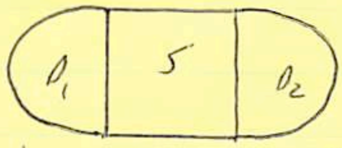
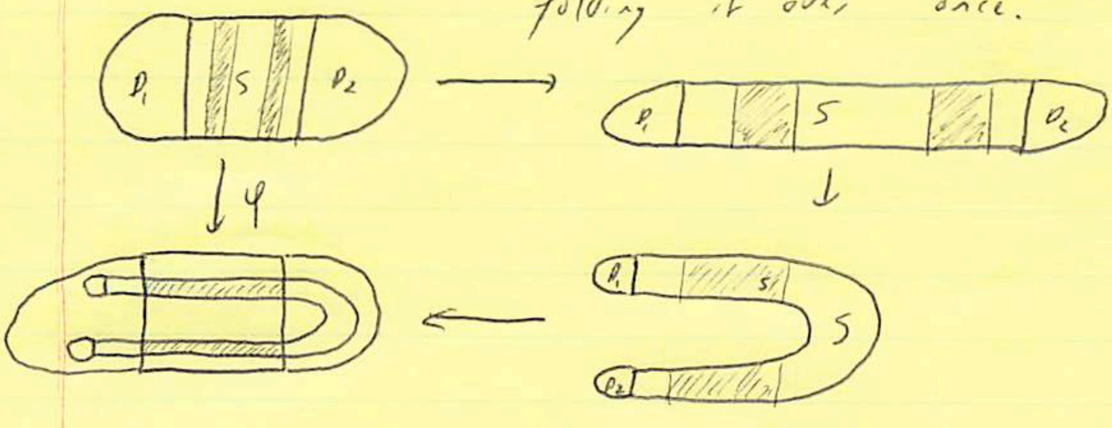
Smale's Horseshoe.

This example is the basis for the rest of Moser's presentation. What Moser does, is take the Horseshoe map and embed it in the 3-body problem, thus showing that the 3-body problem has chaos.

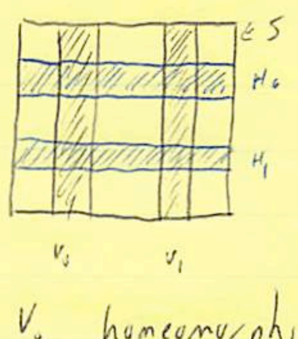
The horseshoe map φ is defined on the region below. S is a square of side 1, D_1 and D_2 are semicircles.

$D =$ 

φ is defined geometrically by contracting D ~~in the~~ down and stretching it out. Then we take that strand and put it back inside of D, after folding it over once.



What we are interested in are the iterates of the horseshoe map. Note that the two shaded ~~vert~~ vertical strips in S are the parts that get mapped back into S by φ. So φ itself takes these two vertical strips to two horizontal strips. So φ gives us the following picture on S

