# Casey's Geometrical Derivation of Lagrange's Equations

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Many different derivations of Lagrange's equations have been done over the years. Some have made use of variational principles, the principal of virtual work, or Newton's second law. James Casey [1, 2] has done a derivation of Lagrange's equations for the case of a finite set of particles and a separate derivation for the case of a rigid continuum. Both were done using a geometrical approach and placing no restrictions on the types of forces on the systems. Synge and Schild [4] had previously done a derivation for a set of particles using a geometrical viewpoint, but it was heavy in tensor calculus, whereas Casey's derivation is much simpler, and manages to avoid the use of Christoffel symbols altogether.

Consider a body consisting of a finite set of particles $X_1, X_2, \ldots, X_N$ in three-dimensional space. Let $(x_i^1, x_i^2, x_i^3)$ be the Cartesian components of the position vector $\underset{\sim}{x}_i$ of particle $X_i$. Also let $\underset{\sim}{v}_i$ be the velocity and $M_i$ be the mass of $X_i$.

In [1, 2] Casey represents the entire set of particles by a single particle $Z$ in a $3N$-dimensional Euclidean space such that the mass $m$ of $Z$ is given by $m = \sum_{i=1}^{N} M_i$. In the absence of holonomic constraints, this $3N$-dimensional space is the configurational manifold. The addition of $l$ holonomic constraints will create a time dependent configuration manifold $Q$ of dimension $3N - l = n$. Nonholonomic constraints will not affect the dimension of $Q$.

Letting $(u^{3i-2}, u^{3i-1}, u^{3i}) = (x_i^1, x_i^2, x_i^3)$, we assign $3N$ coordinates to $Z$, and consider the general case of $l$ holonomic constraints of the type

$$f_j(u^1, u^2, \ldots, u^{3N}, t) = 0. \qquad (j = 1, \ldots, l < 3N)$$

The manifold $Q$ will then have $n$ generalized coordinates $q^\gamma$. The position vector $\underset{\sim}{r}$ of $P$ is then

$$\underset{\sim}{r} = \underset{\sim}{r}(q^\gamma, t)$$

and we can span the tangent space $T_P Q$ by vectors

$$\underset{\sim}{a}_\gamma = \frac{\partial \underset{\sim}{r}}{\partial q^\gamma} \qquad (\gamma = 1, \ldots, n = 3N - l)$$

that are tangent to the coordinate curves on Q. Then the velocity of $\Xi$ is given by

$$\underset{\sim}{v} = \dot{q}^{\alpha} \underset{\sim}{a}_{\alpha} + \frac{\partial \underset{\sim}{r}}{\partial t}$$

and the kinetic energy is

$$T = T_2 + T_1 + T_0$$

where (defining $a_{\alpha\beta} = \underset{\sim}{a}_{\alpha} \cdot \underset{\sim}{a}_{\beta}$)

$$T_2 = \tfrac{1}{2} m\, a_{\alpha\beta}\, \dot{q}^{\alpha} \dot{q}^{\beta} ,$$

$$T_1 = m \frac{\partial \underset{\sim}{r}}{\partial t} \cdot \underset{\sim}{a}_{\alpha}\, \dot{q}^{\alpha} ,$$

$$T_0 = \tfrac{1}{2} m \frac{\partial \underset{\sim}{r}}{\partial t} \cdot \frac{\partial \underset{\sim}{r}}{\partial t} .$$

Using a metric on Q such that

$$d^2(P,O) = \frac{1}{m} \sum_{i=1}^{N} M_i\, \underset{\sim}{x}_i \cdot \underset{\sim}{x}_i$$

we get the length of a line element on Q to be

$$ds^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r} = \left(\frac{\partial \underset{\sim}{r}}{\partial q^{\alpha}} dq^{\alpha}\right) \cdot \left(\frac{\partial \underset{\sim}{r}}{\partial q^{\beta}} dq^{\beta}\right) = \frac{2T_2}{m} dt^2 .$$

Thus the geometry of Q is Riemannian. If $\frac{\partial \underset{\sim}{r}}{\partial t}$ is zero then Q is fixed. Otherwise the manifold is free to move and deform.

