Bifurcations of Periodic Orbits

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Bifurcations have received a great deal of interest in recent years along with the study of chaos and non-linear dynamical systems. In general, a system described by a differential equation, time map, or Hamiltonian can have parameters in the equations which affect the dynamics the system. Obviously, in most cases, a small change in one of the parameters will have little effect on the system, but only slightly perturb it. However, at certain values these parameters a small change can have dramatic effects and cause a change in the qualitative behavior of the system. The values at which these changes occur are known as *bifurcation points.* I plan to discuss in this paper some theorems about the existence of bifurcations at fixed points of Poincaré maps. In particular I will look at extremal versus elementary fixed points and also look at the existence of bifurcations in period doubling and k-period orbits.

One Dimensional Example

In a one dimensional system, described by a differential equation, bifurcations usually refers to changes in the stability of a point. As an example, consider the one dimensional case of Hamiltonian : $H = (1/2)y^2 + (1/3)x^3 + \mu x$, where y is the first time derivative of x. The solution involves a bifurcation when $\mu = 0$. If we look at phase diagrams of y versus x on the level sets of the energy, we see a qualitative change as μ varies. See figure one for the phase portraits, y versus x. The curves in the diagram are level sets of the Hamiltonian. We get the critical points by setting the first partial derivative of H with respect to x equal to zero: $dH/dx = x^2 + \mu = 0$. So we get critical points at $x = \pm\sqrt{-\mu}$. The second derivative is 2x. So when μ is negative, we get two critical points, with the $x = \pm\sqrt{-\mu}$ being a stable point, (the second derivative being positive) and $x = \mu$ being unstable (the second derivative is negative). However, for μ positive, there are no stable points. So we go from having an unstable and stable equilibrium point to having no equilibrium points as μ passes through 0. When $\mu=0$, the two critical points collide, with the stability being determined by the direction of the perturbation in x.

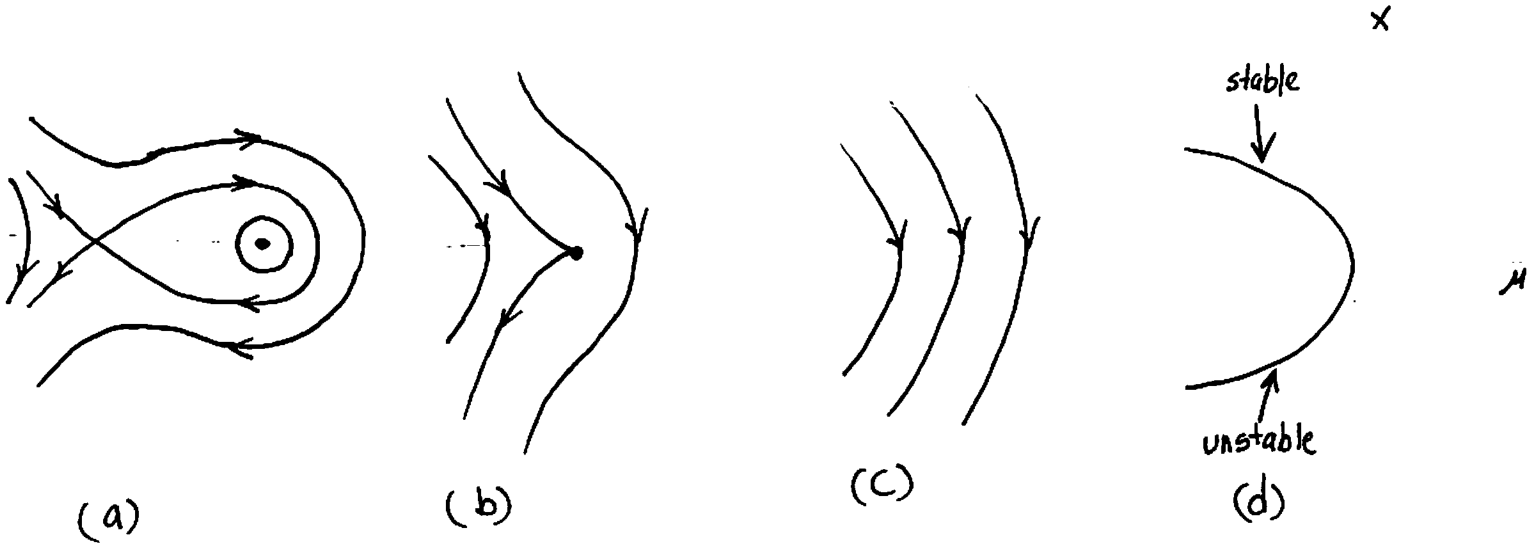


Figure 1. A One-dimensional example of a bifurcation.
$H = (1/3)x^3 + (1/2)y^2 + \mu x$. The level sets of the Hamiltonian in the three cases: (a) $\mu<0$. There are two equilibrium points, one stable and one unstable. (b) $\mu=0$. The two points converge. (c) $\mu>0$. There no stable points. (d) The graph shows the locations of stability as x varies. Later we will show how this corresponds to an extremal fixed point.