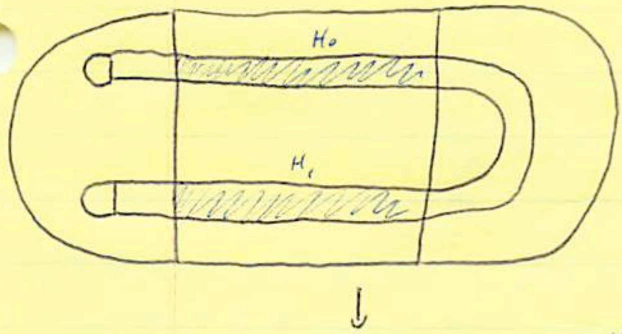


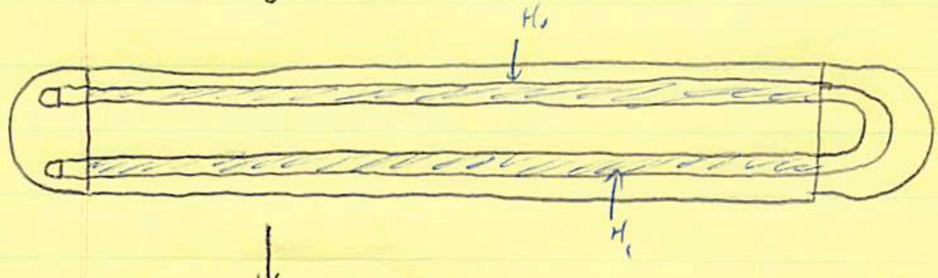
φ sends V_0 homeomorphically onto H_0, and V_1 onto H_1. We are interested in the set

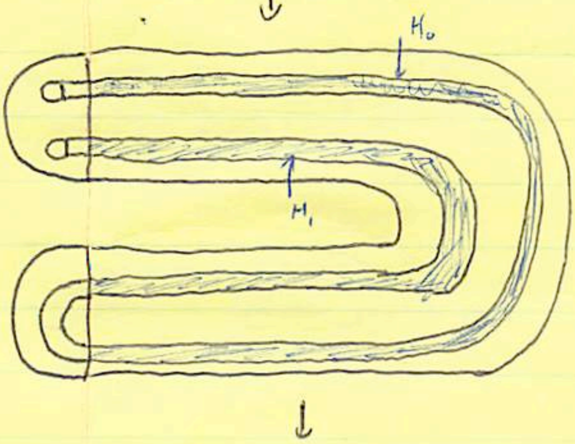
$$\Lambda = \{ q \in S \mid \varphi^k(q) \in S, \ \forall k \in \mathbb{Z} \}.$$

If you draw out the pictures, then you can see that φ^2, restricted to S, will send H_0

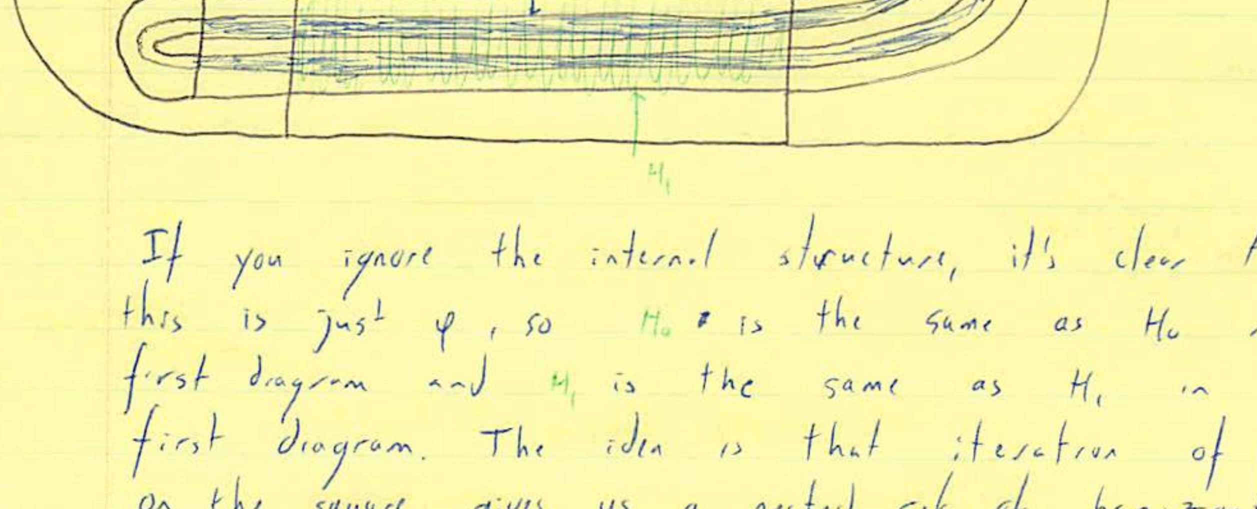
into two strips, H_{00} and H_{01}, with $H_{00} \subset H_0$, $H_{01} \subset H_1$, and the widths of H_{00} and H_{01} being less than the width of H_0. Likewise H_1 is taken to H_{10} and H_{11}.