Casey defines a 3N-dimensional force vector $\underset{\sim}{\Phi}$ such that

$$\underset{\sim}{\Phi} = m \dot{\underset{\sim}{v}} .$$

The covariant components of $\underset{\sim}{F}$ are then

$$Q_\gamma = \underset{\sim}{F} \cdot \underset{\sim}{a}_\gamma = m \underset{\sim}{\dot{v}} \cdot \underset{\sim}{a}_\gamma .$$

Thus any portion of the force vector $\underset{\sim}{F}$ that is orthogonal to $T_p Q$ will not contribute to $Q_\gamma$. In the context of our class, $\underset{\sim}{F}$ would be considered an element of $T^*Q$, and $Q_\alpha$ would be the pairing of $\underset{\sim}{F}$ and $\underset{\sim}{a}_\gamma$ rather than the dot product Casey uses. Casey does not speak of the cotangent bundle at all. Later he introduces vectors $\underset{\sim}{a}^\alpha \in T_p Q$ satisfying

$$\underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}_\beta = \delta^\alpha{}_\beta .$$

We would rather define covectors $\underset{\sim}{a}^\alpha \in T_p^* Q$ such that

$$\langle \underset{\sim}{a}^\alpha, \underset{\sim}{a}_\beta \rangle = \delta^\alpha{}_\beta .$$

The derivation of Lagrange's equations is now quite simple. From $T = \frac{1}{2} m \underset{\sim}{v} \cdot \underset{\sim}{v}$ we get

$$\frac{\partial T}{\partial q^\gamma} = m \underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{v}}{\partial q^\gamma} \quad \text{and} \quad \frac{\partial T}{\partial \dot{q}^\gamma} = m \underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{v}}{\partial \dot{q}^\gamma} .$$

But note that

$$\frac{\partial \underset{\sim}{v}}{\partial \dot{q}^\gamma} = \underset{\sim}{a}_\gamma$$

and
$$\frac{\partial \underset{\sim}{v}}{\partial q^\gamma} = \dot{q}^\alpha \frac{\partial \underset{\sim}{a}_\alpha}{\partial q^\gamma} + \frac{\partial}{\partial q^\gamma}\left(\frac{\partial \underset{\sim}{r}}{\partial t}\right)$$

$$= \dot{q}^\alpha \frac{\partial}{\partial q^\gamma}\left(\frac{\partial \underset{\sim}{r}}{\partial q^\alpha}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \underset{\sim}{r}}{\partial q^\gamma}\right)$$

$$= \dot{q}^\alpha \frac{\partial}{\partial q^\alpha}\left(\frac{\partial \underset{\sim}{r}}{\partial q^\gamma}\right) + \frac{\partial \underset{\sim}{a}_\gamma}{\partial t}$$

$$= \frac{\partial \underset{\sim}{a}_\gamma}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial \underset{\sim}{a}_\gamma}{\partial t}$$

$$= \dot{\underset{\sim}{a}}_\gamma$$

Therefore we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\gamma}\right) - \frac{\partial T}{\partial q^\gamma} = \frac{d}{dt}\left(m \underset{\sim}{v} \cdot \underset{\sim}{a}_\gamma\right) - m \underset{\sim}{v} \cdot \dot{\underset{\sim}{a}}_\gamma$$

$$= m \dot{\underset{\sim}{v}} \cdot \underset{\sim}{a}_\gamma$$

$$= \underset{\sim}{\Phi} \cdot \underset{\sim}{a}_\gamma$$

$$= Q_\gamma \quad .$$

Note these equations hold for any choice of generalized coordinates and any force $\underset{\sim}{\Phi}$. This includes dissipative forces and forces due to constraints. If it is assumed that the constraint forces are vertical, i.e. if they are normal to the constraint surfaces then they will not contribute to Lagrange's equations. Actually, if a force is perpendicular to any of the constraint surfaces then it will have no contribution, since the configuration manifold is the intersection of the individual constraint surfaces. The normal to a constraint surface $f(q^1, \ldots q^n, t)$ is $\nabla f$.

To write the equations in terms of the Lagrangian function $L = T - V$, let

$$\underset{\sim}{\Phi} = \underset{\sim}{\Phi}^{V} + \underset{\sim}{\Phi}'$$

where $\underset{\sim}{\Phi}^{V}$ represents the portion of the force that is derivable from a potential. Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^{\gamma}}\right) - \frac{\partial L}{\partial q^{\gamma}} = \underset{\sim}{\Phi}' \cdot \underset{\sim}{a}_{\gamma} \quad .$$

To relate this notation with that used in Jerrold Marsden's book [3], here we are referring to $L_N$, the Lagrangian restricted to the submanifold $TN \subset TQ$ created by the holonomic constraints. The $Q$ used in discussing Casey's paper corresponds to Marsden's $N$. The unrestricted $3N$-dimensional manifold in Casey's paper would correspond to Marsden's $Q$.

