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CDS 205

final project.

project: to study the Kelvin-Noether theorem and its applicability to fluid mechanics.

Sources:

1) Holm, D.D., Marsden, J.E., and Ratiu, T.S.
"The Euler-Poincaré Equations and Semidirect Products with Applications to Continuum Theories"
Advances in Mathematics 137, 1-81 (1998)

2) Saffman, P.G. Vortex Dynamics. 1992.

3) Arnold, V.I. Mathematical Methods of Classical Mechanics. 1989.

4) Arnold, V.I. Topological Methods in Hydrodynamics. 1998.

Good Project!
A few bits were a little hard to follow.

Arnold: " To every one-parameter group of diffeomorphisms of the configuration manifold of a Lagrangian system which preserves the Lagrangian function, there corresponds a first integral of the equations of motion.

Suppose M is a smooth manifold, $L: TM \to \mathbb{R}$ a smooth fcn on its tangent bundle TM, and $h: M \to M$ a smooth map.

Noether's theorem. If (M, L) is such that for

$$h^s: M \to M, \quad s \in \mathbb{R},$$

$$L(h^s_* v) = L(v),$$

then the Lagrangian system of eqns corresponding to L has a first integral $I: TM \to \mathbb{R}$.

where $I(q, \dot q) = \frac{\partial L}{\partial \dot q} \left. \frac{d h^s(q)}{ds} \right|_{s=0}$ in local coordinates.

For example, consider a system of pt masses with masses m_i:

$$L = \sum m_i \frac{\dot x_i^2}{2} - U(x) \qquad x_i = x_{i_1} e_1 + x_{i_2} e_2 + x_{i_3} e_3$$

$f_j(x) = 0$ constraints.

suppose $L(h^s_* \dot x_i) = L(v)$ for $h^s: x_i \to x_i + s e_1$ (translation)

So its center of mass by Noether's thm, moves uniformly as projected onto the e_1-axis:

$$\frac{d}{ds} h^s(x_i)\Big|_{s=0} = e_1$$

and $$I = \Sigma \frac{\partial L}{\partial \dot{x}_i} e_1 = \Sigma m_i \dot{x}_i,$$

which is a restatement of the conclusion from Noether's thm.

Consider the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = ad_\xi^* \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a$$

where $ad_\xi^* : \mathfrak{g}^* \to \mathfrak{g}^*$

$ad_\xi^* = \frac{d}{dt} Ad^*_{g(t)}\Big|_{t=0}$

$Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ $\qquad g \in G$

$(Ad_g^* \xi)(\omega) = \xi(Ad_g \omega)$ $\qquad \xi \in \mathfrak{g}^*$, $\omega \in \mathfrak{g}$

$Ad_g a = (A_{g^{-1}}|_e) a$ $\qquad a \in \mathfrak{g} = T_e G$

$= T_e(R_{g^{-1}} L_g)\, a$

$v \diamond a = \underset{\text{dual}}{\rho_v^*} a \in \mathfrak{g}^*$

$\rho_v(\xi) = \xi v$

Suppose in an infinite dimensional system we consider solutions of the Euler-Poincaré equations. What is an equivalent version of Noether's theorem?

now consider a Lagrangian $L_{a_0} : TG \to \mathbb{R}$

$$L_{a_0}(v_g) = L(v_g, a_0) \qquad a_0 \in V^*$$

by left-invariance, define

$$l : \mathfrak{g} \times V^* \to \mathbb{R}$$

$$l(g^{-1} v_g, g^{-1} a_0) = L(v_g, a_0)$$

and a manifold M and an equivariant map $h : M \times V^* \to \mathfrak{g}^{**}$

now define the Kelvin-Noether quantity $I : M \times \mathfrak{g} \times V^* \to \mathbb{R}$

$$I(m, \xi, a) = \left\langle h, \frac{\delta l}{\delta \xi} \right\rangle \qquad m \in M$$

we seek a theorem of the form:

<u>Thm 4.1</u> Kelvin-Noether.

fix $m_0 \in M$ and let ξ, a satisfy the E-P equations.

with $g(t)$ the solution of $\dot{g}(t) = g(t)\xi(t)$, $g(0) = e$

$m(t) = g^{-1}(t) m_0$

then $\frac{dI}{dt} = \left\langle h, \frac{\delta l}{\delta a} \diamond a \right\rangle$

<u>Proof</u>. set $a(t) = g^{-1}(t) a_0$

and $\left\langle h(m(t), a(t)), \frac{\delta l}{\delta \xi}(\xi(t), a(t)) \right\rangle$

$$= \left\langle h(m_0, a_0), g(t) \left[\frac{\delta l}{\delta \xi}(\xi(t), a(t)) \right] \right\rangle$$

this is possible through equivariance of h. note use of left representation.