so $H_{00} = H_0 \cap H_0$
$H_{01} = H_0 \cap H_1$
$H_{10} = H_1 \cap H_0$
$H_{11} = H_1 \cap H_1$



If you ignore the internal structure, it's clear that this is just φ, so H_0 * is the same as H_0 in the first diagram and H_1 is the same as H_1 in the first diagram. The idea is that iteration of φ on the square gives us a nested set of horizontal strips whose widths are going to zero. So if we take $\cap\, \varphi^k(H_0)$ we get a set of horizontal lines. Likewise, $\varphi^{-1}(S)$ gives us V_0 and V_1 as on page 1.

$\varphi^{-1}(V_0)$ will ~~give us~~ be all the points in V_0 and V_1 that get sent to H_{00} and H_{01}, so $\varphi^{-1}(V_0)$ is a pair of vertical strips, one in V_0 and one in V_1. Likewise for $\varphi^{-1}(V_1)$. So if we take $\cap\,\varphi^{-k}(V_0)$, and $\cap\,\varphi^{-k}(V_1)$ we get a set of vertical lines. The horizontal lines ~~at~~ remain in S ~~invariant~~ under forward iterations, and the vertical lines ~~are~~ remain in S ~~invariant~~ under reverse iterations, so the ~~intersect~~ points of intersection of the horizontal and vertical lines is the set Λ. We can identify points in Λ by double infinite sequences. Suppose $p \in \Lambda$. Then for each k, $\varphi^k(p) \in V_0$ or $\varphi^k(p) \in V_1$. So we can define a doubly infinite sequence by

$$s_k = 0 \quad \text{if } \varphi^k(p) \in V_0$$
$$s_k = 1 \quad \text{if } \varphi^k(p) \in V_1$$

If we let Σ_2 = the set of all doubly infinite sequences of 0's and 1's, then this defines a map $S: \Lambda \to \Sigma_2$.

On Σ_2 we define the shift map to be

$$(\sigma(s))_k = s_{k+1}.$$ So σ just shifts each element in the sequence once to the left.

~~Ca~~ We want σ and φ to be topological equivalent or

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varphi} & \Lambda \\ \updownarrow S & & \updownarrow S \\ \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \end{array}$$

this diagram should be commutative with S the map above. With the appropriate topology on Σ_2 we can show S is a homeomorphism.

Suppose $p \in \Lambda$. then $S(p) = s = (s_k)$.

$$(\sigma(s))_k = s_{k+1}$$

We want $S(\varphi(p)) = \sigma(s)$.

but $\varphi^k(\varphi(p)) \in V_i$ iff $\varphi^{k+1}(p) \in V_i$

so $S(\varphi(p)) = (s_{k+1}) = \sigma(s)$.

so φ and σ are topologically equivalent.

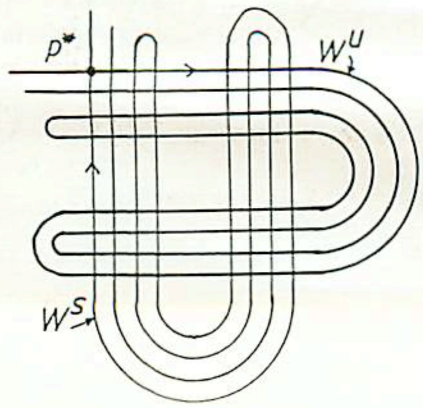


Fig. 3.5. The stable and unstable sets associated to p^*.

The above picture is the stable and unstable sets associated to a point $p^* \in \Lambda$. Note that the horizontal lines are the unstable manifold, while the vertical lines are the stable manifold. In this example, it turns out that the invariant set Λ is a hyperbolic set; Moser will show that in a more general setting, this holds true.

This exposition on Smale's Horseshoe was taken from pages 181-187 of Devaney's "Introduction to Chaotic Dynamical Systems.

