, then to the

other side, then forward again. When landing the general shape of the cat is similar to when released, however the cat has rotated so that its feet are pointed down.

3. The backward bend is mild compared to the forward bend. This reflects physical constraints within the spine.

Their model consists of two bodies connected by a "no-twist" joint. We will consider a simple model in which the two bodies are identical, and can be thought of as cylinders. The no-twist constraint is equivalent to saying that the two bottom rims of the cylinders can roll without slipping on each other. (The no-twist constraint is, however, a holonomic constraint.) The motion to execute a flip consists of one of the two bodies executing a loop in the frame of the other, drawing out a cone. Take the axis of this cone to make an angle α with the "stationary" body, and let the cone have opening angle β. This respects the features above in the following way: The loop to draw a cone clearly is a simple version of the description in feature 2. By bending forwards, sideways, and backwards to draw the cone, it does not need to twist its spine. Beginning with some forward bend in its spine, by an angle $\beta + \alpha$, and then drawing a cone with opening angle β, the forward bend is $\beta + \alpha$ but the backward bend is $\beta - \alpha$. So, as long as α is positive (the cat starts out bent forwards, as described), the forward bend is bigger than the backward bend. See Fig. 1 for a photo of a falling cat with the Kane-Scher solution superimposed.

Rather than go through the derivation of the differential equation that the cat's shape must satisfy to create an overall rotation, I will simply lay out the Kane-Scher coordinate system and quote their final results. We will see later that these results are consistent with, and derivable from, a more tractable geometric argument. See Fig. 2 in reference to the coordinates for the Kane-Scher solution. I have not burdened this discussion with the full coordinate system of Kane and Scher because it is only necessary in the derivation, which I do not reproduce here. It's only purpose here would be to show how awkward it is.

Let J and I be the axial and transverse moments of inertia of the bodies. Let α be as above, the angle between the principal axis of body A and the center direction of the cone swept out by the principal axis of body B, and let β be the opening angle of this cone. Parameterize the motion around the cone by an angle θ. ψ is the orientation angle of the system in a fixed external frame (the frame of the person who just dropped the cat). Kane and Scher showed that the orientation depends on the shape by the following differential equation:

$$\frac{d\psi}{d\theta} = \frac{(J/I)S}{(T-1)[1-T+(J/I)(1+T)](1+T)^{1/2}} \tag{1}$$

where

$$S = -\sqrt{2}\ \sin\beta\ (\cos\alpha\sin\beta + \sin\alpha\cos\beta\cos\theta) \tag{2}$$

and

$$T = \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\theta. \tag{3}$$

One can determine the overall reorientation of the cat by fixing α and J/I, then integrating θ from 0 to 2π. As evidenced by Fig. 1, the Kane-Scher model is capable of reproducing the motion of a real falling cat with remarkable accuracy.

3 Geometry

Since the reorientation of the cat is a purely geometric effect, it makes sense now to turn to a geometric framework for studying the same problem. First let us take a few steps in a more general direction. We have a cat, moving through space under free fall evolution, with two rigid body halves. Therefore the overall configuration space of the body $Q = SO(3) \times SO(3) \times \mathbb{R}^3$. The Lie group $G = SE(3)$ of rigid body rotations and translations acts on the body without changing the cat's shape. Therefore the shape space $S = SO(3) \times SO(3) \times \mathbb{R}^3/SE(3) \cong SO(3)$ [2]. An element in $SO(3)$ describes the relative orientation of one body relative to the other, which is sufficient to determine the shape of the model cat. The projection map $\pi : Q \to S$ provides the structure to define a principal bundle, with S as the base and the group G as the fiber. Elements in the configuration space are denoted q and elements in the shape space are denoted x.

A connection on this principal bundle defines a horizontal and vertical subspace of the tangent space of configurations at q, T_qQ. In particular, conservation of angular momentum defines a connection. Since were dealing only with the angular momentum = 0 case in this paper, it is easy to see that configuration changes that preserve angular momentum = 0 are orthogonal to configuration changes that correspond to rotations of the body. A tangent vector that corresponds to an infinitesimal rigid rotation is in the vertical subspace, and a tangent vector with angular momentum zero (an infinitesimal shape change) is horizontal.