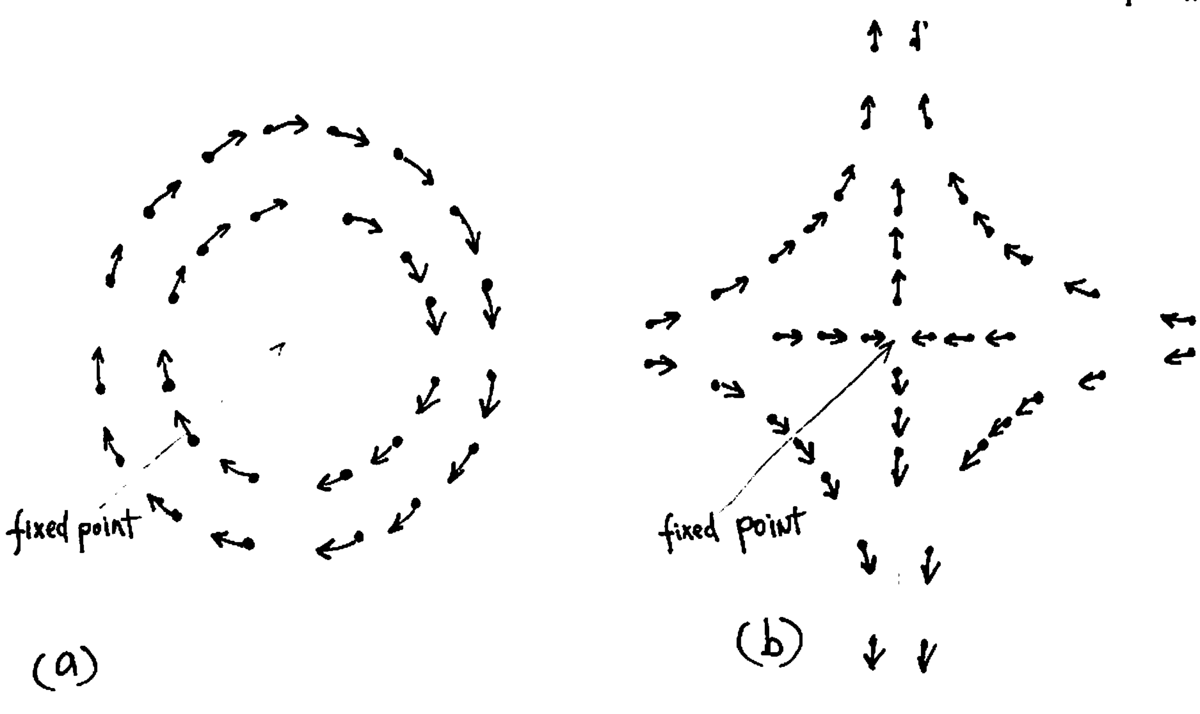


Figure 2. Hyperbolic versus Elliptic orbits in Poincare maps.
The Poincare maps of periodic points are a series of nearby points. (a) In the elliptic case the points rotate about the fixed point at the origin. The arrows indicate the mapping of one iteration of the Poincare map to the next. (b) The hyperbolic case. The points diverge from the fixed point.

Poincare maps, multipliers, and elliptic and hyperbolic fixed points

This example serves to illustrate the general nature of bifurcations but the examples and theorems I will examine in this paper involve systems in two dimensions. Specifically I will be looking bifurcations near periodic orbits. If f(v,t) is a function of time and a vector, $v \in \Re^2$ then a periodic orbit would be a point v such that f(0,v) = f(T,v), where T is the period.

A powerful tool used to study the behavior of systems about such points is the *Poincaré map.* I will describe the Poincaré map in the system we are currently looking at, which, since it involves points in $\Re^2$, is a four dimension Hamiltonian system (two spatial and two momentum coordinates). The Poincaré map always looks at a cross section, called Σ, of co-dimension one, meaning one dimension less than the system under investigation. We denote the Poincaré map by P(v), where v is a point in Σ. The point P(v) is the point at which the point v will cross Σ on its next trip around. Relating this back to periodic points, it is obvious that if f(0,v) = f(T,v) and x Σ then P(v) = v and in the Poincaré map, we refer to v as a *fixed point* of P. We use the Poincaré map to study the behavior of points near fixed points. We see that in any application to a Hamiltonian system of degree four, the Poincare section reduces the dimension by one, and looking at level sets of the energy would reduce the dimension by one more, allowing us to describe the Poincare map with two variables. Since we are only dealing with local results in this discussion, it is always possible to shift our coordinates such that the fixed point in question is at the origin. When we look the Poincaré map as a 2x2 matrix, consider the Jacobian of this matrix at the origin :

$$\begin{pmatrix} dP_x(0,0)/dx & dP_x(0,0/dy \\ dP_y(0,0)/dx & dP_y(0,0)/dy \end{pmatrix}$$

and call it A. The eigenvalues of this matrix are known as the multipliers of the fixed point.

Before I begin the discussion of bifurcations, I would like to mention several important mathematical results which we rely on. First, it can be shown (Meyer and Hall, V.E) that at a fixed point of a periodic orbit, one of the multipliers is always +1. For this we revert back to the f(v,t) notation.

<u>Proof:</u> Consider the fixed point v and the map f(t,v) where f(T,v) = f(0,v).

We know f (T, f (v,t)) = f (t + T, v).

Differentiating with respect to v we and setting t = 0 we get

$$\frac{\partial f}{\partial v}(T,V)\,\frac{df}{dt}(0,V) = \frac{df}{dt}(T,V)$$

$$\frac{\partial f}{\partial v}(T,V)\,\frac{df}{dt}(0,V) = \frac{df}{dt}(0,V)$$

so df/dt(0,v) is an eigenvector corresponding to +1.

The second result I would like to mention is that the eigenvalues of a Hamiltonian system, the eigenvectors always come in conjugate and reciprocal sets: If λ is an eigenvalue of P, then so is λ^{-1}, $\bar{\lambda}$, and $\bar{\lambda}^{-1}$ (see Meyer and Hall, II.C). This applies to our problem, as I will explain below. There is another proof (Meyer and Hall V.E) which states that if we follow integral surfaces, i.e., level sets of the Hamiltonian, then the multiplier +1 has multiplicity of at least 2. This is then applied to Poincaré maps and it is shown that if the multipliers of a period solution are 1,1, $\lambda_3, \ldots \lambda_m$ then the multipliers in the Poincare map on integral surfaces of the fixed point are $\lambda_3, \ldots \lambda_m$. So the two remaining eigenvalues in out problem are the two that manifest themselves as multipliers of the matrix dP(0,0)/d(x,y)=0. Since these eigenvalues are subject to the same restrictions I mentioned at the beginning of this paragraph, we see that they must either be reciprocals on the real line or conjugate pairs on the unit circle (to satisfy the reciprocal and conjugate conditions). The real line case could also allow both of them to be +1, and we will consider this as a separate case.

We will also use the fact the in two dimensions, symplectic maps are area preserving, meaning their Jacobians have determinant 1.

The last important definitions are the distinctions between *elliptic* and *hyperbolic* orbits. Consider a differential equation such $v = f(v) = Bv + g(v)$ where B is the linearized system of equations at $v=0$ and $g(v)$ is the correction, which goes to zero at $v=0$. Then the solution is elliptic if there is an eigenvalue of B with a real part equal to zero. Intuitively this makes sense, as the there would then be no exponential decay or gain with time. If all the eigenvectors of B have a non-zero real part then the orbit is hyperbolic since its distance from the fixed point will grow or shrink exponentially in time. However, we are here discussing a mapping from point to point at discrete times: v --> P(v). A solution satisfying the equation $v = B(v)$ is $v = e^{BT}(v)$, where the orbit has period T. If the eigenvalues of B have zero real part 0, the eigenvalues of e^{BT} have modulus one. They are rotations in the complex plane, corresponding to elliptic orbits. Or if the eigenvalues have a real part, then e^{BT} would have eigenvalues with values greater or less than one, corresponding to hyperblic orbits. So in the case of Poincaré maps, we look at the Jacobian of the mapping and the same concept leads us to the conclusion if the eigenvalues have modulus 1, i.e., they lie on the unit circle, we consider them *elliptic points*. Intuitively, this again means that the distance from the fixed point does not increase or decrease in time. If they are not on the unit circle, which in our case would mean they are reciprocals on the real line, the points on the mapping to not rotate about the fixed point but change distances from it. We call these *hyperbolic points*. Figure 2 shows the basic examples.

