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'M 189
Final Project.

1990 (?) ①

A Simple Calculation of the Born-Pauli-Infeld Bracket for Maxwell's Equations, Paying Attention to Boundary Terms.

Following the notes, I consider Q = vector fields A on $\mathbb{R}^3$. Since this is a linear space, I can identify $T^*Q \approx Q \times Q$ by pairing an element Y in Q^*, also a vector field in $\mathbb{R}^3$, with an element A via the dot product and integration,

$$\langle Y, A \rangle \equiv \int_V A \cdot Y \, d^3x .$$

Now specify Ω to be the canonical symplectic form (1.1.81)

$$\Omega\left[(A_1, Y_1), (A_2, Y_2)\right] = \int_V A_1 \cdot Y_2 \, d^3x - \int A_2 \cdot Y_1 \, d^3x$$

And specify the Hamiltonian

$$H \equiv \frac{1}{2}\int_V \left(\|Y\|^2 + \|\nabla \times A\|^2\right) d^3x .$$

Then the Hamiltonian vector field is determined by

$$\Omega\left[(X_A, X_Y), (\delta A, \delta Y)\right] = dH \cdot \delta Z = \left\langle \frac{\delta H}{\delta A}, \delta A \right\rangle + \left\langle \delta Y, \frac{\delta H}{\delta Y} \right\rangle$$

Evaluating both sides gives

$$\int_V X_A \cdot \delta Y \, d^3x - \int_V X_Y \cdot \delta A \, d^3x = \int_V Y \cdot \delta Y \, d^3x + \int_V (\nabla \times A) \cdot (\nabla \times \delta A) \, d^3x$$

$$= \int_V Y \cdot \delta Y \, d^3x - \int_V (-\nabla \times (\nabla \times A)) \cdot \delta A \, d^3x - \int_V \nabla \cdot [(\nabla \times A) \times \delta A] \, d^3x$$

$$= \int_V Y \cdot \delta Y \, d^3x - \int_V [-\nabla \times (\nabla \times A)] \cdot \delta A \, d^3x - \int_S [n \times (\nabla \times A)] \cdot \delta A \, d^2x$$

Thus we see that provided

$$\int_S [n \times (\nabla \times A)] \cdot \delta A \, d^2x = 0 \Rightarrow n \times (\nabla \times A) = 0 \quad \text{on the boundary,}$$

$$X_A = \frac{\partial A}{\partial t} = Y \quad , \quad X_Y = \frac{\partial Y}{\partial t} = -\nabla \times (\nabla \times A)$$

Identifying $B = \nabla \times A$, $Y = -E$ gives

$$E = -\frac{\partial A}{\partial t} \quad , \quad \nabla \times B = \frac{\partial E}{\partial t} .$$

Note: Generally, $E = -\nabla \Phi - \frac{\partial A}{\partial t}$, where Φ is the scalar potential.

As things stand, the first equation is clearly gauge dependent. Therefore one has to take the curl of the first equation to obtain the correct Maxwell's equations

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad , \quad \nabla \times B = \frac{\partial E}{\partial t} .$$

Why is this extra step necessary? Since H is gauge invariant, why aren't the naive equations of motion?.

To examine the implications of the boundry condition $n \times (\nabla \times A) = 0$, I note first that if I consider V to be all of space and demand $A \to 0$ at ∞, then the condition is satisfied. Consider now the case of a conducting cavity. Here the boundry conditions are

$$n \times (\nabla \times A) = 4\pi K \qquad n \cdot (\nabla \times A) = 0 \qquad K = \text{surface current}$$

$$-n \cdot Y = 4\pi\sigma \qquad n \times Y = 0 \qquad \sigma = \text{surface charge}$$

in conductor.

Thus, in a conducting cavity, the B-P-I bracket will yield the correct equations of motion only if $K = 0$. This makes perfect sense, since there will be energy associated with such a surface current, namely

$$H_{surf} = \frac{1}{2} \int_s K \cdot A \, d^2x,$$

which has not been included in the Hamiltonian (Why should the Hamiltonian be the total energy?). Including such a surface current in the model would require modelling the current, perhaps as a 2-D plasma, and the inclusion of a coupling between the currents and the fields (i.e., the Lorentz force law). This procedure would be somewhat complicated by the fact that the plasma is 2-D, so that motion is restricted to a surface. Unfortunately, this project is due in 45 minutes, which makes working out the details of such a model impractical.