The angular momentum is a defined as a vector valued one-form on Q [4]. Each deformation δq yields a vector $M(q)\delta q$ which is the angular momentum vector resulting from that configuration change. By this definition,

$$M(q)dq = g(I(x)g^{-1}dg + m(x)dx). \tag{4}$$

Here $m(x)$ is the angular momentum due to the shape change $s(x) \longrightarrow s(x + dx)$, and $I(x)$ is the inertia tensor for the shape $s(x)$, if all joints were locked, called the "locked inertia tensor". $g \in G$ is an element of the symmetry group $G = SE(3)$. The form $g^{-1}dg$ is the angular velocity with respect to a frame fixed in the body (rather than from the perspective of an outside observer).

Clearly, the kernel of the angular momentum one-form corresponds to shape changes paired with angular velocities so that the angular momentum caused by the shape change is compensated by angular velocity of the body as a whole. This is exactly what the cat needs to do: change its shape so that it acquires angular velocity to turn over.

If we set $M(q)dq = 0$, and multiply the angular momentum expression by $gI^{-1}g^{-1}$, we find that

$$dg + gI(x)^{-1}m(x)dx = 0. \tag{5}$$

This looks just like a control problem, which can be rewritten as

$$\begin{aligned} \dot{x} &= u \\ \dot{g} &= -gI(x)^{-1}m(x)(u), \end{aligned} \tag{6}$$

where u is the control – in our case the cat's decision to change its shape with derivative u. Montgomery [3] points out that the control problem is probably the natural framing of the problem from the cat's perspective. In particular, a blindfolded cat which cannot sense its orientation fails to turn over.

We can now define $\Gamma(x) = I(x)^{-1}m(x) : T_xS \rightarrow Lie(SO(3))$, the connection on the shape space that corresponds to the angular momentum connection on the configuration space. This is the "natural mechanical connection", which we will use in its explicit form for the model cat in the following section.

4 The Geometric Cat

In keeping with the common pattern that geometric ways of framing problems have simpler form, the coordinates used by Montgomery are much simpler than Kane and Scher's coordinates. The yz plane is defined by the principal axes of the two bodies that make up the cat, with the y direction defined as the direction of the bisector of the two body axes ("vertical" from the observers standpoint). The z direction is orthogonal to this, and horizontal from the observer's perspective. The shape of the cat can be determined by three angles (in keeping with the shape space being three dimensional, like $SO(3)$). The first is the angle ψ between the two body halves. The others are the angles θ_f and θ_b, which measure the orientation of the feet relative to the yz plane. The zeroes of θ_f and θ_b are set by requiring that when $\theta_b = \theta_f = 0$, the feet are pointing "up". See Fig. 3 for a sketch of this coordinate system.

Following Montgomery [4], I will start by assuming that the two bodies are free to rotate relative to each other (that is, it is a ball and socket joint, not a special no-twist joint). We first need to calculate $\Gamma = I^{-1}m$, the natural mechanical connection. For notational convenience I follow Montgomery and denote

$$\sin(\psi/2) = s \tag{7}$$

and

$$\cos(\psi/2) = c. \tag{8}$$

Then the angular momentum of the front body due to rotations along its central axis, where I_3 is the moment of inertia along this axis, is

$$m_f = I_3(s\mathbf{e}_3 + c\mathbf{e}_2)d\theta_f. \tag{9}$$

And similarly for the back body:

$$m_b = I_3(s\mathbf{e}_3 - c\mathbf{e}_2)(-d\theta_b). \tag{10}$$

The locked inertia tensor for the shape determined by ψ, θ_f and θ_b is:

$$I = 2diag(I_1 + ml^2c^2, I_1s^2 + I_3c^2 + ml^2s^2, I_1c^2 + I_3s^2), \tag{11}$$

where $I_1 = I_2$ are the moments of inertia for one of the bodies along axes perpendicular to the symmetry axis through the center of mass of that body. m is the mass of the body half (not to be confused with the angular momentum form $m(x)$), and l is the distance of the center of mass of a body half from the pivot point.