Throughout this paper I will be looking at maps in normal form. This is basically a Taylor expansion of the map, keeping only terms that are needed to see the behavior of the system. While linearized equations are often good enough to do this, bifurcations often occur at points where the linearized equations to not tell us the behavior of the system and the next order term is needed. I do not have the background or space to discuss how the normal forms are arrived at, however, in most of the following theorems, we simply assume the a form of the Poincaré map in normal form, and make calculations from this point. When equations are given in normal form, it is assumed that all constants are non-zero unless otherwise noted.

Elementary versus External Fixed Points

Elementary Points

Now we are ready to study the Poincaré maps of these periodic points. As before, let us make the origin in $\Re^2$ a fixed point. Denote a vector in the plane as (x,y). The Poincaré map is $P:\Re^2 \times I \dashrightarrow \Re^2 : (x,y,\mu) \dashrightarrow P(x,y,\mu) = (x',y')$. The Jacobian of the map P at the fixed point, $dP(x,y,\mu)/d(x,y)|(0,0,0)$, we will call A. The eigenvalues of A will be the two multipliers. Here μ is a parameter in the equations for P which may be varied over the interval I. We wish to look at changes in behavior with changes in this parameter.

Theorem : Our first claim is that if the multipliers are not +1, then the fixed point can be *continued* in the following sense : If (x,y) = (0,0) is a fixed point so P(0,0,0) = (0,0) there is a neighborhood $(-\mu_0,\mu_0)$ over which μ can vary with $P(x(\mu),y(\mu),\mu) = (x,y)$. Furthermore, if the origin is a elliptic, then so is $(x(\mu),y(\mu))$ and likewise for hyperbolic points. So no bifurcation is present.

Proof : Define $G(x,y,\mu) = P(x,y,\mu) - (x,y)$. Assuming (0,0) is a fixed point, we have $G(0,0,0) = (0,0)$ and $dG(x,y,\mu)/d(x,y)|(0,0,0) = A - I$, which is non singular, since we assumed the eigenvalues of A were non equal to +1. With these conditions we can apply the implicit function theorem, which allows us to conclude that if μ is varied continuously, there exists a vector $(x(\mu),y(\mu))$ which continues to satisfy $G(x(\mu),y(\mu),\mu) = 0$. Thus $P(x(\mu),y(\mu),\mu) = (x(\mu),y(\mu))$ and so these are fixed points as well. For the second part of the theorem, we rely on the fact that the eigenvalues of a matrix vary continuously with a parameter so they can not jump from the real line to the unit circle or vice versa as μ is varied. □

Extremal Fixed Points

However, when the multipliers are +1, the proof breaks down, because A-I is then singular. Intuitively, this means that the eigenvalues are going to depend on the nonlinear terms, particularly ones involving μ. So the behavior could qualitatively change at $\mu=0$. These points are called *extremal fixed points*. We define them for this discussion as fixed points with the map $P(x,y,\mu) = (x',y')$ given by:

$x' = x + \alpha y + \mu\gamma + \ldots$ (higher order terms)

$y' = y + \mu\delta + \beta x^2 + \ldots$

We see that the Jacobian at the origin, $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$

and so the eigenvalues are both +1.

Theorem: (1) If (0,0) is a fixed point at $\mu=0$, then there exists a map T: $(-\tau_1,\tau_1) \dashrightarrow I \times \Re^2 : \tau \dashrightarrow (\mu(\tau),\tau,y(\tau))$ such that $(\tau,y(\tau))$ is a fixed point for $\mu(\tau)$. Note that x is simply τ in this mapping. Also $T(\tau = 0) = (0,0,0)$.

(2) For this mapping, the fixed points are elliptic in one direction and hyperbolic in the other.

(3) μ achieves a maximum at 0. So fixed points exists for either μ negative or μ positive, but not both, and there is both an elliptic and hyperbolic fixed point for each μ.

(See figure 3).

Proof: In order for (x,y) to be a fixed point, we must have :

$x' = x + \alpha y + \mu\gamma + \ldots = x \implies x' - x = \alpha y + \mu\gamma + \ldots = 0$

$y' = y + \mu\delta + \beta x^2 + \ldots = y \implies y' - y = \mu\delta + \beta x^2 + \ldots = 0$

We solve the last equalities for y and μ as functions of x to get:

$\mu(x) = (-\beta/\delta)x^2 + \ldots$ and $y(x) = (\gamma\beta/\alpha\delta)x^2$

So $T(\tau) = (\tau,y(\tau),\mu(\tau))$ with $y(\tau)$ and $\mu(\tau)$ given in the previous line, proving (1).

First we see that μ is a parabola and has a maximum at $\tau = 0$, which is a maximum if $\beta\delta > 0$ and a minimum if $\beta\delta < 0$. Also it has two solutions on one side of $\tau = 0$ and never attains values on the side $\tau=0$, proving (3).

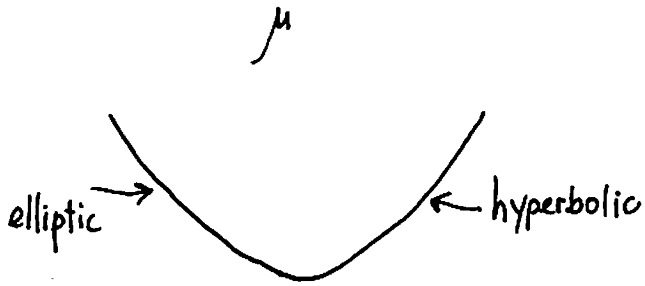


Figure 3. Extremal fixed point.
An extremal fixed point has both an elliptic and hyperbolic for each τ. Recall that $\tau = x$ in the solution so the τ axis can also be thought of as a spatial axis. This is the case $\beta\delta<0$ and $\alpha\beta>0$.

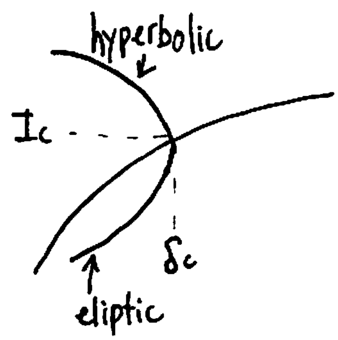


Figure 4. Duffings equation.
We see how the solutions to I for varying δ behaves much like an extremal fixed point, with the critical point corresponding to the fixed point. The three solutions are not drawn like actual solutions to the equation, but merely demonstrate the behavior of the solutions' multiplicity and type.

The Jacobian of the mapping P is $d(x',y')/d(x,y) = \begin{pmatrix} 1 & \alpha \\ 2\beta\tau & 1 \end{pmatrix}$.

