# ANOTHER TWIST TO BERRY'S BEAD ON A HOOP

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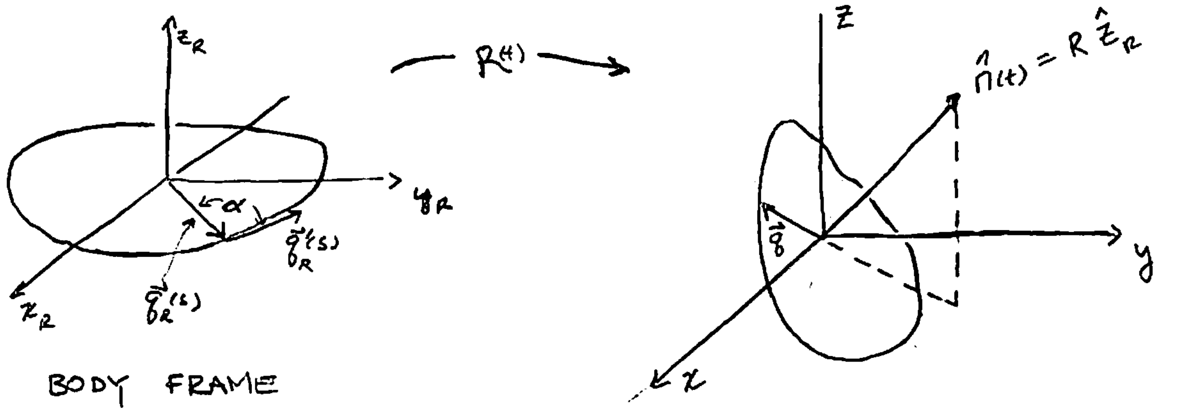
Very Nice!

I HAVE FOUND THAT THE HANNAY-BERRY PHASE FOR THE BEAD ON A PLANAR HOOP WHICH IS ROTATED ARBITRARILY ABOUT A FIXED POINT WITHIN THE HOOP TO BE:

$$\langle s(T) - s_0 - v_0 T \rangle = -\frac{2A}{L}\int_0^T \hat{n}(t')\cdot\vec{\omega}(t')\,dt' \qquad (1)$$

HERE $\hat{n}(t')$ IS THE UNIT VECTOR NORMAL TO THE HOOP AT THE ORIGIN IN THE INERTIAL FRAME. AS THE HOOP ROTATES $\hat{n}$ SWEEPS OUT A PATH ON A UNIT SPHERE. $\vec{\omega}$ IS THE ANGULAR VELOCITY IN THE INERTIAL FRAME AS WELL. WE HAVE $\frac{d\hat{n}}{dt} = \vec{\omega}\times\hat{n}$.

I NOW BEGIN VERIFICATION OF THE ABOVE FORMULA WITH SOME PICTURES AND NOTATIONAL DEFINITIONS.




BODY FRAME

INERTIAL FRAME

IN THE INERTIAL FRAME THE BEAD'S POSITION IS GIVEN BY $\vec{q} = R(t)\,\vec{q}_R(s(t))$. $R(t)$ IS AN ARBITRARY ROTATION. IN THE BODY SYSTEM $\vec{q}_R$ LOCATES THE BEAD.

DIFFERENTIATING $\vec{q} = R\vec{q}_R$ AND USING $\hat{\omega} = \dot{R}R^{-1}$ I GET AN EXPRESSION FOR $\dot{\vec{q}}$:

$$\dot{\vec{q}} = \dot{R}\,\vec{q}_R + R\,\vec{q}_R{}'\,\dot{s} = \dot{R}R^{-1}R\,\vec{q}_R + R\,\vec{q}_R{}'\,\dot{s} = \vec{\omega}\times(R\vec{q}_R) + R\,\vec{q}_R{}'\,\dot{s}$$

$$= R\left[(R^{-1}\vec{\omega})\times\vec{q}_R\right] + R\,\vec{q}_R{}'\,\dot{s} = R\left\{\vec{\omega}_R\times\vec{q}_R + \vec{q}_R{}'\,\dot{s}\right\} \qquad (2)$$

THE LAGRANGIAN IS:

$$L = \tfrac{1}{2}M\left\|\vec{\omega}_R\times\vec{q}_R + \vec{q}_R{}'\,\dot{s}\right\|^2$$

AND EULER'S EQUATIONS ARE: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{s}} = \dfrac{\partial L}{\partial s}$.

