ROTATING SYSTEMS IN

GOLDSTEIN AND LANDAU AND LIFSHITZ

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The purpose of this paper is to compare different expositions of the dynamics of a rotating body in a non-inertial frame. I have chosen Goldstein's Classical Mechanics and Landau and Lifshitz's Mechanics as the classical texts. I will restrict the paper to the development up to the formulation of the equations of motion given in rotating coordinates. To make this project germaine to the course, I will compare the classical texts to the development in Marsden's lectures.

A proper analysis of any physical system requires an adequate choice of coordinate systems and configuration space. In their texts, Goldstein and Landau and Lifshitz detail how to reduce the number of general coordinates to the proper degrees of freedom. For a rigid body, the degrees of freedom total six: three to place the center of mass and three to determine the movement of the body around the center of mass. This argument is given in L. and L. Goldstein makes a similar, though more general argument, formally starting with 3N degrees (N representing the number of particles), and then applying the condition that the relative distances between particles in the body remain constant. Taking the special case where the center of mass stays at rest, i.e., no translations, we get a total of three degrees of freedom.

By setting an inertial frame with origin at the body's center of mass, the rotations become equivalent to a set of orthogonal matrices. Goldstein devotes a section to orthogonal matrices and proves that a physical rotation corresponds to 3x3 orthogonal matrix with determinant +1 and at least one eigen value equal to +1. L. and L. refer to their arguments on angular momentum, where they argue geometrically $|\delta r| = |r||d\Phi|\sin\theta$. Hence $\delta r = \delta\Phi \times r$, where r is the radius vector of a given particle or point mass, and $\delta\Phi$ is the infinitesimal angular rotation vector perpendicular to the plane of rotation, i.e. in the direction of the +1 eigen vector of the orthogonal transformation. Marsden avoids this bare hands approach and introduces the space SO(V) of orthogonal transformations on the inertial frame V. Non-homogeneous rotations then correspond to a curve $\psi_t \in SO(V)$.

The key to deriving the equations of motion is obtaining the velocity in terms of the orthogonal transformation. As just mentioned, L. and L. present $\delta r = \delta\Phi \times r$ directly from the geometry of the problem. Goldstein takes a much longer and subtle approach . After introducing the Eulerian angles as the standard set of generalized coordinates, he introduces the Cayley-Klein parameters as another option. I see this as a prelude to the association of the cross product with matrix multiplication. He introduces the 2x2 matrix which takes its values in $\mathbb{C}^2$. By requiring his matrix Q to be unitary and to have determinant +1, the eight real values are reduced to three; a correspondence is naturally sought with the rotating system.

Taking $P = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$; $x,y,z \in \mathbb{R}$, then P is transformed by Q according to the relation $P'=QPQ^*$. The Hermitian property and the trace of P reamian unchanged by a similarity transformation, so $P' = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix}$; $x', y', z' \in \mathbb{R}$. The determinant of P is also invariant, so $|P| = -(x^2+y^2+z^2) = -(x'^2+y'^2+z'^2) = |P'|$ Hence for each orthogonal transformation that takes (x,y,z) to (x',y',z') there is a matrix Q in $\mathbb{C}^2$.

While this analysis does not directly come into the computation of the equations of motion, pedagogically it serves a purpose. It introduces, albeit in a relatively complicated manner, the notion of different mathematical objects, in this case complex matrices and transformations, both representing the same event. He returns later linking real matrices with real vectors.

Goldstein then derives geometrically the change in the radius vector under a finite rotation around the vector n, by an angle Φ :

(*) $r' = r\cos\Phi + n(n\cdot r)(1 - \cos\Phi) + (r \times n)\sin\Phi$. Here and later the ' represents the rotated vector.

