ALGORITHMIC IMPLEMENTATION OF ENERGY METHODS FOR CLASSIFYING STABILITY OF RELATIVE EQUILIBRIA

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June 2003

Abstract

This paper addresses the use of the well established Energy-Casimir and Energy-Momentum Methods for determining stability of relative equilibria. In particular, these methods will be compared and analyzed in the context of a possible algorithmic implementation for automated classification of the stability of relative equilibria. This analysis will complete the first step towards constructing a software package, built on the powerful Energy methods, to systematically determine and classify the stability of relative equilibria in a broad class of physical systems.

CONTENTS:

1. INTRODUCTION

In this paper, I will discuss the Energy-Casimir and Energy-Momentum methods, constructing each from basic theorems and definitions. I will further investigate an algorithmic implementation of these methods. This will allow for computer automated classification of the stability of a large class of physical systems. As the Energy-Momentum method is a general technique that may be applied to nearly all physical systems with symmetry, new modules may be added to the program, including topological information about the configuration space and expressions for the Lagrangian and for constraints, allowing for the automated analysis of entire classes of new physical systems.

Sections 1 through 3 of this paper involve basic definitions and theories regarding stability and relative equilibria, whereas in Sections 4 & 5, the two Energy methods are described and analyzed in the context of a potential computer implementation. For this analysis, it is important to keep in mind the capabilities and limitations of available mathematical software packages such as Mathematica and MAGMA computer algebra software.

2. RELATIVE EQUILIBRIA

In the study of dynamical systems there are few features more important than the *equilibrium positions* of the system. Aside from the obvious implications of knowing the states of equilibrium of a physical system, the dynamics surrounding these positions provide invaluable information about systems that are too complex to solve directly. Equilibrium positions are positions corresponding to critical points of the dynamical system—these particles are both at rest and have no forces acting on them, hence they are in a state of equilibrium.

When analyzing dynamical systems with inherent symmetry, it is often valuable to look at equilibrium points of a reduced phase space, which are called *relative equilibria* of the original system. Given the Poisson action of a Lie group G on a symplectic manifold P, we construct the reduced phase space by taking the manifold of orbits of the action of G_p on P_p, where P_p corresponds to the level set of the momentum of the original symplectic manifold P. In other words, P_p is the set given by the inverse image of some $p \in \mathfrak{g}^*$ under the standard momentum mapping J [11]. The relative equilibria of a system in motion, such as an asteroid pair, are often of great importance to understanding the overall dynamics of the system and to resolving issues of control. More formally, *relative equilibria* are phase curves

in P that project to equilibrium positions in the reduced system and on the reduced phase space.

There are many illustrative examples of relative equilibria—for instance, the stationary rotations of a rigid body that is fixed at is center of mass, or the two solutions to the double spherical pendulum:

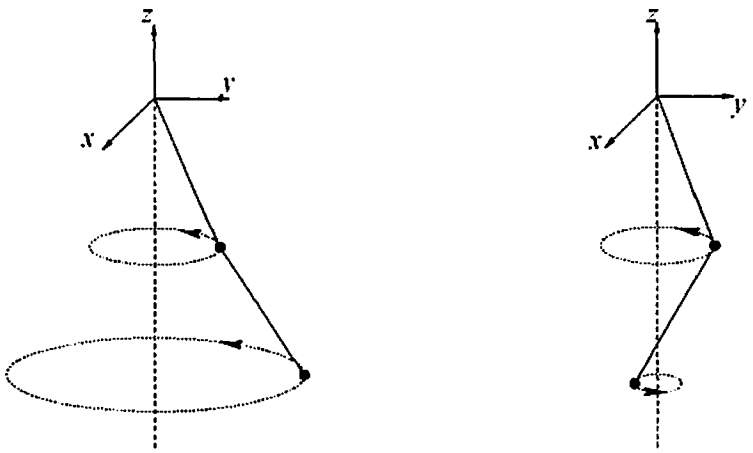

Figure 1: The solution on the left is the stretched-out equilibrium and the solution on the right is the cowboy equilibrium. This figure was taken from [7].