I will now directly calculate $\Gamma(x)$. First, for notational convenience, I will follow Montgomery and define $\alpha = I_1/I_3$ and $\beta = I_1/(ml^2)$ (not to be confused with the angles α and β in the discussion of Kane and Scher's model). Using this notation, we have

$$I = 2I_1 diag(1 + \beta c^2, s^2 + \alpha c^2 + \beta s^2, c^2 + \alpha s^2). \quad (12)$$

So,

$$\begin{aligned} I^{-1}m_f &= \frac{1}{2}\left[\frac{\alpha s}{c^2 + \alpha s^2}\mathbf{e}_3 + \frac{\alpha c}{s^2 + \alpha c^2 + \beta s^2}\mathbf{e}_2\right] d\theta_f \quad (13) \\ &= (\Phi_+\mathbf{e}_2 + \Phi_-\mathbf{e}_3)d\theta_f \quad (14) \end{aligned}$$

where we use this to define Φ_+ and Φ_-. Similarly,

$$I^{-1}m_b = (-\Phi_+\mathbf{e}_2 + \Phi_-\mathbf{e}_3)(-d\theta_b), \quad (15)$$

and therefore

$$\Gamma = (\Phi_+\mathbf{e}_2 + \Phi_-\mathbf{e}_3)d\theta_f + (-\Phi_+\mathbf{e}_2 + \Phi_-\mathbf{e}_3)(-d\theta_b). \quad (16)$$

The no-twist constraint corresponds to the cat's front and back feet always being turned the same angle from upright. That is, $\theta = \theta_f = -\theta_b$. We could add a constant here, since physiologically what that cat cannot do is move the front and back with different angular velocities, but we choose to let that constant equal zero, in keeping with the picture of the feet turned the same angle from the vertical. Adding this constraint into the calculation of the connection Γ, we see that for the fully characterized "realistic" cat,

$$\Gamma = 2\Phi_-\mathbf{e}_3 d\theta = \frac{\alpha s}{c^2 + \alpha s^2}\mathbf{e}_3 d\theta. \quad (17)$$

Note that Γ points only along the z direction, so the cat can only rotate around the z axis. This fact follows also if one calculates the angular momentum vectors for the two bodies, rotating in a way that preserves the no-twist constraint. The y components of the vectors cancel, leaving only a z component. Therefore, going back to thinking about the problem from the standpoint that the body rotates to keep zero total angular momentum, the cat's rotating of its body parts has only a z component of angular momentum and therefore the cat as a whole can only rotate around the z axis.

Now, using the translation between notation that Montgomery provides, we can show that this natural mechanical connection results in exactly the same motion as Kane and Schers model. Here is a translation table:

Kane and Scher	Montgomery
ψ	χ,angular measure of rotation around the z axis
θ	t, parameter value along curve
T	$\cos(\psi+\pi) = z$ component of curve
J/I	α
S	$\sqrt{2}\sin^2\psi\frac{d\theta}{dt}$
$1+T$	$2s^2$
$1-T$	$2c^2$

Starting with Kane and Scher's differential equation, we have:

$$\frac{d\psi}{d\theta} = \frac{(J/I)S}{(T-1)[1-T+(J/I)(1+T)](1+T)^{1/2}} \tag{18}$$

$$\Longleftrightarrow \frac{d\chi}{dt} = \frac{\alpha\sqrt{2}\sin^2\psi\frac{d\theta}{dt}}{(-2c^2)[2c^2+2\alpha s^2](2s^2)^{1/2}}. \tag{19}$$

Working through the algebra, we see that

$$d\chi = \frac{-\alpha\sin^2\psi d\theta}{(c^2+\alpha s^2)(4\cos^2(\psi/2)\sin(\psi/2))} \tag{20}$$

$$= \frac{-\alpha d\theta}{c^2+\alpha s^2}\frac{1-\cos\psi}{2\sin(\psi/2)} \tag{21}$$

$$= \frac{-\alpha d\theta}{c^2+\alpha s^2}\left[\frac{1}{\sqrt{2}}(1-\cos\psi)^{1/2}\right] \tag{22}$$

$$= \frac{-\alpha s d\theta}{c^2+\alpha s^2} \tag{23}$$

$$= -\Gamma, \tag{24}$$

which is precisely the dynamics from optimal control of our control system. So, we see that the outwardly complex Kane-Scher dynamics is simply a result of the natural geometric nature of the problem.