CALCULATING THESE DERIVATIVES I GET:

$$\frac{\partial L}{\partial \dot{s}} = M(\vec{\omega}_R\times\vec{q}_R + \vec{q}_R{}'\,\dot{s})\cdot\vec{q}_R{}' = M\left[\vec{\omega}_R\times\vec{q}_R\cdot\vec{q}_R{}' + \dot{s}\right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = M\left[\ddot{s} + (\dot{\vec{\omega}}_R\times\vec{q}_R)\cdot\vec{q}_R{}' + \vec{\omega}_R\times\vec{q}_R\cdot\vec{q}_R{}''\,\dot{s}\right]$$

$$\frac{\partial L}{\partial s} = M(\vec{\omega}_R\times\vec{q}_R + \vec{q}_R{}'\,\dot{s})\cdot(\vec{\omega}_R\times\vec{q}_R{}' + \vec{q}_R{}''\,\dot{s})$$

$$= M\left[(\vec{\omega}_R\times\vec{q}_R)\cdot(\vec{\omega}_R\times\vec{q}_R{}') + \vec{\omega}_R\times\vec{q}_R\cdot\vec{q}_R{}''\,\dot{s}\right]$$

$$= M\left[\tfrac{1}{2}\frac{d}{ds}\left\|\vec{\omega}_R\times\vec{q}_R\right\|^2 + \vec{\omega}_R\times\vec{q}_R\cdot\vec{q}_R{}''\,\dot{s}\right]$$

Substitution INTO EULER'S EQUATION YIELDS

$$\boxed{\ddot{s} = -(\dot{\vec{\omega}}_R\times\vec{q}_R)\cdot\vec{q}_R{}' + \tfrac{1}{2}\frac{d}{ds}\left\|\vec{\omega}_R\times\vec{q}_R\right\|^2} \qquad (3)$$

NOW I CAN RE WRITE $(\dot{\vec{\omega}}_R\times\vec{q}_R)\cdot\vec{q}_R{}' = (\vec{q}_R\times\vec{q}_R{}')\cdot\dot{\vec{\omega}}_R = |\vec{q}_R|\,|\vec{q}_R{}'|\sin(\pi-\alpha)\,\hat{z}_R\cdot\dot{\vec{\omega}}_R$

$$(\dot{\vec{\omega}}_R\times\vec{q}_R)\cdot\vec{q}_R{}' = q_R\sin\alpha\;\hat{z}_R\cdot\dot{\vec{\omega}}_R$$

HERE I'VE CHOSEN THE BEAD TO TRAVEL COUNTER CLOCKWISE WHEN VIEWED FROM THE POSITIVE $z_R$ AXIS.

SUBSTITUTION OF THE PREVIOUS EXPRESSION INTO THE $\ddot{s}$ EQUATION GIVES:

$$\boxed{\ddot{s} = -g_R \sin\alpha \;\hat{z}_R \cdot \dot{\vec{\omega}}_R + \frac{1}{2}\frac{d}{ds}\|\vec{\omega}_R \times \vec{g}_R\|^2} \qquad (4)$$

NOW USE TAYLOR'S FORMULA WITH REMAINDER:

$$s(t) = s_0 + \dot{s}_0 t + \int_0^t (t-t')\left\{\frac{1}{2}\frac{d}{ds}\|\vec{\omega}_R(t') \times \vec{g}_R(s(t'))\|^2 - g_R(s(t'))\sin\alpha(s(t'))\;\hat{z}_R \cdot \dot{\vec{\omega}}_R(t')\right\} dt'$$

AVERAGING ON S DEPENDENT QUANTITIES:

$$S(T) \cong s_0 + \dot{s}_0 T + \int_0^T (T-t')\,dt' \left\{\frac{1}{2L}\int_0^L \frac{d}{ds}\|\vec{\omega}_R(t') \times \vec{g}_R(s)\|^2 ds - \frac{\hat{z}_R \cdot \dot{\vec{\omega}}_R(t')}{L}\int_0^L g_R(s)\sin\alpha(s)\,ds\right\}$$

but $\int_0^L \frac{d}{ds}\|\vec{\omega}_R \times \vec{g}_R(s)\|\,ds = 0$

AND $\int_0^L g_R(s)\sin\alpha(s)\,ds = 2A$, A = AREA OF HOOP.