3. Stability Criterion

When studying the equilibrium points of a dynamical system, a natural question arises—do points that start in a small neighborhood of an equilibrium point stay in the vicinity of that point? This is a question of the stability of a given equilibrium point and is of significant importance when analyzing systems experiencing perturbations. Considerable time and effort have been applied to developing the theory of stability of relative equilibria and perturbation theory. To this end, we will make use of Liapunov's Theorem and the Lagrange-Dirichlet Criterion.

3.1 Liapunov's Theorem

Theorem: If all eigenvalues of $\mathbf{d}f(x_e)$ have real part that is strictly negative, then the fixed point x_e is stable—conversely, if the real part of any eigenvalue is positive then the fixed point is unstable. —Note: $\mathbf{d}f(x_e)$ is the Jacobian of f at the point x_e, which is defined as the matrix of partial derivates $\left[\frac{\partial f^i}{\partial x^j}\right]_{x=x_e}$.

Sketch of Proof: This is a relatively elementary result from the theory of Ordinary Differential Equations. Solutions to the system $\dot{x} = F(x)$ may be written as a linear combination of $e^{\lambda_i t}$'s, where λ_i is the i^{th} eigenvalue of F. Thus, if any of the eigenvalues of $\mathbf{d}f(x_e)$ have a positive real component, then there will be a blowup, and the fixed point will be unstable in time. Likewise, if the eigenvalues all have negative real components, then each component will decay to zero as time grows sufficiently large, meaning that if we start in a neighborhood of x_e, then we will remain in some finite neighborhood of x_e, and eventually approach x_e. Thus the fixed point is stable.

This form of Liapunov's Theorem cannot be applied to determine the stability of fixed points in a Hamiltonian system as eigenvalues are always in quartets distributed symmetrically about the origin, and thus always have eigenvalues in both the left and right half planes. Thus we turn to the Lagrange-Dirichlet criterion for fixed points of a Hamiltonian system.

3.2 Lagrange-Dirichlet Criterion

Theorem: If the $2n \times 2n$ matrix of second partial derivates, i.e. the second variation $\delta^2 H$, is either positive or negative definite at (q_e, p_e) then it is a stable fixed point.

Proof: We simply consider the case when $\delta^2 H$ is positive definite. For a fixed point (q_e, p_e) then $\frac{\partial H}{\partial q}(q_e, p_e) = \frac{\partial H}{\partial p}(q_e, p_e) = 0$. Thus, at $z_e = (q_e, p_e)$, H has a nondegenerate minimum, which allows us to apply Taylor's theorem. Thus, we find that near z_e, level sets of H are bounded on both sides by spheres of arbitrarily small size. Finally, by the conservation of energy, all solutions are restricted to level surfaces of H, implying that any solu tion beginning in a small neighborhood of z_e will stay near this minimum.

This criterion is very useful for Hamiltonians of the form $H=T+V$ since critical points are found when $p_e=0$ and q_e is a critical point of the potential V. Thus, when $H=T+V$, one needs only find a non-degenerate minimum of the potential.

4. ENERGY-CASIMIR METHOD

The Energy-Casimir Method is a generalization of the method obtained from the Lagrange-Dirichlet criterion. This method tests the stability of a given equilibrium z_e, which satisfies $X_H(z_e)=0$, by choosing some conserved function C such that z_e is a critical point of $H+C$ and computing the second variation $\delta^2(H+C)(z_e)$. If the resulting matrix is either positive or negative definite then the equilibrium z_e is stable. Broken up into steps, the Energy-Casimir Method may be stated as follows [4]:

Step 1. Find some conserved function C such that it satisfies the relationship:

$$\delta(H+C)(z_e)=0.$$

Step 2. We then calculate the second variation $\delta^2(H+C)(z_e)$.

Step 3. If the second variation $\delta^2(H+C)(z_e)$ is either positive or negative definite, then z_e is formally stable.

The third step requires explanation, and the conserved functions in the first step deserve further treatment.