5 Related Problems

How a cat turns over in free fall with no angular momentum is only one of a class of related problems. Although I did not discuss it in detail, the geometric method described, when thought of as a control problem, is the problem of turning the cat with minimal power expenditure. This can be generalized as the "Cat's Problem": Given a deformable body in free fall with constant angular momentum, find the most efficient way to deform it and achieve a desired reorientation. This is also equivalent to the problem addressed by Shapere and Wilczek [5]. They studied the motion of amoeba in an infinite-viscosity limit, where shape changes can result in linear motion. Here the infinite viscosity of the fluid plays the role of angular momentum conservation by defining a connection.

Montgomery [2] also places the cat's problem in a larger context, as an "isoholonomic" problem: Among all the loops with a given holonomy, find the loop of minimum length. In the cat's case, the length is determined by the energy cost of motion, and the given holonomy is a rotation by π radians. If we have a metric for determining lengths, which we can restrict to the horizontal subspace, we then also have a "sub-Riemannian geodesic" problem: Find the horizontal curve joining two points whose length, as measured by the metric restricted to the horizontal subspace, is a minimum.

The cat's problem as posed by Kane and Scher is a complicated dynamical problem, requiring pages of algebra. However, by placing it in its geometric context, and doing the calculations as much as possible geometrically rather than in coordinates, the same result for the cat flip is obtained. In addition, the geometric method, by allowing the problem to be stated as a control problem, allows one to show that the motion is optimal.

References

[1] T.R. Kane and M.P. Scher. *A dynamical explanation of the falling cat phenomenon*, Int'l J. Solids and Structures 5, 663-670, 1969.

[2] R. Montgomery, *Isoholonomic Problems and Some Applications*, Comm. Math. Phys., vol. 128, 565-592, 1990.

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[4] R. Montgomery, *Gauge Theory of the Falling Cat*, Fields Inst. Comm., vol. 1, 193-218, 1993

[5] A. Shapere and F. Wilczek, *Self-Propulsion at Low Reynolds Number*, PRL 58 20, 2051-2054, 1987