Marsden's approach to treating holonomic constraints differs from Casey's in the sense that he uses the same coordinates on the constrained submanifold $N$ as he does on the unconstrained manifold $Q$, whereas Casey reduces the number of coordinates on the submanifold to equal the dimension of the constrained manifold. Marsden, considering the forces as lying in the cotangent manifold, has a mapping $\varphi$ from $Q$ to the dual space of a vector bundle over $Q$, and the forces due to the constraints $\varphi_a(q(t)) = 0$ contribute to Lagrange's equations as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) - \frac{\partial L}{\partial q^i} = \lambda^a \frac{\partial \varphi_a}{\partial q^i} .$$

Here the $q^i$ represent Marsden's coordinates on $N$ (and $Q$), and $\lambda(q,t)$ is a function resulting from the Lagrange multiplier theorem. If Casey had not used generalized coordinates on $N$, he would have gotten a similar right-hand side term. Even in the case where the constraint force $\Phi^c_\ell = \lambda_\ell \nabla f_\ell$ (i.e. when it is normal to the corresponding constraint surface), he would get the right hand side looking like

$$\lambda_\ell \nabla f_\ell \cdot a_\alpha .$$

Identifying $\nabla f_\ell \cdot a_\alpha$ with $\frac{\partial \varphi_\ell}{\partial q^\ell}$, the two approaches match.

The derivation of Lagrange's equations for the case of a rigid continuum has similarities but is somewhat more involved. I will not go into all the details of the derivation, but will summarize the key facts.

Each configuration of the continuum can be represented by a point P in an infinite-dimensional configuration space Q. The rigidity constraint requires the motion to be of the form

$$x = R(t)X + q(t)$$

where $X$ is the reference position of a point in the continuum, $R(t)$ is a proper orthogonal rotation tensor, and $q$ is a translation vector $q(t)$. $R$ lies in a 9-dimensional Euclidean space, and $q$ lies in a 3-dimensional Euclidean space, but proper orthogonality of $R$ allows $R$ to be represented by only 3 distinct variables, usually taken to be the Euler angles. Thus $N \subset Q$ can be thought of as a 6-dimensional configuration space for $(\underline{q}, \hat{Q})$, ~~f~~ but Casey uses $(\bar{x}, Q)$ instead, where $\bar{x}$ is the center of mass of the continuum.

~~For~~ The continuum, the manifold is still Reimannian in geometry. The metric is

$$d^2(P,O) = \frac{1}{m}\int \rho\, \underset{\sim}{x}\cdot\underset{\sim}{x}\, dv .$$

The kinetic energy, linear momentum, and angular momentum are given by

$$T = \frac{1}{2}\int \rho\, \underline{v}\cdot\underline{v}\, dv ,$$

$$\underset{\sim}{G} = \int \rho\, \underset{\sim}{v}\, dv ,$$

$$\underset{\sim}{H}^{o} = \int \rho\, \underset{\sim}{x}\times\underset{\sim}{v}\, dv .$$

The tangents to the coordinate curves are

$$\bar{a}_\gamma = \frac{\partial\bar{x}}{\partial\eta^\gamma} \quad , \quad A_\gamma = \frac{\partial Q}{\partial\xi^\gamma}$$

where $\bar{x} = \bar{x}(\eta^1, \eta^2, \eta^3)$ and $Q = Q(\xi^1, \xi^2, \xi^3)$.

The result is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{z}^\gamma}\right) - \frac{\partial T}{\partial z^\gamma} = [m\dot{v}, a_\gamma]$$

where $(z^1, z^2, z^3) = (\eta^1, \eta^2, \eta^3)$ and $(z^4, z^5, z^6) = (\xi^1, \xi^2, \xi^3)$, $(a_1, a_2, a_3, a_4, a_5, a_6) = (\bar{a}_1, \bar{a}_2, \bar{a}_3, A_1, A_2, A_3)$ and $[\ ,\ ]$ is an inner product on $E^{12}$.

For the case when $\underset{\sim}{q} = \underset{\sim}{0}$, the configuration space $N$ is $SO(3)$, which is a Lie group. $N$ is also a symmetry group, since the same motion is obtained by rotating the coordinate frame.

References:


[1] J. Casey, Geometrical derivation of Lagrange's equations for a system of particles. Am. J. Phys. 62, 836-847 (1994)

[2] J. Casey, On the advantages of a geometrical viewpoint in the derivation of Lagrange's equations for a rigid continuum. ZAMP 54 143 (1995)

[3] J.E. Marsden & T.S. Ratiu, An Introduction to Mechanics and Symmetry Volume 1, January 1994.

[4] J.L. Synge & A. Schild, Tensor Calculus, University of Toronto Press 1949, Dover reprint 1978.