now, $$\frac{dI}{dt} = \frac{d}{dt}\left\langle h(m(t), a(t)),\ \frac{\delta l}{\delta \xi}(\xi(t), a(t))\right\rangle$$

$$= \frac{d}{dt}\left\langle h(m_0, a_0),\ g(t)\left[\frac{\delta l}{\delta \xi}(\xi(t), a(t))\right]\right\rangle$$

$$= \left\langle h(m_0, a_0),\ \frac{d}{dt}\left\{ g(t)\left[\frac{\delta l}{\delta \xi}(\xi(t), a(t))\right]\right\}\right\rangle$$

now, $$\frac{d}{dt}(g\mu) = g(t)\left[-\mathrm{ad}^*_{\xi(t)}\mu(t) + \frac{d}{dt}\mu(t)\right]$$

$$\xi(t) = g^{-1}(t)\,\dot{g}(t)$$

so $$\frac{dI}{dt} = \left\langle h(m_0, a_0),\ g(t)\left[-\mathrm{ad}_\xi \frac{\delta l}{\delta \xi} + \frac{d}{dt}\frac{\delta l}{\delta \xi}\right]\right\rangle$$

assuming that the Euler-Poincaré eqs are satisfied,

$$\frac{dI}{dt} = \left\langle h(m_0, a_0),\ g(t)\left[-\mathrm{ad}_\xi \frac{\delta l}{\delta \xi} + \left(\mathrm{ad}_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a\right)\right]\right\rangle$$

$$\frac{dI}{dt} = \left\langle h(m_0, a_0),\ g(t)\left[\frac{\delta l}{\delta a} \diamond a\right]\right\rangle$$

$$\frac{dI}{dt} = \left\langle g^{-1}(t)\, h(m_0, a_0),\ \frac{\delta l}{\delta a} \diamond a\right\rangle$$

and by equivariance,

$$\frac{dI}{dt} = \left\langle h(m(t), a(t)),\ \left[\frac{\delta l}{\delta a} \diamond a\right]\right\rangle$$

Kelvin's circulation theorem.

Consider the circulation $\Gamma(t)$ around a closed curve $C(t)$.

$$\Gamma(t) = \oint_{C(t)} u \cdot ds = \oint u_\sigma \cdot \frac{\partial s_\sigma}{\partial \sigma}\, d\sigma$$

velocity

$$\text{so} \quad \frac{d\Gamma}{dt} = \oint \underbrace{\frac{du_\sigma}{dt}}_{\frac{\partial u}{\partial t} + u \cdot \nabla u} \cdot \frac{\partial s_\sigma}{\partial \sigma}\, d\sigma + \oint u_\sigma \cdot \underbrace{\frac{d}{dt}\left(\frac{\partial s_\sigma}{\partial \sigma}\right)}_{d(ds_\sigma/dt)} d\sigma$$

$$\frac{d\Gamma}{dt} = \oint \left(-\frac{1}{\rho}\nabla p + F\right) \cdot ds + \oint (u \cdot (ds \cdot \nabla) u)$$

from conservation of momentum

$$= \oint \left(-\frac{1}{\rho}\frac{\partial p}{\partial \sigma} + \frac{\partial}{\partial \sigma}\left(\tfrac{1}{2}u^2\right)\right) d\sigma + \oint F \cdot ds$$

barotropic

$$\text{so} \quad \frac{d\Gamma}{dt} = \oint F \cdot ds \qquad \left(= 0 \text{ if } F = \frac{\partial \nu}{\partial \sigma}\right)$$

so the circulation around a material circuit in an ideal fluid is an invariant of the motion.

we seek to view this theorem in the light of the Kelvin - Noether theorem.

<u>Theorem 6.2</u> Kelvin circulation theorem

now, the momentum eqn is equivalent to the velocity $v(x,t)$

satisfying the Euler - Poincaré equations:

$$\frac{\partial}{\partial t}\left(\frac{\delta l}{\delta v}\right) = - \mathcal{L}_v\left(\frac{\delta l}{\delta v}\right) + \frac{\delta l}{\delta a} \diamond a$$

Lie derivative

$$\left(\text{note,} \quad \frac{\partial}{\partial t}\frac{\delta l}{\delta v} + \mathcal{L}_v \frac{\delta l}{\delta v} = \frac{\delta l}{\delta a} \diamond a \quad \approx \quad ma = F. \right)$$

where $\frac{\partial a}{\partial t} + \mathcal{L}_v a = 0$.