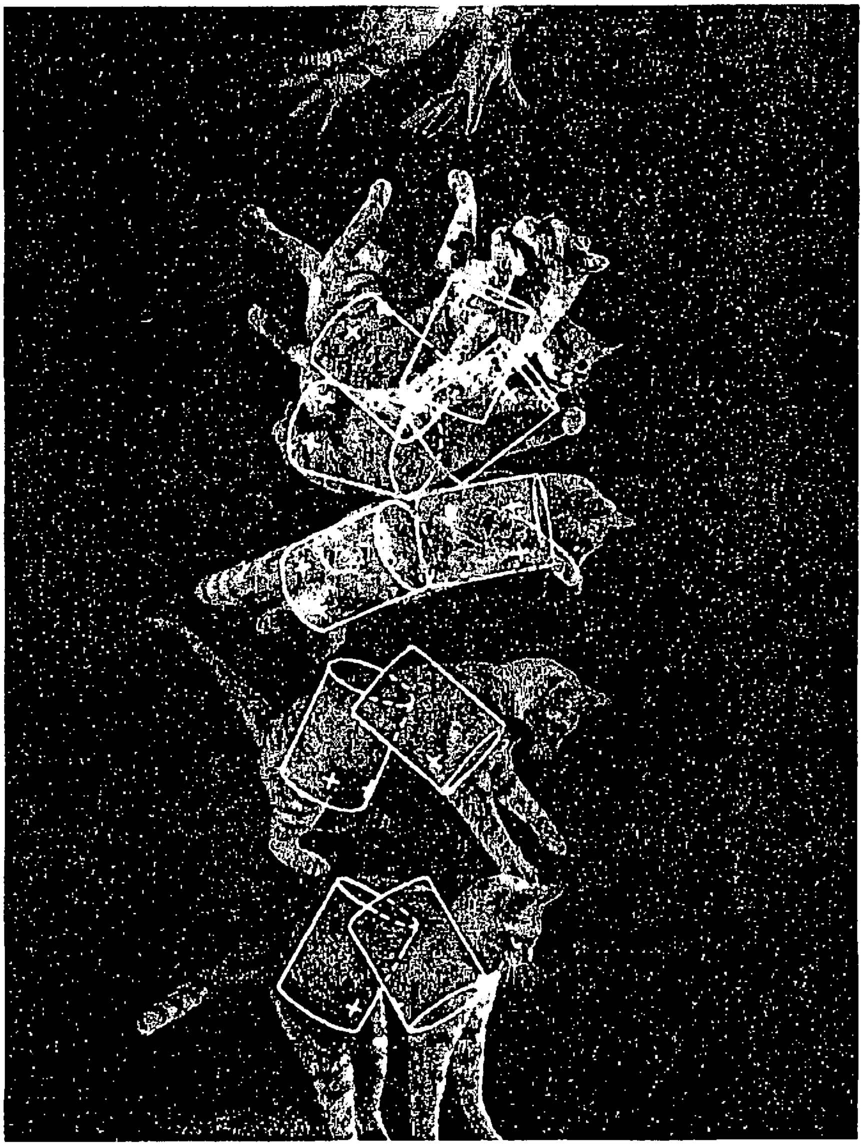

Figure 1. Reproduced figure from Kane and Scher [1], showing the success of their model in reproducing cat dynamics

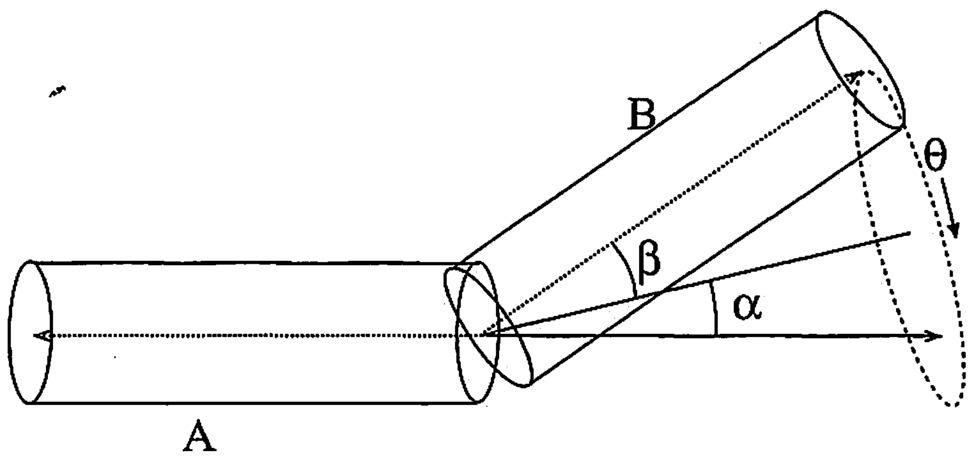

Figure 2. Kane-Scher coordinates (as much as needed for the discussion in this project)

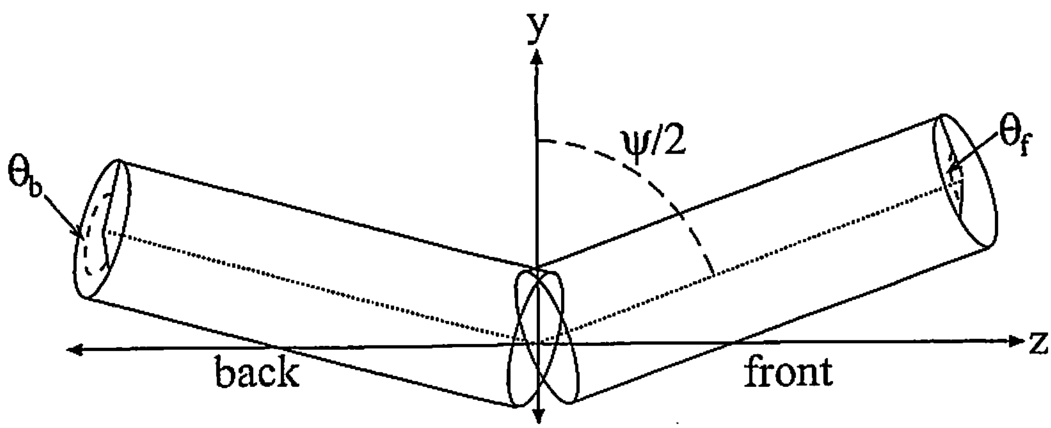

Figure 3. Montgomery's coordinates