$$\Rightarrow \quad S(T) \simeq s_0 + \dot{s}_0 T - \int_0^T (T-t')\;\hat{z}_R \cdot \dot{\vec{\omega}}_R(t')\left(\frac{2A}{L}\right) dt'$$

NOW INTEGRATING BY PARTS:

$$\int_0^T (T-t')\;\hat{z}_R \cdot \dot{\vec{\omega}}_R\,dt' = \int_0^T \left[\frac{d}{dt'}\left\{(T-t')\;\hat{z}_R \cdot \vec{\omega}_R\right\} + \hat{z}_R \cdot \vec{\omega}_R\right] dt'$$

$$= -T\,\hat{z}_R \cdot \vec{\omega}_R(0) + \int_0^T \hat{z}_R \cdot \vec{\omega}_R(t')\,dt'$$

$$\Rightarrow \quad \boxed{S(T) \simeq s_0 + \dot{s}_0 T + \frac{2AT}{L}\hat{z}_R \cdot \vec{\omega}_R(0) - \frac{2A}{L}\int_0^T \hat{z}_R \cdot \vec{\omega}_R(t')\,dt'} \qquad (5)$$

NOW I DEFINE THE UNIT VECTOR $\hat{n}$ BY $\hat{n}(t) = R(t)\,\hat{z}_R$. THIS UNIT VECTOR $\hat{n}$ IS NORMAL TO THE HOOP IN THE INERTIAL FRAME:


BODY FRAME INERTIAL

SINCE THE 'DOT PRODUCT' IS A SCALAR WITH RESPECT TO ROTATIONS:

$$\int_0^T \hat{z}_R \cdot \vec{\omega}_R(t')\,dt' = \int_0^T \hat{n}(t') \cdot \vec{\omega}(t')\,dt'$$

SO AS $\hat{n}$ TRACES OUT A PATH ON A UNIT SPHERE IN THE INERTIAL FRAME ONLY COMPONENTS OF $\vec{\omega}$ ALONG $\hat{n}$ CONTRIBUTE TO THE BERRY'S PHASE.

IF WE CALCULATE THE INITIAL VELOCITY OF THE BEAD ALONG THE HOOP RELATIVE TO THE INERTIAL FRAME WE GET:

$$V_0 = \left[\vec{\omega}_R(0) \times \vec{g}_R(s_0) + \vec{g}_R{}'(s_0)\,\dot{s}_0\right] \cdot \vec{g}_R{}'(s_0)$$

$$= \vec{\omega}_R(0) \cdot \vec{g}_R(s_0) \times \vec{g}_R{}'(s_0) + \dot{s}_0 = g_R(s_0)\sin\alpha(s_0)\;\hat{z}_R \cdot \vec{\omega}_R(0) + \dot{s}_0 \qquad (6)$$

AVERAGING OVER INITIAL CONDITIONS:

$$\langle S(T) - S_0 - V_0 T \rangle = \frac{2AT}{L}\,\hat{z}_R \cdot \vec{\omega}_R(0) - T\,\hat{z}_R \cdot \vec{\omega}_R(0)\,\frac{1}{L}\int_0^L g_R(s_0)\sin\alpha(s_0)\,ds_0 \qquad (7)$$
$$- \frac{2A}{L}\int_0^T \hat{z}_R \cdot \vec{\omega}_R(t')\,dt'$$

$$\langle S(T) - S_0 - V_0 T \rangle = -\frac{2A}{L}\int_0^T \hat{z}_R \cdot \vec{\omega}_R(t')\,dt' = -\frac{2A}{L}\int_0^T \hat{n}(t') \cdot \vec{\omega}(t')\,dt'$$