4.1 Casimir Functions

A Casimir function is one that Poisson commutes with every other function defined on the phase space of a given Hamiltonian system. In other words $\{C,F\}=0$ for all functions F in the phase space P. Since $\dot{C}=\{C,H\}=0$, we see that Casimir functions are constant along the flow of all Hamiltonian vector fields; i.e., $X_C=0$. Also, since Casimir functions commute with all functions defined on the phase space by definition, the set of all Casimir functions forms the center of the Poisson Algebra—this follows from definition, as the center of a group is the set of all elements that commute with every element of the group.

Thus we see, upon examination of Casimir functions, that Step 3 of the Energy-Casimir method truly is a simple extension of the Lagrange-Dirichlet criterion. As long as we verify that z_e is a critical point of $H+C$ then we see that if

$\delta^2(H+C)(z_e)$ gives a definite matrix, then z_e is stable, by a simple variation on Lagrange-Dirichlet criterion. This may be stated strongly as the proof of the criterion holds for all of the above modifications.

4.2 Example: Rigid Body [12]

Although the previous section outlined the general concept of the Energy-Casimir method, it is far more revealing to apply this method to a simple problem. We now analyze the stability of the rigid body:

For the free rigid body, the Hamiltonian will simply correspond to the Kinetic Energy of the body. Letting Ω and Π represent the angular velocity and angular momentum, respectively, we see the natural relationship $\Pi_i = I_i \Omega_i$, where I is the standard moment of inertia tensor. Thus we have the following

$$\dot{\Pi} = \Pi \times \Omega \qquad (4.2\ .1)$$

$$H(\Pi) = \tfrac{1}{2}\, \Pi \cdot \Omega \qquad (4.2\ .2)$$

which are the equation of motion and the Hamiltonian. From [11] & [12] we see that given some smooth function $\Phi : \mathbb{R} \to \mathbb{R}$, then there exists a Casimir function:

$$C_\Phi(\Pi) = \Phi\left(\tfrac{1}{2}\, \|\Pi\|^2\right)$$

We now apply the three steps of the Energy-Casimir method, directly as described in section 4:

Step 1. We find some C_Φ such that the first variation of $H_{c_\Phi} = H + C_\Phi$ is zero at some equilibrium point of equation (4.2.1). Upon inspection we see that these equilibrium points occur when Π is parallel to Ω. We normalize and assume that Π and Ω are pointing in the x-direction, for ease of calculation. Thus $\Pi_e = (1, 0, 0)$ is an equilibrium solution. Taking the derivative of H_{C_Φ} we see that:

$$H_{C_\Phi}(\Pi) = \tfrac{1}{2}\, \Pi \cdot \Omega + \Phi\left(\tfrac{1}{2}\, \|\Pi\|^2\right)$$
$$\Longrightarrow D H_{C_\Phi}(\Pi) \cdot \delta\Pi = \left(\Omega + \Phi'\left(\tfrac{1}{2}\, \|\Pi\|^2\right)\Pi\right) \cdot \delta\Pi.$$

This derivative is zero precisely when $\Omega + \Phi'\left(\tfrac{1}{2}\, \|\Pi\|^2\right)\Pi = 0$. At $\Pi_e = (1, 0, 0)$, we see that the derivative is zero when $\Phi'\left(\tfrac{1}{2}\right) = -\tfrac{1}{I_1}$.

Step 2. Now we use the expression for $\mathbf{D}H_{C_\Phi}(\Pi)\cdot\delta\Pi$ and take the second derivative of H_{C_Φ} at the equilibrium $\Pi_e = (1, 0, 0)$:

$$\begin{aligned}\mathbf{D}^2 H_{C_\Phi}(\Pi)\cdot(\delta\Pi, \delta\Pi) &= \delta\Omega\cdot\delta\Pi + \Phi'\left(\tfrac{1}{2}\,\|\Pi_e\|^2\right)\|\delta\Pi\|^2 + (\Pi_e\cdot\delta\Pi)^2\,\Phi''\left(\tfrac{1}{2}\,\|\Pi_e\|^2\right)\\ &= \sum_{i=1}^{3}\frac{(\delta\Pi_i)^2}{I_i} - \frac{\|\delta\Pi\|^2}{I_1} + \Phi''\left(\tfrac{1}{2}\right)(\delta\Pi_1)^2\\ &= \left(\tfrac{1}{I_2} - \tfrac{1}{I_1}\right)(\delta\Pi_2)^2 + \left(\tfrac{1}{I_3} - \tfrac{1}{I_1}\right)(\delta\Pi_3)^2 + \Phi''\left(\tfrac{1}{2}\right)(\delta\Pi_1)^2.\end{aligned}$$