Along the solutions we found this is

and so we find the eigenvalues are : $1 \pm \sqrt{(2\alpha\beta\tau)}$. So if $\alpha\beta\tau > 0$, the solutions are on the real line and we get a hyperbolic solution and if $\alpha\beta\tau < 0$ we get an elliptic solution (the non-linear terms will put the solution exactly on the unit circle). Thus, when τ goes through zero, the fixed point solutions switch from being hyperbolic to elliptic, proving (2).□

If we were to look at this solution in only the two dimensions of the Poincaré map, and associate hyperbolic with unstable equilibrium points and elliptic with stable equilibrium points, we see the bifurcation is much like the one dimension example we saw, with there being a two fixed points on one side of $\mu=0$, one hyperbolic and one elliptic, and no fixed points on the other side of $\mu=0$. And the two solutions again collide at the origin and thus the origin is like an extremal fixed point in this equation. Of course, in two dimensions, the orbits look much more complicated but the Poincaré map has this behavior and captures the essential behavior of the orbit over time. So the origin in the one dimension example we looked at is an extremal fixed point.

Resonance in Duffing's equation

A very common example applied to bifurcation theory is Duffing's equation, which is of the form $x + \omega_n^2 x + \gamma x^3 = A\cos(\omega_f t)$. This is a variation on a harmonic oscillator, with ω_n acting as the natural frequency of oscillation. The non-linear γ term is a non-linear perturbation and the term on the right is an oscillating applied force at a different frequency, ω_f. We can put this in the form of a Hamiltonian by setting y to the time derivative of x, making x and y canonically conjugate variables and then setting the Hamiltonian $H = -(\omega_n/2)(x^2 + y^2) + (\gamma/\omega_n)x^4/4 - (A/\omega_n)x\cos(\omega_f t)$. If $\gamma=0$ this has a solution $x = [A/(\omega_e^2 - \omega_f^2)] \cos(\omega_f t)$. Note that the frequency of the solution is that of the forced oscillator. We are going to look at solutions corresponding to different relations between the forced and natural frequencies. Without the forced or non-linear term (γ=A=0), Hamilton's equations become $dx/dt = dH/dy = \omega_n y$ and $dy/dt = -dH/dx = -\omega_n x$ which has the solution is simply $x = x\cos(\omega_n t) + y\sin(\omega_n t)$ and $y = -x\sin(\omega_n t) + y\cos(\omega_n t)$. Thus, if we compute the period map with the forced oscillation period, $2\pi/\omega_f$, as we computed before, we get the matrix:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(2\pi\omega_n/\omega_f) & \sin(2\pi\omega_n/\omega_f) \\ -\sin(2\pi\omega_n/\omega_f) & \cos(2\pi\omega_n/\omega_f) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The multipliers are $\exp(+-2\pi i\omega_n/\omega_e)$. Note that these are +1 if ω_n/ω_e is equal to an integer. If this is not the case, then we have elementary multipliers (not equal to +1) and by our first theorem, we see that the solution can be continued with small variations in the parameter A or γ. However when $\omega_n/\omega_f = i$, i=0,+1,-1,2,-2,. . . then we can not apply the theorem. So let us consider the case where the ratio is almost an integer. We normalize $\omega_f = 1$ and then assume $\omega_e^2 = 1 - \mu\delta$ where μ is a small parameter. We can then look at the solutions when all the non-linear terms are small by multiplying them by μ also. Thus we have $dx^2/dt^2 + x = \mu(\delta x + \gamma x^3 + A\cos t)$, which has the Hamiltonian :

$H = (1/2)(x^2 + y^2) - \mu(\delta x^2/2 + \gamma x^4/4 + Ax\cos t)$

We make the change to action-angle variables by setting $x = (2I)^{1/2}\cos\phi$, and $y = (2I)^{1/2}\sin\phi$. This makes the Hamiltonian :

$H = I - \mu(\delta I\cos^2\phi + \gamma I^2\cos^4\phi + A(2I)^{1/2}\cos\phi\cos t)$

Now we rely on the theory of normal forms, which tells us that the only the terms in I and (ϕ+t) are necessary to the solution. This motivates making the following the substitutions:

$\cos^2\phi = (1 + \cos 2\phi)/2$
$\cos^4\phi = (3 + 4\cos 2\phi + \cos 4\phi)/8$
$\cos\phi\cos t = [\cos(\phi+t) + \cos(\phi-t)]/2$

So, keeping only terms in I and (ϕ+t) and we get:

$H = I - \mu[I\delta/2 + 3\gamma I^2/8 + AI/2\cos(\phi+t)] + ...$

Then our equations become:
$dI/dt = dH/d\phi = -\mu AI/2\sin(\phi+t)$
$d\phi/dt = -dH/dI = -1 + \mu[3\gamma I/4 + A2^{-3/2}I^{-1/2}\cos(\phi+t)$

We integrate these from t=0 to 2π (we assume $\omega_f = 2\pi/T = 1$) to get the period mapping :
$I' - I = \mu\pi(2I)^{1/2}A^2\sin\phi + \ldots$
$\phi' - \phi = -2\pi + \mu\pi[\delta + 3\gamma I/2 + (2AI)^{-1/2}\cos\phi] + \ldots$

We then divide the first equation by $\mu\pi(2I)^{1/2}$ and solve the first equation $(I'-I)/\mu\pi(2I)^{1/2} = 0 = \sin\phi$ so our solutions are $\phi=0 + \ldots$ and $\phi=\pi + \ldots$ We then plug this into the second equation so $\cos\phi$ becomes +-1 + ... Then the second equation $(\phi'-\phi+2\pi)/\mu\pi = 0 = \delta + 3\gamma I/2 + (2AI)^{-1/2}$. Thus we get a fixed point for $\delta = -3\gamma I/2 \pm (2AI)^{-1/2}$. For simplicity let us combine constants so $\delta = -aI \pm bI^{-1/2}$. Now we set $d\delta/dI = -a \pm (1/2)bI^{-3/2} = 0$ so we get a critical point of δ at $I_c = |b/2a|^{2/3}$. And so $\delta_c = k|ab^2|$ when we plug back into the equation. k is a combination of numerical constants. This equation has one solution for I when $\delta > \delta_c$ and three solutions when $\delta < \delta_c$. Two of these solutions collide at δ_c. The Jacobian elements are :

$dI'/dI = 1$ ($\sin\phi = 0$ at the solution)
$dI'/d\phi = \pm\, \mu\pi(2I)^{1/2}A$
$d\phi'/dI = \mu\pi[3\gamma/2 - (1/2)(2A)^{-1/2}I^{-3/2}] = \mu\pi(d\delta/dI)$ (we calculated this above)
$d\phi'/d\phi = 1$

So the characteristic polynomial is $(\lambda-1)^2 \pm \sqrt{(\mu^2\pi(2I)^{-1/2}A)}$ and the multipliers are then $\lambda = 1 \pm \sqrt{([redacted]) -\mu^2\pi(2I)^{-1/2}A(d\delta/dI)}$. Since $d\delta/dI$ changes sign at the critical point, we see that the stability type changes from elliptic to hyperbolic for the two points near the critical point δ_c. Thus, other than for the fact that we have an extra solution on each side, we have an extremal point, where an elliptic and hyperbolic fixed point appear on one side of the critical point. See figure 4.