Goldstein sets up the problem of again reducing the number of degrees of freedom to 3. Put into matrix form (*) has too many. Again he wants to associate mathematical objects to simplify computations. Associating the transformation with a vector will not do: the composition of matrices corresponds to an addition of vectors, and vector addition is commutative while matrix composition is not. (Here again, Goldstein makes use of the oft-maligned group jargon.) However, the transformations associated with infinitesimal rotations do commute. Goldstein argues:

$r' = (I + \varepsilon)r$, ε is infinitesimal and higher orders are discarded.

$(I + \varepsilon_1)(I + \varepsilon_2) = I + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2 = I + \varepsilon_1 + \varepsilon_2$.

Hence, $I - \varepsilon = (I+\varepsilon)^{-1} \doteq (I+\varepsilon)^T$ since $I+\varepsilon$ is orthogonal. So $\varepsilon^T = -\varepsilon$ and $\varepsilon = dr$ is an anti-symmetric matrix: $\begin{pmatrix} 0 & -d\Omega_3 & d\Omega_2 \\ d\Omega_3 & 0 & -d\Omega_1 \\ -d\Omega_2 & d\Omega_1 & 0 \end{pmatrix}$

Comparing with the cross product we get

$dr = d\Omega \times r$, $d\Omega = n\, d\Phi$

Marsden begins with the velocity vector field coupled to the roation group: $X_t(\psi_t(v)) = \frac{d}{dt}\psi_t(v)$ or $X_t = \dot{\psi}_t \psi_t^{-1}$. Then we have the claim: X_t is anti-symmetric.

Proof: $\psi_t \psi_t^T = I \;\rightarrow\; \frac{d}{dt}\psi_t \cdot \psi_t^T + \psi_t \cdot \frac{d}{dt}(\psi_t^T) = 0$

$$X_t \cdot \psi_t \cdot \psi_t^T + \psi_t \cdot (X_t \cdot \psi_t)^T = 0$$

$$X_t \cdot (\psi_t \cdot \psi_t^T) + ((X_t \cdot \psi_t) \cdot \psi_t^T)^T = 0$$

$$X_t + X_t^T = 0 \quad \blacksquare.$$

HENCE:

$$X_t = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$X_t(r) = \hat{\omega}(r) = \omega \times r .$$

Already we see a variety of techniques to arrive at the same simple equation $dr = d\Omega x r$. The similarity is in the use of the infinitesimal. Though Goldstein's use of infinitesimal ε's is a bit sloppy, this general idea is used in all three developments.

Deriving the equations of motion in non-inertial coordinates now requires a modification of the velocity equation. L. and L., in keeping with their concise format, just compose vectors:

$v = v' + \left(\frac{d\Omega}{dt} x\, r\right) = v' + (\omega x\ r)$. (L. and L. actually have a translational velocity term that is carried through the calculations. For our purposes we can set it to zero.)

Goldstein also composes vectors, but then justifies his result using orthogonal transformations. Taking any r:

$$r_i = a_{ij}{}^T r_j' = a_{ji} r_j' \quad ;$$

taking differentials and readjusting the inertial coordinates so that they line up with the rotating frame:

$$dr_i = a_{ji}\, dr_j' + da_{ji}\, r_j' = dr_i' + (\varepsilon^T)_{ij}\, r_j$$

$$= dr_i' - \varepsilon_{ij} r_j = dr_i' + (d\Omega \times r)_i$$

Arguing that the choice of the vector r is arbitrary, Goldstein arrives at a general time derivative operator:

$$\frac{d}{dt}(\cdot) = \left(\frac{d}{dt}(\cdot)\right)' + \omega \times (\cdot)$$

Marsden makes a similar argument: active orthogonal transformations correspond to a change of basis, from the inertial to the rotating frame. So $r = Ar'$; $\hat{\omega} = \dot{A}A^{-1}$ for a time dependent matrix A associated with the orthogonal transformation. Hence,

$$\frac{dr}{dt} = A\frac{dr'}{dt} + \mathring{A}r' = A\frac{dr'}{dt} + \hat{\omega}(Ar') = A\frac{dr'}{dt} + \hat{\omega}(r)$$
$$= A\frac{dr'}{dt} + \omega \times r \ .$$