let η_t be the flow of v:

$$v_t = \frac{d\eta_t}{dt} \circ \eta_t^{-1}$$

for $\gamma_t = \eta_t \circ \gamma_0$,

$$I(t) = I(\gamma_t, v_t, a_t),$$

then $\frac{d}{dt} I = \oint \frac{1}{\rho} \frac{\delta l}{\delta a} \diamond a$

$$\left(\text{compare:} \quad \frac{d\Gamma}{dt} = \oint F \cdot ds \right)$$

$\longrightarrow$

Proof of Thm 6.2

let C be the space of continuous loops $\gamma : S^1 \longrightarrow D$

and $(\eta,\gamma) \in \mathrm{Diff}(D) \times C \mapsto \eta \circ \gamma = \eta\gamma \in C$

define the circulation map

$$K : C \times V^* \longrightarrow \mathfrak{X}(D)^{**}$$

$$\langle K(\gamma, a), \alpha \rangle = \oint_\gamma \frac{\alpha}{\rho} \quad \vdots \quad \text{circulation.}$$

note that this map is equivariant:

$$\langle K(\eta\circ\gamma, \eta_* a), \eta_* \alpha \rangle$$

$$= \langle \eta \cdot K(\gamma, a), \eta \cdot \alpha \rangle$$

$$= \langle K(\gamma, a), \alpha \rangle$$

now, $I(t) = \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta l}{\delta v}$

$$= \oint_{\gamma_0} \eta_t^* \left[\frac{1}{\rho_t} \frac{\delta l}{\delta v} \right]$$

$$= \oint_{\gamma_0} \rho_0 \, \eta_t^* \left[\frac{\delta l}{\delta v} \right]$$

now, $\frac{d}{dt} (\eta_t^* \alpha_t) = \eta_t^* \frac{d\alpha_t}{dt}$

$$= \eta_t^* \left(\frac{\partial \alpha_t}{\partial t} + \pounds_v \alpha_t \right)$$

so $$\frac{dI}{dt} = \frac{d}{dt} \oint_{\gamma_0} \frac{1}{\rho_0} \eta_t^* \left[\frac{\delta l}{\delta v}\right]$$

$$= \oint_{\gamma_0} \frac{1}{\rho_0} \frac{d}{dt} \left(\eta_t^* \left[\frac{\delta l}{\delta v}\right] \right)$$

$$= \oint_{\gamma_0} \frac{1}{\rho_0} \eta_t^* \left[\frac{\partial}{\partial t} \frac{\delta l}{\delta v} + \mathcal{L}_v \left(\frac{\delta l}{\delta v} \right) \right]$$

now, $\frac{\partial}{\partial t} \frac{\delta l}{\delta v} = - \mathcal{L}_v \frac{\delta l}{\delta v} + \frac{\delta l}{\delta a} \diamond a$ by the Euler-Poincaré equations:

$$\frac{dI}{dt} = \oint_{\gamma_0} \frac{1}{\rho_0} \eta_t^* \frac{\delta l}{\delta a} \diamond a$$

so $$\frac{dI}{dt} = \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta l}{\delta a} \diamond a$$

now, $$\frac{d}{dt} \oint_{\gamma_t} \frac{1}{\rho} \frac{\delta l}{\delta v} = \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta l}{\delta a} \diamond a$$

so $$\oint_{\gamma_t} \frac{d}{dt} \frac{1}{\rho} \frac{\delta l}{\delta v} = \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta l}{\delta a} \diamond a$$

so $$\frac{d}{dt} \frac{1}{\rho} \frac{\delta l}{\delta v} = \frac{1}{\rho} \frac{\delta l}{\delta a} \diamond a$$

or $$\left(\frac{\partial}{\partial t} + \mathcal{L}_v \right) \frac{1}{\rho} \frac{\delta l}{\delta v} = \frac{1}{\rho} \frac{\delta l}{\delta a} \diamond a .$$

Kelvin-Noether form of the Euler-Poincaré equations

An ideal incompressible fluid.

consider the reduced action R and reduced Lagrangian $l(v,D)$:

$$R = \int dt\, l = \int dt \int d^n x \left[\tfrac{1}{2} D|v|^2 - p(D-1) \right]$$

note,

$$\frac{1}{D}\frac{\delta l}{\delta v} = v$$

$$\frac{\delta l}{\delta D} = \tfrac{1}{2}|v|^2 - p$$

$$\frac{\delta l}{\delta p} = -(D-1)$$

From the Kelvin-Noether form,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_v\right)\left(\frac{1}{D}\frac{\delta l}{\delta v}\cdot dx\right) - \nabla\left(\frac{\delta l}{\delta D}\right)\cdot dx = 0$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_v\right)(v\cdot dx) - \nabla\left(\tfrac{1}{2}|v|^2 - p\right)\cdot dx = 0$$

$$\text{so} \quad \frac{\partial v}{\partial t} + (v\cdot\nabla)v + \nabla p = 0 \qquad \text{Euler equation.}$$

Constrain $D=1$ for incompressibility:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_v\right) D\, d^n x = 0 \implies \frac{\partial D}{\partial t} = -\nabla\cdot Dv$$

$$=$$

$$\implies \nabla\cdot v = 0 .$$