Period Doubling and Period-k Bifurcation Points

Period 2

One of the most important kinds of bifurcations being studied currently are those of period doubling. Consider a discrete mapping that has a periodic fixed point. That is, each iteration brings a point back to itself $P(\mu,x,y) = (x,y)$. Occasionally, as μ passes through a particular value, the iteration brings the point back to itself only after two iterations, so $P(P(x,y,\mu)) = (x,y)$. This is known as period doubling. We can generalize this to the concept of a k-period point, meaning that it comes back to original point after k iterations: $P^k(x,y,\mu) = (x,y)$, where P^k is the map P composed k times. Period doubling is an extremely important concept in chaos theory, as a solution will bifurcate into a period 2 solution, then each of its branches will in turn bifurcate, and so on. These bifurcations approach a limit past which the behavior is said to be chaotic and completely unpredictable. Here we only deal with only with period-2 and period-k solutions to the Poincaré map (see figure 5).

Again let A be the Jacobian of the P at the origin, a fixed point. Note that $A^k(x,y) = (x,y) + \ldots$(non-linear terms) which has only one solution : $(x,y) = (0,0)$ unless an eigenvalue of A is the kth root of unity. We first consider the period doubling case. Because -1 is the second root of +1, then if it was a double multiplier of A, then $A^2(x,y) = (x,y)$, where (x,y) is an eigenvector corresponding to -1. We now make another definition:

<u>Definition:</u> The origin is a *transitional fixed point* of P at $\mu=0$ if $P(x,y,\mu) = (x',y')$ is given by:

$x' = -x + \alpha y + \mu(ax + by) + \ldots,\qquad \alpha = \pm 1$

$y' = -y + \mu(cx + dy) + \beta x^3 + \ldots$

We see that the multipliers are -1 at $\mu=0$ since then $A = \begin{pmatrix} -1 & \alpha \\ 0 & -1 \end{pmatrix}$

<u>Theorem:</u> Let the origin be a transitional fixed point for P given above at $\mu=0$. Then the following hold:

(1) If $\alpha c > 0$, the origin is hyperbolic when $\mu>0$ and is elliptic when $\mu<0$. The situation is reversed for $\alpha c < 0$.

(2) If $\beta c > 0$, there are period 2 points at $\mu<0$ and no periodic points for $\mu>0$. Again, the opposite is true for $\beta c<0$. The period 2 points tend to the origin as μ approaches zero.

(3) The stability type (elliptic or hyperbolic) of the origin and the period 2 points are opposite.

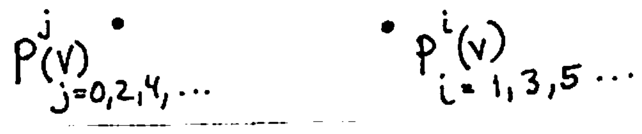


Figure 5. Period Doubling
A Poincare map of a period 2 map will alternate between two points in the plane.

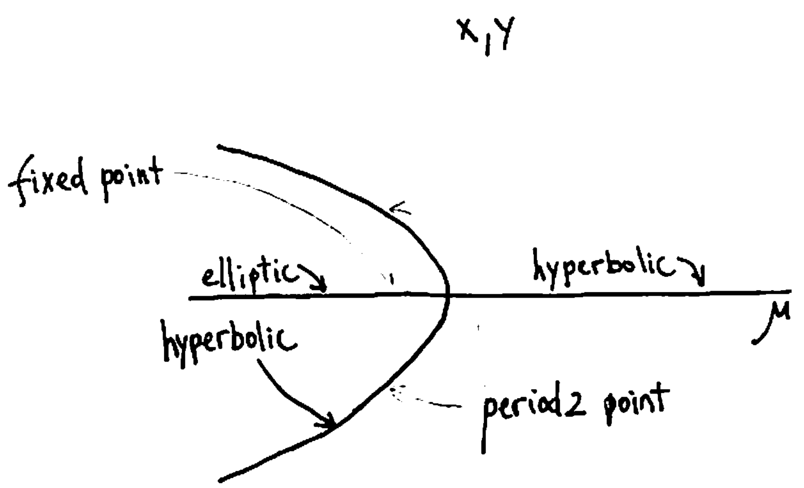


Figure 6. Period Doubling Bifurcation
The vertical axis is the x,y plane suppressed into one dimension. As μ goes through 0, we see that the type of the origin's stability changes and a period 2 point appears on one side. Lines are drawn both directions as the period 2 point will intersect the plane in two points, even though it is only one particular orbit. This is the case $\alpha c>0$, $\beta c>0$

(See figure 6)

<u>Proof :</u>
(1)
$dx' / dx = -1 + \mu a + \ldots$
$dx'/dy = \alpha + \mu b + \ldots$
$dy'/dx = \mu c + 3\beta x^2 + \ldots$
$dy'/dy = -1 + \mu d$
So the Jacobian at the origin is $A = \begin{pmatrix} -1+\mu a & \mu b \\ \mu c & -1+\mu d \end{pmatrix}$

Before computing the eigenvalues we note that the determinant of this matrix at the origin is $1 - \mu (a + d + \alpha c) + \ldots$ and this must be equal to +1 for the mapping to be symplectic and thus area preserving, Then $a + d = -\alpha c$ is required. Now we compute the eigenvalues from the characteristic equation :

$(\lambda-1+\mu a)(\lambda-1+\mu d) - (\mu c)(\alpha+\mu b) = 0$
Throwing out terms of order μ^2 we get
$(\lambda-1)^2 + (\lambda-1)(\mu\alpha + \mu d) - \mu c\alpha = 0$
We replace $\mu a+\mu d = -\alpha c\mu$ to get:
$\lambda^2 - 2\lambda + 1 - \lambda\alpha c\mu + \alpha c\mu - \mu c\alpha = \lambda^2 - 2\lambda + 1 - \lambda\alpha c\mu = 0$
$\lambda = \left[2 + \alpha c\mu \pm \sqrt{(2 + \alpha c\mu)^2 - 4}\right]/2$

We already know the values are either on the real line or the unit circle so we only need to look at the discriminant to determine if the eigenvalue is real or complex.

The discriminant is $(\alpha c\mu)^2 + 4\alpha c\mu$ and so we conclude that the origin is elliptic if $\alpha c\mu < 0$ and hyperbolic if $\alpha c\mu > 0$, proving the first statement.