Devaney also gives some properties of the shift, but without proof. One is that periodic points are dense in the sequence space, and another is that there is a dense orbit. So not only can any sequence be approximated by a sequence of periodic sequences, but it can also be approximated by a ~~set of~~ sequence of sequences from the orbit of one, nonperiodic orbit.

The Moser Shift

A) definition

Let A be a countable, possibly finite set.
a doubly infinite sequence from A is a sequence

$$s = (\cdots\cdots s_{-2}, s_{-1}, s_0, s_1, s_2 \cdots)$$

where $s_k \in A$.

Define A^* = the set of all such doubly infinite sequences.
Define a topology on A^* by defining a neighborhood basis for $s^* \in A$ by the sets

$$U_j = \{ s \in S \mid s_k = s_k^*, \text{ for } |k| < j \}$$

for $j = \{1, 2, \cdots\}$.

Note that if $j' < j$, then $U_j \subset U_{j'}$.
So two sequences are close if they match for a large number of terms around ~~so~~ the zero term.

Moser defines the shift homeomorphism by $(\sigma(s))_k = s_{k-1}$. This is the inverse of the shift as defined by Devaney. Since the shift is a homeomorphism, I don't think this causes any trouble. It reverses the direction of the stable and unstable manifolds, in Devaney's example.

~~B) Compactification.~~

B) Subsystems.

$$\begin{tikzcd} T \arrow[r, "\phi"] & T \\ S \arrow[u, "\tau"] \arrow[r, "\psi"] & S \arrow[u, "\tau"'] \end{tikzcd}$$

S, T topological spaces.

ψ is a subsystem of ϕ if there exists an embedding τ (so $\tau: S \to \tau(S)$ is a homeomorphism) such that the above diagram commutes, ie

$$\cancel{\tau\psi} \quad \tau \circ \psi = \phi \circ \tau$$

What Moser will do is define conditions that a map ϕ has to satisfy to have the shift as a subsystem.

Moser's shift.

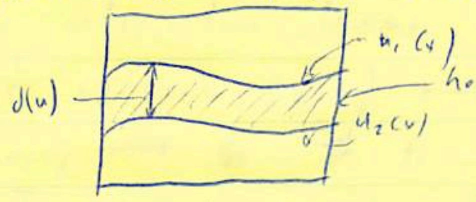
Now that we know what the shift is, we need to define conditions under which a function φ has the shift as a subsystem.

To generalize the shift on 2 symbols $(0, 1)$ to a shift on infinite symbols, we need some definitions.

Definition 1: horizontal curve

given $\mu \in]0,1[$, a curve $y = u(x)$ is a horizontal curve if $0 \le u(x) \le 1$, for $x \in [0,1]$ and $|u(x_1) - u(x_2)| \le \mu |x_1 - x_2|$

So a horizontal curve is basically a more-or-less straight path from one vertical boundary of the square to the other



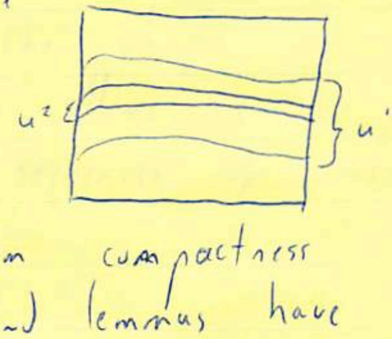
definition 2: horizontal strip

if u_1 and u_2 are horizontal curves with $u_2(x) < u_1(x)$ for all $x \in [0,1]$, Then the area between the two is a horizontal strip. If u is a horizontal strip, then $d(u) = \max_{0 \le x \le 1} (u_1(x) - u_2(x))$ is the diameter of u

Note that all these definitions are for a fixed μ. The set of horizontal curves you get depends on the choice of μ, and therefor the set of horizontal strips also depends on μ.

lemma 1: nested strips

if $u^1 \supset u^2 \supset u^3 \cdots$ is a sequence of horizontal strips u^k, and if $d(u^k) \to 0$ as $k \to \infty$, then $\bigcap_{k=1}^{\infty} u^k = u(x)$, a horizontal curve



This follows from compactness

All these definitions and lemmas have exact parallels for vertical strips $x = v(y)$.