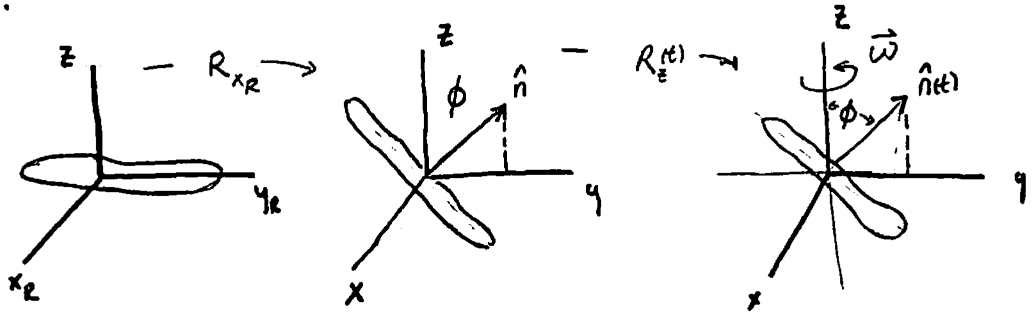
THUS, I ARRIVE AT EQUATION (1):

$$\boxed{\langle S(T) - S_0 - V_0 T \rangle = -\frac{2A}{L} \int_0^T \hat{n}(t') \cdot \vec{\omega}(t')\, dt'}$$

NOTE: $\hat{n} = R\hat{z}_R$

$$\dot{\hat{n}} = \dot{R}\hat{z}_R = \dot{R}R^{-1}R\hat{z}_R = \vec{\omega} \times \hat{n} \quad \Rightarrow \quad \frac{d\hat{n}}{dt} = \vec{\omega} \times \hat{n}$$

NOW LET'S LOOK AT A SPECIAL CASE. LET'S ROTATE THE BODY FRAME BY A CONSTANT ANGLE $\phi$ ABOUT THE $x$ AXIS [ASSUME $x$ + $x_R$ ARE INITIALLY ALIGNED]. THEN ROTATE THE SYSTEM IN TIME ABOUT THE Z-AXIS ONCE:




THE ROTATION R IS $R = R_z(t) R_{x_R}$ WHERE $R_{x_R}$ IS A CONSTANT ROTATION WHICH PLACES BODY IN INITIAL ORIENTATION.

THE INTEGRAL OF EQUATION (1) IS:

$$\int_0^T \hat{n}(t') \cdot \vec{\omega}(t')\, dt' = \int_0^T \cos\phi \; \omega(t')\, dt' = \cos\phi \int_0^T \frac{d\theta}{dt'}\, dt' = 2\pi \cos\phi$$

THUS AFTER ONE COMPLETE ROTATION:

$$\langle S(T) - S_0 - V_0 T \rangle = -\frac{4\pi A}{L} \cos\phi \qquad (8)$$

FOR $\phi = 0$ THIS REDUCES TO THE EXPECTED RESULT FOUND IN CLASS.

ONE OTHER CASE WORTH EXAMINING IS THAT OF A CIRCULAR HOOP. LOOKING OVER THE PREVIOUS CALCULATIONS WE SEE THAT THE AVERAGING OVER INITIAL CONDITIONS IS NOT NECESSARY IN EQUATION (7). WE CAN WRITE FOR A CIRCLE OF RADIUS $R$:

$$S(T) - S_0 - V_0 T = -R \int_0^T \hat{n}(t') \cdot \vec{w}(t')\, dt' \tag{9}$$

IF WE ROTATE AS IN THE PREVIOUS EXAMPLE THIS BECOMES:

$$S(T) - S_0 - V_c T = -R 2\pi \cos\phi = -L \cos\phi \tag{10}$$

THIS GIVES AN ANGULAR SHIFT ON THE HOOP AFTER ONE ROTATION OF:

$$\theta = -\frac{L}{R}\cos\phi = -2\pi\cos\phi \tag{11}$$

IF WE TAKE A BICYCLE WHEEL FOR EXAMPLE AND SET IT SPINNING RAPIDLY AND THEN ROTATE ITS AXIS ONCE ABOUT A CONE WITH APEX ANGLE OF $2\phi$ I would expect [IF IGNORING GRAVITY IS PERMISSIBLE] THE wheel to suffer AN ANGULAR SHIFT of $-2\pi\cos\phi$.

EQUATION (11) looks A lot LIKE THE PHASE SHIFT of THE FOUCAULT PENDULUM. A BEAD ON A circular Hoop IS much Like A pendulum SWEEPING out Horizontal circles. However the Foucault pendulum bob would lie IN A PLANE WHICH UNDERGOES ADDITIONAL TRANSLATIONAL MOTION ABOUT THE EARTH.

What I MIGHT TRY NEXT IS to CONSIDER WHAT HAPPENS TO A PARTICLE CONFINED TO A PLANE WHICH ROTATES ABOUT A POINT WITH A CENTRAL POTENTIAL.