Assuming that the boundry condition $n \times (\nabla \times A)$ is met, we see that

$$X_H = \left(\frac{\delta H}{\delta Y}, -\frac{\delta H}{\delta A} \right).$$

Then the poisson bracket is

$$\{F, K\} = \Omega(X_F, X_K) = \int_V \frac{\delta F}{\delta Y}\left(-\frac{\delta K}{\delta A}\right) d^3x - \int_V \left(-\frac{\delta F}{\delta A}\right)\cdot\frac{\delta K}{\delta Y}\, d^3x$$

$$\boxed{\{F, K\} = \int_V \left(\frac{\delta F}{\delta A}\cdot\frac{\delta K}{\delta Y} - \frac{\delta F}{\delta Y}\cdot\frac{\delta K}{\delta A}\right) d^3x}$$

To produce the B-P-I bracket, one uses reduction as described in the notes:

H has a symmetry, namely $G = \mathcal{F}(\mathbb{R}^3)$, the space of functions on $\mathbb{R}^3$, and

$$\Phi_g : (A, Y) \mapsto (A + \nabla g, Y).$$

$\mathcal{G}$ is also $\mathcal{F}(\mathbb{R}^3)$. Letting $\xi \in \mathcal{G}$,

$$\xi_p = \frac{d}{dt}\left[(A + \nabla e^{t\xi}, Y)\right]_{t=0}$$

$$= \frac{d}{dt}\left[(A + (t\nabla\xi)e^{t\xi}, Y\right]_{t=0}$$

$$= (\nabla\xi e^{t\xi} + \xi(t\nabla\xi)e^{t\xi}, 0)\big|_{t=0}$$

$$= (\nabla\xi, 0)$$

So $\langle J(z), \xi\rangle = i_{\xi_p}\Theta = \int_V Y\cdot\nabla\xi\, d^3x$ (since Ω is canonical).

$$= \int_V \nabla\cdot(\xi Y) - \int_V (\nabla\cdot Y)\xi\, d^3x$$

$$= \int_V (-\nabla\cdot Y)\xi\, d^3x + \int_S \xi Y\cdot n\, d^2x$$

Now we see that provided $\int_S \rho Y \cdot n \, d^2x = 0 \Rightarrow Y \cdot n = 0$.

$$J(z) = -\nabla \cdot Y = \nabla \cdot E = \text{const.} \equiv \rho.$$

Again considering a conducting cavity, we see that $\sigma = 0$. This can also be attributed to the fact that there would be a energy contribution from such a surface charge, namely

$$H_{surf} = \frac{1}{2} \int_S \sigma \Phi \, d^2x.$$

Note also that this boundry condition is satisfied for $V =$ all space, $E \to 0$ at ∞.

Assuming that the boundry conditions are met, J can now reduce to Max_ρ, the space of E's and B's with $\nabla \cdot E = \rho$, $\nabla \cdot B = 0$. The map $(A, Y) \longmapsto (-Y, \nabla \times A) = (E, B)$ will be Poisson with respect to the canonical bracket from T^*Q evaluated on Max_ρ. To find this, Let F, K be functions of E and B, i.e.

$$\{F, K\} = \int_V \left(\frac{\delta F(-Y, \nabla \times A)}{\delta A} \cdot \frac{\delta K(-Y, \nabla \times A)}{\delta Y} - \frac{\delta F(-Y, \nabla \times A)}{\delta Y} \frac{\delta K(-Y, \nabla \times A)}{\delta A} \right) d^3x$$

$$= \int_V \left(\frac{\delta F(-Y, \nabla \times A)}{\delta(\nabla \times A)} \frac{\delta(\nabla \times A)}{\delta A} \cdot (-1) \frac{\delta K(-Y, \nabla \times A)}{\delta(-Y)} \right.$$

$$\left. + \frac{\delta F(-Y, \nabla \times A)}{\delta(-Y)} \cdot \frac{\delta K(-Y, \nabla \times A)}{\delta(\nabla \times A)} \frac{\delta(\nabla \times A)}{\delta A} \right) d^3x$$

Now

$$\left\langle \frac{\delta(\nabla\times A)}{\delta A}, \delta A \right\rangle = \int_V \nabla\times\delta A \, d^3x$$

And

$$\left\langle \frac{\delta F(\nabla\times A)}{\delta A}, \delta A \right\rangle = \int_V \left(\frac{\delta F}{\delta(\nabla\times A)}\right)\cdot\left(\nabla\times\delta A\right) d^3x$$

$$= \int_V \nabla\cdot\left(\frac{\delta F}{\delta(\nabla\times A)}\times\delta A\right) d^3x + \int_V \left(\nabla\times\frac{\delta F}{\delta(\nabla\times A)}\right)\cdot\delta A$$

$$= \int_V \left(\nabla\times\frac{\delta F}{\delta(\nabla\times A)}\right)\cdot\delta A + \int_S \left(n\times\frac{\delta F}{\delta(\nabla\times A)}\right)\cdot\delta A$$

Thus, provided that

$n \times \dfrac{\delta F(E,B)}{\delta B} = 0$, $n \times \dfrac{\delta K(E,B)}{\delta B} = 0$, the P-B-I bracket becomes

$$\{F, K\} = \int_V \left[\frac{\delta F}{\delta E}\cdot\left(\nabla\times\frac{\delta K}{\delta B}\right) - \frac{\delta K}{\delta E}\cdot\left(\nabla\times\frac{\delta F}{\delta E}\right)\right] d^3x$$

Again, if $E, B \to 0$ at ∞ this condition will be satisfied. Otherwise, for a conducting cavity the surface terms are again presumably related to the surface energy.