(2) We look at the second iterate of the map and again keep only to first order in μ, β:

$x'' = (-1 +\mu a)x' + (\alpha + \mu b)y' + \ldots$
$\quad = (-1 + \mu a)(-x+\mu ax - \alpha y + \alpha\mu dy + \alpha\mu cx + \alpha\beta x^3) + (\alpha + \mu b)(-y + \mu dy + \mu cx + \beta x^3)$
$\quad = x - \mu ax + \alpha y - \alpha\mu dy - \alpha\mu cx - \alpha\beta x^3 - \mu ax - \mu a\alpha y - \alpha y + \alpha\mu dy + \mu\alpha cx + \alpha\beta x^3$
$\quad\quad - \mu by$
$\quad = x - 2\alpha y + O(\mu) + \beta x^3 + \ldots$
$y'' = (-1 + \mu d)\, y' + \mu cx' + \beta x'^3 + \ldots$
$\quad = (-1 + \mu d)\,(-y + \mu dy + \mu cx + \mu c\alpha y + \beta x^3) - \mu cx + \mu c\alpha y - \beta x^3 + \ldots$
$\quad = y - 2\mu cx - 2\mu dy - 2\beta x^3 + \ldots$

In order for this to be period 2 we must have $x'' - x = 0 = -2\alpha y + \beta x^3 + O(\mu) + \ldots$ so a solution $y(\mu,x)$ is of order $O(\mu + x^3)$. This we can plug into our other equation:

$y'' - y = 0 = -2\mu c x - 2\mu d y + \beta x^3$

In this equation we can drop the y term as it is of order $O(\mu^2 + \mu x^3)$. We first divide by x and get $(y'' - y)/x = 0 = -2\mu c + \beta x^2$. This is justified since we are not interested in the origin as a solution. So, solving for x as a function of μ we have $x(\mu) = -2c\mu/\beta$ as a solution corresponding to a period 2 map. From this we conclude that if $\beta c > 0$, there are two solutions for $\mu<0$ and none for $\mu>0$, and vice versa for $\beta c < 0$, proving statement (2).

(3) Now we look at the Jacobian of the period 2 solution:

$dx''/dx = 1 + \ldots$ $\quad dx''/dy = -2\alpha + \ldots$ $\quad dy''/dx = -2\mu c - 6\beta x^2 + \ldots$
$dy''/dy = 1 - 2\mu d + \ldots$

Plugging in the solution $x^2 = -2c\mu/\beta$ we have the Jacobian matrix as:

$$\begin{pmatrix} 1 & -2\alpha \\ 10\alpha c\mu & 1. \end{pmatrix}$$

The characteristic equation is then $(\lambda-1)^2 + 20\alpha c\mu = 0$. So $\lambda = 1$ $1 \pm \sqrt{-20\alpha c\mu}$

So the solution is hyperbolic if $\alpha c\mu < 0$ and elliptic of $\alpha c\mu > 0$. This is the opposite of the fixed point, proving (3). □

So we have shown that near a transitional fixed point, we have period 2 orbits appearing on one side of $\mu=0$ and they have the opposite stability type as the fixed point. The fixed point itself is remains fixed on either side, though it changes stability type at $\mu=0$ (see figures 6 and 7).

<u>Period 3</u>

We now turn our attention to period k points. For this, we use action-angle coordinates by the transformation $I = (1/2)(x^2 + y^2)$ and $\phi = \arctan(y/x)$. We noted that -1, being a second root of +1, caused a period doubling to occur. Therefore, we define k bifurcation points which have multipliers that are k-roots of +1.

<u>Definition:</u> The origin is a *k-bifurcation point* at $\mu=0$ if the map $P(I,\phi,\mu) = (I',\phi')$ where