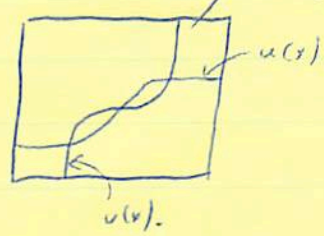
lemma 2: intersections of curves

A horizontal curve and a vertical curve intersect in one point.

This lemma is also intuitively clear. The restriction that

$$\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|} \leq \mu < 1 \text{ for } x_1 \neq x_2$$

is basically putting a bound on the derivative of u, for differentiable functions. This restriction keeps us from having situations like the below



The ~~lemm~~ curve intersection lemma gives us a map from pairs of curves $(u, v) \longmapsto I \times I$ = square, by sending (u, v) ~~to~~ to their point of intersection. In Smale's Horseshoe, we had 2 horizontal ~~curves~~ strips, and 2 vertical ~~curves~~ strips, and a map φ that mapped the vertical ~~curves~~ strips homeomorphically onto the horizontal strips. Iterations of φ produced nested horizontal strips, which resulted in a set of horizontal curves. Iterations of φ^{-1} produced nested vertical strips, which resulted in a set of vertical curves. These iterations gave us two sets of infinite sequences, one for vertical curves and one for horizontal curves. ~~These curves~~ Each pair of curves had one intersection, which could be given a unique doubly infinite sequence of 0's and 1's, constructed by glueing together the ~~two~~ sequences of its parents curves. Then the map φ on the intersection was equivalent to the shift on the sequences. Moser's generalization allows us to follow the same method, but for sequences of integers, instead.

Assumptions for chaos, for a map ϕ on the square.

i) The Joining Assumption

let $A = \{1, 2, \ldots N\}$ if $N < \infty$

or $A = \{1, 2, \ldots\}$ if $N = \infty$.

Assume: $\forall a \in A$, there are horizontal strips U_a, vertical strips V_a, and that $U_a \cap U_b = \phi$ for $a \neq b$

$V_a \cap V_b = \phi$ for $a \neq b$. We need $\phi(V_a) = U_a$, homeomorphically, and that the vertical boundaries of V_a get sent to the vertical boundaries of U_a, while the horizontal boundaries get sent to horizontal boundaries.

I call this the joining assumption because this allows us to stick the two infinite sequences that we will generate together. Note that Smale's Horseshoe does meet this assumption. ~~Also note that all the~~

Intuitively, what we are going to do is define a set by applying ϕ^{-1} to V_a, and ϕ to U_a, and then attach sequences to the set. When we shift on the sequences, we need to be able jump ~~nicely~~ "smoothly" from the vertical strips (the ϕ^{-1} of V_a) to the horizontal strips (the ϕ of U_a). This assumption guarentees that.

ii) The Iteration ~~Lemma~~ Assumption

if V is a vertical strip in $\bigcup_{a \in A} V_a$, then $\forall a \in A$, we need $\phi^{-1}(V) \cap V_a = \tilde{V}_a$, where $\tilde{V}_a$ is a nonempty vertical strip and for a fixed $\nu \in]0, 1[$ we need $d(\tilde{V}_a) \leq \nu\, d(V_a)$. Likewise for horizontal strips under ϕ.

This allows us to get nested sequences of vertical strips with strictly decreasing diameters by using ϕ^{-1}. We can get nested sequences of horizontal strips with strictly decreasing diameters by using ϕ. These iterations will give us infinite sequences in the following manner. Suppose $\phi^{-k}(p) \in \bigcup_{a \in A} V_a$ for all k. Define a sequence by: ~~s_k~~ if $\phi^{-k}(p) \in V_{a_k}$ then let $s_k = a_k$.