Step 3. Finally, we test this quadratic form for definiteness. By looking at each term, we see that this form is positive definite if and only if $I_1 > I_2$, $I_1 > I_3$, and $\Phi''\left(\frac{1}{2}\right) > 0$. Taken with the condition that $\Phi'\left(\frac{1}{2}\right) = -\frac{1}{I_1}$, we integrate Φ'' twice, obtaining

$$\Phi(x) = -\tfrac{1}{I_1}\,x + \left(x - \tfrac{1}{2}\right)^2$$

Thus, as this gives a second variation of H_{C_Φ} at (1, 0, 0) which is positive definite, it follows that stationary rotations about the shortest axis is stable in the Liapunov sense. By applying the same method to construct a $\Phi(x)$ that makes the quadric form negative definite, we see that rotations along the long axis is also stable [Liapunov]. The quadratic form is always indefinite if $I_1 > I_2$ and $I_3 > I_1$, or the reverse. By the Energy-Casimir Method, there is no way to prove that the rotation around the middle axis is unstable however. To prove this result, one might use spectral analysis, a treatment of which is given in [12].

Thus we see the usefulness of the Energy-Casimir method in demonstrating that rotations along the shortest and longest axis are Liapunov stable, though it does not provide an instability statement for rotations about the middle axis.

4.3 Limitations

Although the Energy-Casimir method often offers a direct and simple generalization of the Lagrange-Dirichlet criterion, it relies on a wealth of Casimir functions, and only offers stability information: the Energy-Casimir method cannot be directly applied to infer instability. In some examples, such as the dynamics of Geometrically exact rods, no Casimir functions have been found, and we cannot get past the first step of the process. Also, problems arise when one tries to apply the

Energy-Casimir method to systems with infinitely many degrees of freedom (such as fluids or plasmas) [12].

5. ENERGY-MOMENTUM METHOD

The method developed to overcome the limitations described in section 4.3 is known as the Energy-Momentum method, and is closely linked to the method of reduction. The Energy-Momentum method also uses conserved quantities, but rather than Casimir functions that may or may not exist, it uses the conservation of energy and the momentum map, which are more readily available than Casimir functions. Similar to the Energy-Casimir method, the Energy-Momentum method makes use of the augmented Hamiltonian, which is of the form

$$H_\xi(q,p)=H(q,p) - \xi \bullet \mathbf{J}(q,p)$$

Where **J** is the momentum map and ξ may be thought of as a Lagrange multiplier. One begins by equating the first variation of H_ξ with zero, obtaining a relative equilibria. Stability is then determined by taking the second variation $\delta^2 H_\xi$ and checking for definiteness. More formally, we may construct the method for a mechanical system with symmetry as follows:

> Let Q be the configuration space with phase space T^*Q and some symmetry group G acting with the standard momentum map **J**: $T^*Q \rightarrow \mathfrak{g}^*$, with $\mathfrak{g}^*$ as the Lie algebra of G. We begin by constructing the problem on the unreduced space, where relative equilibria associated with an element ξ of the Lie algebra are simply critical points of the augmented Hamiltonian $H_\xi := H - \langle \mathbf{J}, \xi \rangle$. We now calculate the second variation of H_ξ at some relative equilibrium z_e with momentum μ, subject to the condition that $J = \mu$ and on the space transverse to G_μ, which is the subgroup of G that fixes μ. Thus our space $\boldsymbol{S}$ is a subspace of the kernel of $\mathbf{DJ}(z_e)$ and is transverse to the G_μ orbit inside the kernel of $\mathbf{DJ}(z_e)$. The energy momentum method requires us to simply:
>
> (1) Find some ξ in $\mathfrak{g}$ such that the variation of H_ξ at the equilibrium point is zero, and
>
> (2) Test the second variation of H_ξ for definiteness on our space $\boldsymbol{S}$.