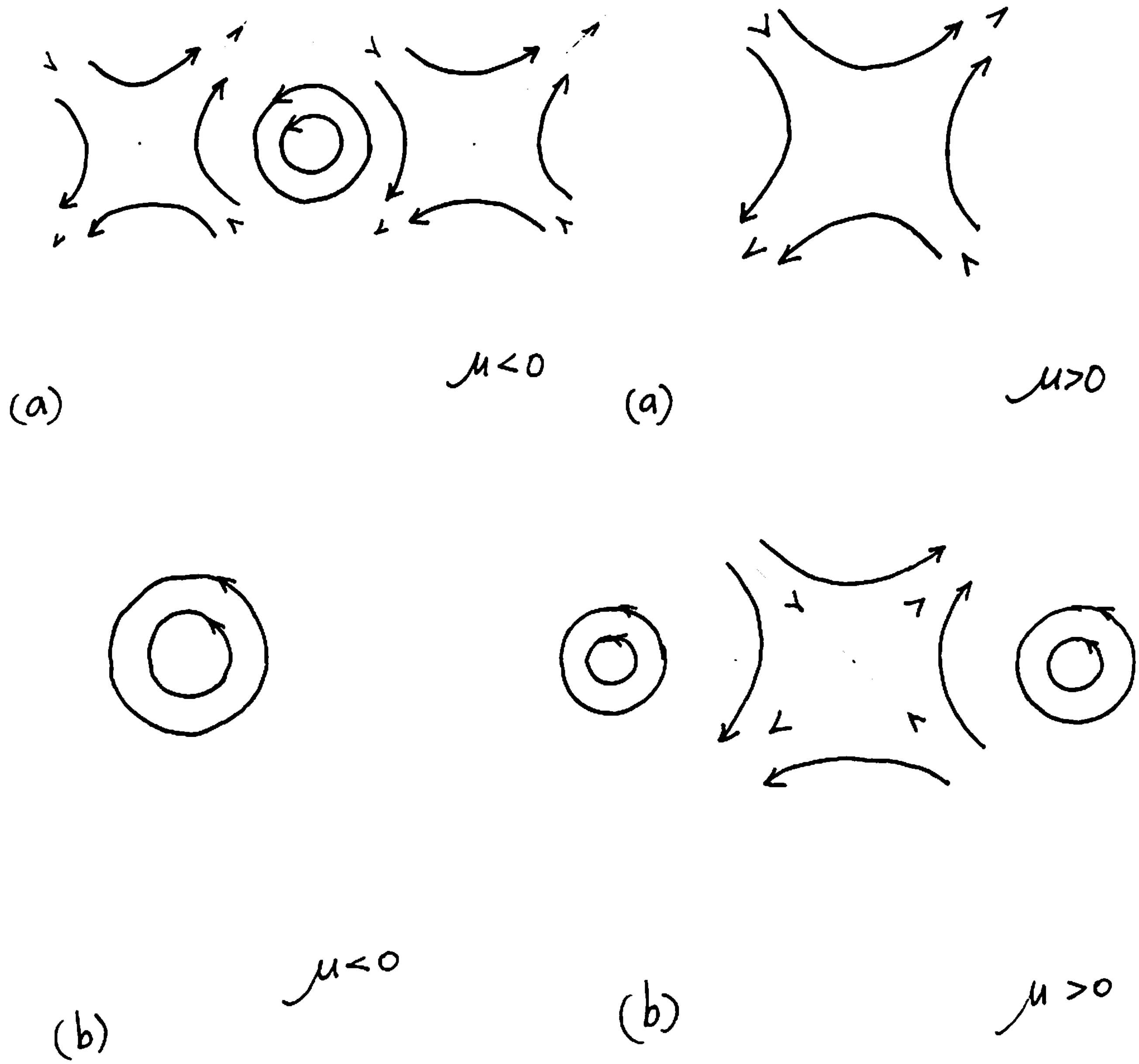


Figure 7. Phase Diagrams of Period Doubling Bifurcation
The period 2 point is only of one side of $\mu=0$ and is opposite of the origin. Again, the period 2 point intersects in two points and so the two points are really only one orbit. This is the case when $\alpha c>0$ and $\beta c>0$. (a) $\mu<0$: The fixed point is elliptic and the period 2 point is hyperbolic. $\mu>0$: The period 2 point disappears and the origin becomes hyperbolic. (b) $\mu<0$: The fixed point is again elliptic. $\mu>0$: The fixed point becomes hyperbolic and an elliptic period 2 point appears.

$I' = I - (1/k)2\gamma I^{k/2}\sin(k\phi) + \ldots$

$\phi' = \phi + (2\pi h/k) + \delta\mu + \beta I + \gamma I^{(k-2)/2}\cos(k\phi) + \ldots$

The βI term is called the twist term. The γ term is called the resonance term. Notice that, depending on k, these two terms vary in their importance. When k=4, they are of the same order, and in this case, the relative size of δ and β becomes important in the behavior of the bifurcations. I will examine the k=3 case, then the k>4 case, then look at the k=4 as a combination of the previous two.

For k=3 we look at the third iteration of the map given above:

$I^3 = I - 2\gamma I^{(3/2)}\sin(3\phi) + \ldots$

$\phi^3 = \phi + 2\pi h + 3\alpha\mu + 3\gamma I^{1/2}\cos(3\phi) + \ldots$

where we have dropped higher order corrections. In particular the twist term (βI) is of higher order than the resonance term (order $I^{1/2}$) so the twist term can be suppressed in the calculation.

<u>Theorem:</u> Let the origin be 3-bifurcation point as defined above. Then there are hyperbolic orbits of period 3 for both $\mu<0$ and $\mu>0$ and the period points tend to the origin as μ goes to zero.

<u>Proof:</u> We divide the first equation by $2\gamma I^{(3/2)}$ and find a 3 periodic point by the solving $(I^3 - I)/2\gamma^{(3/2)} = 0 = \sin(3\phi)$ which has solution $\phi_i(I) = i\pi/3 + \ldots$, where i=0,1,. . . .5. We plug this into the other equation : $I^3 - I = 0 = 2\pi h + 3\alpha\mu + 3\gamma I^{1/2}\cos(3\phi)$ and approximate $\cos(3\phi_i) = +1$ for i odd and $\cos(3\phi_i) = -1$ for even i. We then solve the equation $(\phi^3 - \phi_i - 2\pi h)/3 = 0 = \alpha\mu\ (p/m)\ \gamma I^{1/2}$ so the solutions are $I_i = +(\alpha\mu/\gamma)^2 + \ldots$

The Jacobian of the period three map is computed as follows at these solutions:

$dI^3/dI = 1 - 3\gamma I^{1/2}\sin(3\phi) = 1 + \ldots(\sin(3\phi) = 0 + \ldots$along the solution).
$dI^3/d\phi = -6\gamma I^{3/2}\cos(3\phi) = +\ 6\gamma I_i{}^{3/2} + \ldots$
$d\phi^3/dI = (3\gamma/2)(\cos 3\phi) = +\ (3\gamma/2)I_i{}^{1/2} + \ldots$
$d\phi^3/d\phi = 1 + 9\gamma I^{1/2}\sin(3\phi) = 1 + \ldots$

So we have the characteristic polynomial $(\lambda-1)^2 - (9/2)\gamma^2 I_i{}^2 = 0$ which has the solution

$\lambda = 1 \pm \sqrt{(9/2)\gamma^2 I_i^2} = 1 \pm \sqrt{(9/2)(\alpha\mu)^2}$. Thus all multipliers and so the 3-periodic points are hyperbolic for all μ and tend to the origin as μ tends to zero. □

We now examine the case k>4. The kth iterate map of a k-periodic point is :

$I^k = I - 2\gamma I^{k/2}\sin(k\phi) + \ldots$

$\phi^k = \phi + 2h\pi + \alpha k\mu + BkI + \ldots$

Here we drop the resonance term, as it of order $I^{3/2}$ and the twist term dominates it.

Theorem: Let the origin be a k-bifurcation point for k>4 of the above mapping for $\mu=0$. Then when $\alpha\beta<0$ there is exists at least pme elliptic and one hyperbolic orbit of period k for $\mu>0$ and no orbits of period k for $\mu<0$. The opposite holds for $\alpha\beta>0$. All the periodic orbits tend to the origin as μ goes to zero (see figure 8).

Proof : We solve the first equation of the k-iterate map by dividing by $-2\gamma I^{k/2}$ and obtaining $(I^k - I)/(-2\gamma I^{k/2}) = 0 = \sin(k\phi)$ so $\phi_i(I) = i\pi/k$, i = 0, ... 2k-1, is a solution. The second equation is $(\phi^k - \phi - 2\pi h)/k = 0 = \alpha\mu + \beta I$. So $I_i = -\alpha\mu/\beta$. Since I must be positive by the definition of action variables, we only have solutions when $\alpha\mu\beta < 0$, proving that no solutions exists for $\alpha\beta<0$ and $\mu < 0$, or $\alpha\beta>0$ and $\mu>0$. If $\alpha\beta\mu < 0$, the Jacobian becomes:

$$\begin{pmatrix} 1 & -2k\gamma I_i^{k/2}\cos(k\phi_i) \\ k\beta & 1 \end{pmatrix}$$

so the equation is $(\lambda-1)^2 + 2k^2\gamma\beta I_i^{k/2}\cos(k\phi_i) = 0$ and the multipliers are :

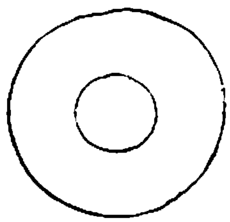
$\lambda = 1 \pm \sqrt{\pm 2k^2\gamma\beta I_i^{k/2}}$ where again $I_i = -\alpha\mu/\beta$. So we get both elliptic and hyperbolic points as the different values of i cause both the negative or positive solution to be taken inside the square root sign. □