Likewise if $\phi^k(p) \in \bigcup_{a \in A} U_a$ for all k, we can get another infinite sequence. Joining them together gives us a doubly infinite sequence. This gives us a map from $I = \bigcap_{k=-\infty}^{\infty} \phi^k\left(\bigcup_{a \in A} V_a\right)$ to the sequence space $A^{\mathbb{Z}}$. These assumptions will make this map be a homeomorphism, ~~with~~ and ϕ and σ = shift map ~~top~~ will be topologically equivalent.

Theorem: chaos for ϕ

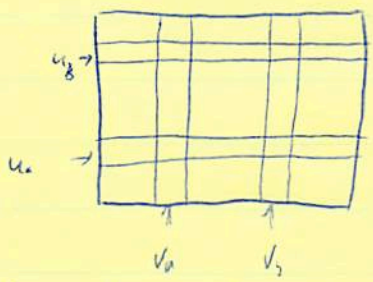
If ϕ satisfies the Joining Assumption and the Iteration Assumption, then the shift on A is a subsystem of ϕ; ie there is a homeomorphism $\tau : A^{\mathbb{Z}} \to Q$ ($A^{\mathbb{Z}}$ = doubly infinite sequence space from A) with $\phi\tau = \tau\sigma$.

Compactification.

If we have an infinite alphabet, ~~Then~~ A, then the sequence space $A^{\mathbb{Z}}$ is not compact. What Moser does is create a compactification of $A^{\mathbb{Z}}$, where the new points will represent capture and escape orbits in the three body problem.

Assume $A = \{1, 2, 3, \ldots\}$

Assume that if $a<b$, then V_a is to the left of V_b, and if $a<b$, then U_a is below U_b



So the limit of the V_a is the vertical curve $V_\infty = \{x = 1\}$, and the U_a tend to $U_\infty = \{y = 1\}$

Moser says that the problem here is that ϕ is not defined on V_∞, and ϕ^{-1} is not defined on U_∞, in general. This becomes a little clearer in the context of the 3-body problem.

Let $\overline{A^{\mathbb{Z}}}$ = the compactification of $A^{\mathbb{Z}}$

Moser adds elements of the form: for $k \le 0$, $l \ge 1$ integers.

let $s = (\infty, s_{k+1}, \cdots s_{l-1}, \infty)$ for $s_k \in A$.

if $k=0$, $l=1$ $\quad s = (\infty, \infty)$.

$s \in A$ corresponds to $k = -\infty$, $l = +\infty$.

Half-infinite sequences are allowed, ie letting $k = -\infty$, $l < +\infty$, or $k > -\infty$, $l = +\infty$.

The topology defined here corresponds to the previous topology for $s \in A$. If $s^* = (\cdots s^*_{-1}, s^*_0, \cdots s^*_{l-1}, \infty)$ then define a neighborhood around s^* to be all sequences of form

$s_k = s^*_k$ for $-K \le k < l$

$s_l \ge K$

$\quad s^*_{-1}\ s^*_0\ s^*_1\ s^*_2\ s^*_3\ s^*_4\ s^*_5\ s^*_6$

For example if $s^* = (\cdots 1, \underset{\uparrow}{1}, 1, 2, 3, 5, 7, 7, \infty)$

then if $K = 5$ an element in the neighborhood would be $(\cdots 1, 1, 1, 2, 3, 5, 7, 7, 5\ 5\ 5\ 5 \cdots)$

This gives us a topology where $\overline{A^{\mathbb{Z}}}$ is compact although this is not proved.

The shift σ is defined only on

$$D(\bar{\sigma}) = \{ s \in \bar{A}^{c},\ s_0 \neq \infty \}.$$

Note that if we have a sequence that starts at ∞ ie $s = (\infty, s_{k+1}, \text{---} \ \text{---})$ then $\sigma^{k+2}(s) \notin D(\bar{\sigma})$. Repeated iterations of σ will eventually send this out of the domain. Likewise σ^{-1} will eventually send sequences that end at ∞ out of the domain.

