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Robert - This is very good showing progress A in 275 JM

Coadjoint Orbits and Representation Theory

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1. Some Generalities on Coadjoint Orbits

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $O \subset \mathfrak{g}^*$ be a coadjoint orbit of G. Then (O, ω_O) is a symplectic manifold where $(\omega_O)_\mu(\xi \cdot \mu, \eta \cdot \mu) = -\mu([\xi,\eta])$. Here $\mu \in O$, and $\xi \cdot \mu = -\mathrm{ad}^*_\xi \mu$.

In fact ω_O is the unique symplectic form on O which makes the inclusion $O \to \mathfrak{g}^*$ a Poisson map where $\mathfrak{g}^*$ has the − Lie-Poisson structure. Indeed, if ω is a symplectic form on O which makes the inclusion $O \to \mathfrak{g}^*$ a Poisson map, then for all smooth functions F,H on $\mathfrak{g}^*$ $\{F|_O, H|_O\} = \{F,H\}|_O$. This implies that $(dF)_\mu(X_{H|O}(\mu)) = -\mu([\frac{\delta F}{\delta\mu}, \frac{\delta H}{\delta\mu}])$. It follows that $X_{H|_O}(\mu) = \mu \circ \mathrm{ad}_{\frac{\delta H}{\delta\mu}}$. The uniqueness of ω is an immediate consequence of this. The group G acts transitively and symplectically on O and has an equivariant momentum map $J: O \to \mathfrak{g}^*$ given by $J(v) = -v$. Indeed, $\xi_O(v) = \frac{d}{dt}\Big|_{t=0} \exp(t\xi) \cdot v = \xi \cdot v = -\mathrm{ad}^*_\xi v$, and $\hat{J}(\xi)(v) = -v(\xi)$ so $(\omega_O)_\mu(\xi_O(v), \eta \cdot v) = -v([\xi,\eta]) = -(\eta \cdot v) \cdot \xi = d\hat{J}(\xi)_v(\eta \cdot v)$.

Fix $\mu \in O$, and let G_μ be the isotropy subgroup of G at μ. Then G_μ is a closed subgroup of G with Lie algebra $\mathfrak{g}_\mu = \{ \xi \in \mathfrak{g} \mid \xi \cdot \mu = 0 \}$. If G is connected and simply connected, then $O = G\cdot\mu \approx G/G_\mu$ is simply connected if and only if G_μ is connected. It is a non obvious fact that under the assumption that G is connected and compact, $G\cdot\mu$ is simply connected. Consideration of the connected group SL(2,R) shows that the compactness assumption cannot be dropped.

2. Prequantization of (O, ω_O).

Assume that the cohomology class $[\omega_O] \in H^2(O,R)$ lies in the image of the natural map $\varepsilon: H^2(O,Z) \to H^2(O,R)$. Then there exists a line bundle L with connection $\alpha \in \Omega^1(L^*,C^*)$ whose curvature is ω_O, and with Hermitian structure H on L under which parallel translation is an isometry; here L^* is L minus its zero section. If O is simply connected, then (L,α) is unique up to equivalence. Let $\underline{e}(L,\alpha)$ be the Lie algebra of all vector fields η on L^* which are C^*-invariant, and satisfy $L_\eta \alpha = 0$ and $\eta[|H|^2] = 0$. It is shown in [Ko 1970] that there is then a Lie algebra isomorphism $\tilde{\delta}: C^\infty(O,R)^- \to \underline{e}(L,\alpha)$ and an associated representation of $C^\infty(O,R)^-$ on the space S of smooth sections of L given as follows: $\delta(f)\, s = (\nabla_{X_f} + 2\pi i f)\, s$; here $C^\infty(O,R)^-$ is $C^\infty(O,R)$ with the bracket $\{f,h\}^- = -\{f,h\}$.

3. Invariant Complex Polarizations

A G-invariant complex polarization of O is given by a lagrangian subbundle F of the complexified tangent space $TO \otimes C$ such that $g \cdot F_\mu = F_{g\cdot\mu}$ for all $g \in G, \mu \in O$. Fix $\mu \in O$; then the map $\hat{\mu}: G \to O$ given by $g \to g \cdot \mu$ is equivariant. Since $\hat{\mu}$ is also a surjective submersion, it is not hard to see that there is a 1-1 correspondence between subbundles of $TO \otimes C$ on the one hand and subbundles of $TG \otimes C$ that contain $\mathrm{Ker}\, T\hat{\mu}$ and are invariant under $R_h\ \forall\ h \in G_\mu$ on the other. Moreover, this correspondence preserves involutivity as well as (left) G-invariance. Since a left-invariant involutive subbundle of $TG \otimes C$ corresponds exactly to a complex subalgebra of $\mathfrak{g}$, we see that (after fixing $\mu \in O$) specifying a G-invariant complex polarization of O is equivalent to specifying a complex subalgebra $\mathfrak{m}$ of $\mathfrak{g} \otimes C$ satisfying the following conditions.

i) $\mathfrak{m} \supset \mathfrak{g}_\mu \otimes C$,

ii) $\dim(\mathfrak{m}/\mathfrak{g}_\mu \otimes C) = \dim(\mathfrak{g} \otimes C/\mathfrak{m})$,

iii) $\mu([\mathfrak{m},\mathfrak{m}]) = 0$,

iv) $\mathfrak{m}$ is G_μ-invariant.

We note that if G_μ is connected, then condition i) implies condition iv).