Now we are ready to look at the 4-bifurcation point. Now the map is given by

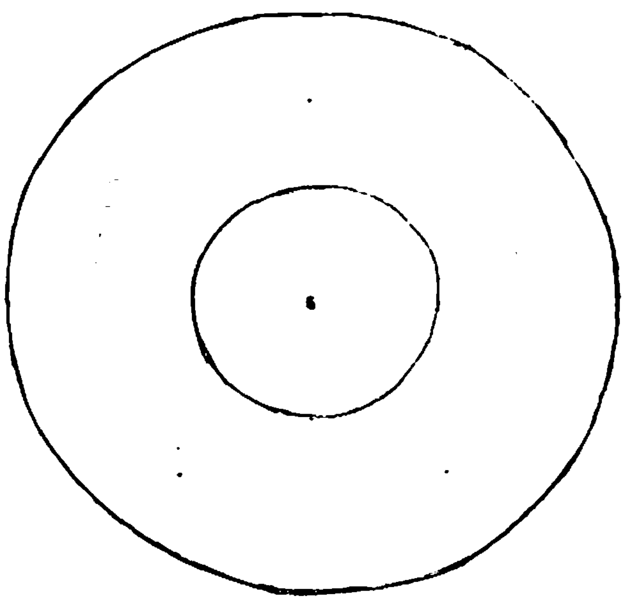
$I^4 = I = 2\gamma I^2\sin(4\phi) + \ldots$

$\phi^4 = \phi + 2h\pi + 4\alpha\mu + 4[\beta + \gamma\cos(4\phi)]I + \ldots$

where we keep both the twist and the resonance terms in the brackets. The prrofs I have been presenting or in Meyer and Hall, however, for the following theorem, my calculations



(a)



(b)

Figure 8. 5-Bifurcation Point

Take the case when $\alpha\beta < 0$ (a) There are no periodic points for $\mu<0$. Here the fixed point is presented as elliptic though the proof did not address this. (b) Ten solutions appear for k=5. Half are elliptic and half are hyperbolic.

disagreed with those of the authors. Therefore I will present what I calculated and mention where the disagreements occur.

<u>Theorem</u>: If $\beta \pm \gamma$ have different signs then there are hyperbolic period 4 orbits for one side of μ=0 and elliptic points on the other side of μ=0. If $\beta \pm \gamma$ have the same sign, then when $\alpha(\beta \pm \gamma) < 0$ there is an elliptic or hyperbolic period orbit of period 4 for μ>0 and no period 4 orbit for μ<0. The opposite holds when $\alpha(\beta \pm \gamma) > 0$.

Meyer claims that if $B \pm \gamma$ have the same sign then there are hyperbolic points on both sides of μ=0. He says that if $\beta \pm \gamma$ have the same sign then both hyperbolic and elliptic solutions are on one side of μ=0 and none are on the othe side. We agree about when the solutions exist (always for the first case, and for one side of μ=0 for the second), however we disagree about the stability types.

<u>Proof</u>: The proof is much the same as the previous two. We divide the first equation by $-2\gamma I^2$ and get $I^4 - I = 0 = \sin(4\phi) + \ldots$ so $\phi_i = i\pi/4$ i = 0,. . .7. The second equation then becomes $(\phi^4 - \phi - 2\pi h)/4 = 0 = \alpha\mu + [\beta + \gamma\cos(4\phi_i)]I + \ldots$ which becomes $\alpha\mu + (\beta + \gamma)I + \ldots = 0$. So a solution is $I_i = -\alpha\mu/(\beta \pm \gamma)$. Note that I has a positive solution only when $\alpha\mu(\beta + \gamma) < 0$. So if $(\beta \pm \gamma)$ have different signs then 4 of the 8 values of i give a good solution. For the ones that do, the period 4 points exists only for μ<0 or μ>0, but not both. As μ passes through zero, the actual values of i that contribute to the solution changes but there are the same number of points on either side. This is like the 3-bifurcation case, where periodic orbits exists on both sides. If, however, $(\beta \pm \gamma)$ have different signs then all 8 solutions exists for the same side of μ and the other side contains no solutions. This is like the k>4 case where there are no periodic solutions on one side of μ. The Jacobian elements are :

$dI^4/dI = 1 - 4\gamma I\sin(4\phi) + \ldots = 1 + \ldots$ along the solution.
$dI^4/d\phi = -8\gamma I^2\cos(4\phi) + \ldots = \mp 8\gamma I_i^2 + \ldots$
$d\phi^4/dI = 4[\beta + \gamma\cos(4\phi)] + \ldots = 4(\beta \pm \gamma) + \ldots$
$d\phi^4/d\phi = 1 + 16\gamma I\sin(4\phi) + \ldots = 1 + \ldots$

So the equation to give the multipliers is $(\lambda-1)^2 \pm \sqrt{32\gamma I_i^2(\beta \pm \gamma)} = 0$. Thus $\lambda = 1 \pm \sqrt{-32\alpha^2\gamma\mu^2/(\beta \pm \gamma)}$ where we substitute the equation for I_i. In the case of $(\beta +- \gamma)$ changing sign, the sign of $(\beta \pm \gamma)$ changes as μ goes through zero since the other 4 solutions are the ones used. Thus the stability type changes from elliptic to hyperbolic as μ passes through zero though solutions exist on both sides. Since μ is squared in the expression this does not change sign as μ passes through 0. Which side of μ has elliptic or hyperbolic points will depend on the sign of γ. In the case of $\beta \pm \gamma$ of the same sign, all solutions are always elliptic or always hyperbolic, depending on γ, but only exist on one side of μ by the same argument as above. □

I am not sure where the origin of the disagreement. Meyer compares the first case to the 3-bifurcation point situation. However, an important difference is that in that case we solved and found $I = (\alpha\mu/\gamma)^2$ whereas we found in the k=4 case $I = \alpha\mu/\gamma$ so the μ term is square in one solution and not in the other. This causes the sign change to be different. Meyer does not explicitly calculate the Jacobian or multipliers so I do not know where the difference occured.

The theorems we proved in the this section have some interesting applications. One is a proof by Birkkhoff applied to Duffing's equation. The theorem is that near a general elliptic points, there exists periodic points of arbitrarily high period. He does this by putting the Hamiltonian like the one in our other example with Duffing's equation. The forced term, though not necessarily the non-linear term are small. He then is able to put the Hamiltonian in normal form and show that a twist term (of the form βI) dominates the ϕ Poincaré periodic mapping. From there, he is able to show that for any k>4, there are solutions corresponding periodic solutions of the Poincaré map.

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