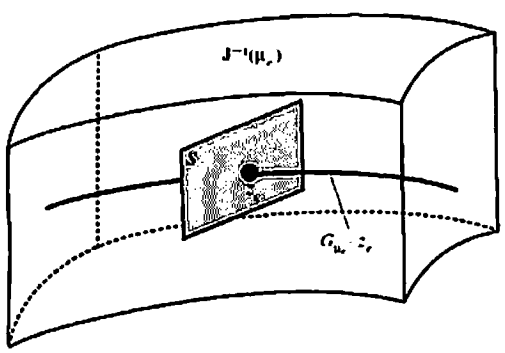

Figure 2: The energy-momentum method tests the second variation of H_ξ for definiteness on $\boldsymbol{S}$. This figure was taken from [11].

5.1 Energy-Momentum Theorem (proof is due to [11])

Theorem: If $\delta^2 H_\xi(z_e)$ is definite, then z_e G_μ-orbitally stable in $\mathbf{J}^{-1}(\mu)$ and G−orbitally stable in P; i.e., then the relative equilibrium z_e is stable.

Proof: The orbit space P_μ has well defined dynamics as induced by the reduced dynamis. Thus we have G_μ−orbital stability within $\mathbf{J}^{-1}(\mu)$; the well defined dynamics on P_μ induces a dynamical system on $\mathcal{S}$ for which H_ξ is an invariant function. As there is a non-degenerate extremum at the equilibrium position z_e, the invariance of its level sets is sufficient for stability. Further, we find G_ν−orbital stability within $\mathbf{J}^{-1}(\nu)$ for ν that are close to μ as the form of Figure 2 changes in a regular way since μ is a regular value and a generic point in $\mathfrak{g}^*$.

5.2 Converse To Energy-Momentum Theorem

Aside from the power of the Energy-Momentum method to handle physical systems with few or no Casimir functions, or systems with infinitely many degrees of freedom, it has the remarkable property that the second matrix may be block diagonalized as described in [11], [13], and [14].

Using this block structure, the converse of the energy-momentum method has been proven, in some sense, as an indefinite second variation implies that the system is unstable. This is not as simple as stated, since many examples are formally unstable with purely imaginary eigenvalues. This converse becomes particularly useful when discussing dissipation-induced instability [5],[8]—the converse can show that with the addition of dissipation, then the system destabilizes.

This converse is perhaps what exhibits most clearly the maturity and power that sets the Energy-Momentum method apart from the Energy-Casimir method.

FINAL THOUGHTS

Having analyzed both the Energy-Casimir and Energy-Momentum methods, we may now address the issue of algorithmic implementation towards automated classification of relative stability. It is simple to see the merits of the Energy-Momentum method over the Energy-Casimir method for a potential program designed to determine stability, as the Energy-Momentum method is both more general and has a converse statement involving dissipation induced instabilities [5]. We must now analyze each step in the Energy-Momentum method, looking both for ways to implement the method in a program, and for difficulties that may arise in the implementation.

From [12], we find that "the second variation $\delta^2 H_\xi$ and the symplectic structure (and therefore the equations of motion) can be explicitly brought into normal form simultaneously" [page 155]. Thus, it seems likely that given minimal topological information regarding the configuration manifold, and a Lagrangian and constraint equations, the Energy-Momentum method could be programmed on a case-by-case basis.

The next step requires us to determine the precise topological data needed by the method from both the configuration manifold and the symmetry group. We must then structure this data for use by the algorithmic implementation of the Energy-Momentum method. Finally, the method used to program examples case-by-case must be abstracted into a general method that allows our classification program to simply accept new conditions defining a physical system in the form of data structures describing the system's Configuration Manifold, Symmetry Group, constraint equations, and Lagrangian. Eventually we may construct a library with different manifolds, constraints, and Lagrangians, and simply pick-and-choose which physical system to apply the Energy-Momentum method to.

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