The polarization F is called totally complex if $F \cap \bar{F} = 0$. The corresponding condition on $\mathfrak{m}$ is that $\mathfrak{m} \cap \bar{\mathfrak{m}} \subset \mathfrak{g}_\mu$ (note that the reverse containment is automatic).

In the case of $G = SU(2)$, we can identify coadjoint orbits with spheres centered at the origin in $\mathbf{R}^3$. The symplectic form on S^2_r is $\frac{-1}{r}$ times the standard area form. Let $\mu = (0,0,r)$. Then $G_\mu = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 = 1 \right\}$. There are exactly 2 G-invariant polarizations on S^2_r corresponding to the holomorphic and anti-holomorphic structures on S^2_r.

4. Induced Representations

Suppose that H is a closed subgroup of the Lie group G and that $\rho: H \to GL(V)$ is a representation of H on the finite-dimensional complex vector space V. Let $F_\rho(G,V) = \{ f \in C^\infty(G,V) \mid f(hg) = \rho(h) \cdot f(g)\ \forall\ h \in H, g \in G \}$. Then $F_\rho(G,V)$ is a complex vector space. Define $\rho^G: G \to GL(F_\rho(G,V))$ by $\rho^G(g) \cdot f = f \circ R_g$. Then ρ^G is a representation of G called the induced representation of ρ to G.

The group H acts freely and properly on $G \times V$ by $h \cdot (g,v) = (hg, \rho(h) \cdot v)$. Let E denote the orbit space. Then E has the structure of a complex vector bundle over the right coset space $H\backslash G$. It is not hard to show that $F_\rho(G,V) \approx$ {smooth sections of E} and that the natural representation of G on {smooth sections of E} is equivalent to ρ^G. Often $H\backslash G$ has the structure of a complex manifold and E has a holomorphic structure. One may then consider only the holomorphic sections of E. This is referred to as holomorphic induction.

It is easy to see that $2\pi i\mu: \mathfrak{g}_\mu \to iR$ is a Lie algebra homomorphism. It is proved in [Ko 1970] that μ is the derivative of a Lie group homomorphism $G_\mu \to T$ if and only if $[\omega_0]$ is an integral cohomology class.

5. Quantization of Coadjoint Orbits

Let $C_F(O,C) = \{ f \in C^\infty(O,C) \mid df \cdot X = 0 \ \forall$ complex vector field X taking values in $F \}$, and let $C_F^1(O,C) = \{ f \in C^\infty(O,C) \mid \{f, C_F(O,C)\} \subset C_F(O,C) \}$. Note that since F is involutive, $f \in C_F(O,C)$ if and only if X_f takes values in F. Since $X_{\{f,h\}} = -[X_f, X_h]$, we see that $C_F(O,C)$ is a subalgebra of the Lie algebra $C^\infty(O,C)$. The Jacobi identity implies that the normalizer $C_F^1(O,C)$ of $C_F(O,C)$ is also a subalgebra of $C^\infty(O,C)$. If O is replaced by a cotangent bundle T^*Q and if F is the vertical polarization, then $C_F(T^*Q,C) = \{ f \in C^\infty(T^*Q,C) \mid f$ factors through $T^*Q \to Q \}$, and $C_F^1(T^*Q,C) = \{ f \in C^\infty(T^*Q,C) \mid \forall q \in Q, f|_{T_qQ}$ is a polynomial function of degree $\le 1\}$. Suppose $f: O \to R$ is such that the flow φ_t of X_f preserves F and suppose $h \in C_F(O,C)$ so that X_h takes values in F. Then $\varphi_t^* X_h$ takes values in F for all t and hence $[X_f, X_h] = \frac{d}{dt}\big|_{t=0} \varphi_t^* X_h$ also takes values in F. Thus $\{f,h\} \in C_F(O,C)$, and hence $f \in C_F^1(O,C)$.

Each vector $\xi \in \mathfrak{g} \otimes C$ gives rise to $\hat{J}(\xi): O \to C$ where $\hat{J}(\xi)(v) = -v(\xi)$. Now $\hat{J}(\xi)$ can be considered to be defined on all of $\mathfrak{g}^*$, and since it is linear, $\frac{\delta \hat{J}(\xi)}{\delta v} = -\xi$. The corresponding Hamiltonian vector field is $X_{\hat{J}(\xi)}(v) = v \circ \mathrm{ad}\, \frac{\delta \hat{J}(\xi)}{\delta v} = -v \circ \mathrm{ad}\xi = \xi \cdot v = \xi_O(v)$. If ξ is real, the time t advance map of this vector field is given by the action of the group element $\exp(t\xi)$. Since F is G-invariant, it follows that $\hat{J}(\xi) \in C_F^1(O,C)$. The same conclusion holds even if ξ is not real.

6. Loose Ends

The irreducible representations of SU(2) are well-known. Up to equivalence, there is exactly one in each positive dimension. It is also known that if T is the tautological line bundle over $CP^1 = P(C^2) \approx S^2$, then T generates the Picard group $H^1(CP^1, O^*) \approx H^2(CP^1, Z) \approx Z$. Moreover, if n is any positive integer, $T^{\otimes n}$ has only one global section, namely the one that is identically 0, while if $n \le 0$, then the global sections of $T^{\otimes n}$ form a $(1-n)$-dimensional complex vector space and the natural representation of SU(2) on these generate all the irreducible representations of SU(2).

This result generalizes to arbitrary compact Lie groups in the form of the Borel-Weil Theorem. This needs to be discussed more thoroughly from the viewpoint of geometric quantization.