The final step is to substitute properly to get the equations of motion. L. and L. use a strict Langrangian formulation.

$$L = \tfrac{1}{2}m|v|^2 - U$$
$$L = \tfrac{1}{2}m|v' + \omega \times r|^2 - U$$
$$L = \tfrac{1}{2}m|v'|^2 + mv' \cdot \omega \times r + \tfrac{1}{2}m(\omega \times r)^2 - U$$
$$\frac{\partial L}{\partial v} = mv' + m\omega \times r$$
$$\frac{\partial L}{\partial r} = mv' \times \omega + m(\omega \times r) \times \omega - \nabla U$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial r} \longleftrightarrow m\dot{v}' + m(\dot{\omega} \times r) + m(\omega \times v) = m(v \times \omega) + m(\omega \times r) \times \omega - \nabla U$$

$$\longleftrightarrow m\ddot{r}' + 2(\omega \times mv)$$
$$+ \ m\omega \times (\omega \times r)$$
$$+ \ m(\dot{\omega} \times r)$$
$$= -\nabla U .$$

Again, L.and L. line up the inertial frame with the rotating frame so that r and ω are the same in both frames.

Goldstein first applies his rate of change operator to the radius vector, getting $v = v' + (\omega \times r)$. Applying the operator again to v, he derives

$$\dot{v} = \left(\frac{d}{dt}(v)\right)' + \omega \times v$$
$$= \left[\frac{d}{dt}(v' + \omega \times r)\right]' + \omega \times (v' + \omega \times r)$$
$$= \dot{v}' + 2(\omega \times v') + \omega \times (\omega \times r)$$

Goldstein here assumes the rotation is homogeneous. Plugging into F = ma :

$$F = ma' + 2m(\omega \times v') + m\omega \times (\omega \times r)$$

Marsden differentiates $\frac{dr}{dt} = \omega \times r + A\frac{dr'}{dt}$ and plugs the result into F = ma:

$$m\ddot{r} = m(\dot{\omega} \times r + \omega \times \dot{r} + \dot{A}\frac{dr'}{dt} + A\ddot{r}')$$

$$= m\ddot{r}' + 2(\omega \times \dot{r}')$$
$$+ m(\omega \times (\omega \times r))$$
$$+ m\dot{\omega} \times r = -\nabla U$$

Marsden also calculates the Langrangian using his velocity equation:

$$L = \tfrac{1}{2}m|q|^2 - U = \tfrac{1}{2}m|\omega \times q + A\dot{q}'|^2 - U$$
$$= \tfrac{1}{2}m|\omega' \times q' + \dot{q}'|^2 - U$$

The main difference here between Marsden's exposition and that in the classical text is the emphasis on the change of basis. For all practical purposes, the variables in the force equation are all expressed in rotational coordinates. The other authors imply this, but do not make it explicit.

Overall, Marsden's development I believe has antecedents in both texts. Landau and Lifshitz keep their arguments concise, in keeping with theri overall philosophy in the text. Goldstein presents the concepts more naively, always providing more formal justification if necessary. However, I find the use of the group SO(V) to aid the computation in a much more graceful fashion. It takes no extra effort to introduce non-homogeneous rotations, and so one gets the Euler forces for free in the development. Goldstein seems to imply by examining only homogeneous rotations that the computations would be unnecessarily bogged down. I did however find the Cayley Klein parameters an interesting side light. Ideally I believe it would serve a good purpose in exposing the utilitarian nature of mathematical objects in defining physical systems. Just as the Lagrangian and Newtonian methods of deriving the equations of motion are equivalent, many other methods of analysis can be put in correspondence. The exploration of the nature of these mathematical relationships provides a keener insight into our general methods of physical abstraction.