Note, actually that σ^{-1} is defined only on

$$R(\bar{\sigma}) = \{ s \in \bar{S},\ s_1 \neq \infty \}.$$

I think ~~points~~ sequences with $s_0 = \infty$ correspond to points on V_∞ while ~~pt~~ sequences with $s_1 = \infty$ correspond to points on U_∞.

Conditions for C^1 mappings.

If we have a C^1 map, then we have a derivative, which means we have a lot more control over the behaviour of the map. What Moser does is replace the iteration ~~terms~~ assumption above by an assumption involving the derivative, which is easier to check. If we represent ϕ by

$$x_1 = f(x_0, y_0) \qquad \text{where } \phi(x_0, y_0) = (x_1, y_1)$$
$$y_1 = g(x_0, y_0)$$

then we have $d\phi$ that sends (ξ_0, η_0) at (x_0, y_0) to (ξ_1, η_1) at (x_1, y_1) by

$$\begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}$$

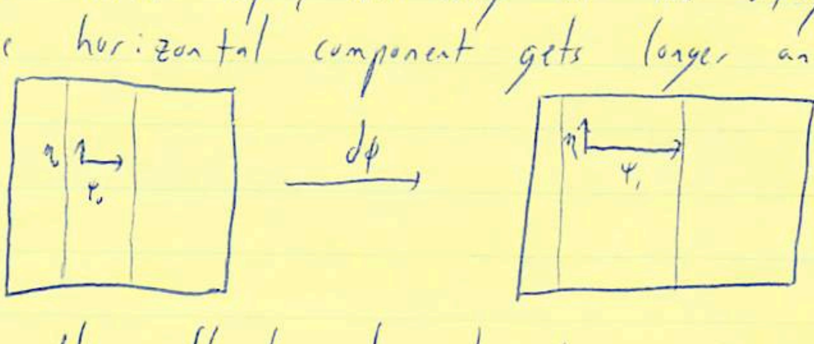
Differential Iteration Assumption

Define S^+ to be tangent vectors with $|\eta| \leq \mu |\xi|$, with $\mu \in]0,1[$ and fixed, and S^+ is defined over $\bigcup_{a \in A} V_a$. The assumption is that ① $(d\phi)(S^+) \subset S^+$, and ② if $d\phi(\xi_0, \eta_0) = (\xi_1, \eta_1)$, with $(\xi_0, \eta_0) \in S^+$, then $|\xi_1| \geq \mu^{-1} |\xi_0|$.

Define S^- to be tangent vectors with $|\xi| \leq \mu |\eta|$ defined over $\bigcup_{a \in A} U_a$. Assume ① $d\phi^{-1}(S^-) \subset S^-$, and if $d\phi^{-1}(\xi_1, \eta_1) = (\xi_0, \eta_0)$ then

$$|\eta_0| \geq \mu^{-1} |\eta_1|$$

What this assumption says is that if we start in S^+, then under $d\phi$, not only do we stay in S^+, but the horizontal component gets longer and longer.



So the effect of iterating with ϕ is to increase in the horizontal direction more than in the vertical. So vertical strips would get wider, and horizontal strips would get narrower. This corresponds to the ~~Itera~~ Iteration assumption, under which φ sends the ~~st~~ vertical strips produced by iteration of φ^{-1} to wider vertical strips, and the horizontal strips to narrower horizontal strips.

The condition on dp^{-1} is the same, but it works in the opposite direction.

Theorem: differential chaos

a C^1 map f that satisfies the joining assumption and the differential iteration assumption satisfies the iteration assumption, and therefor has horseshoe chaos

What Moser does with this is define a map for the restricted 3-body problem, and shows that it satisfies the differential chaos theorem.

Hyperbolicity.

For a C^1 mapping, $\tau(A^Z) = I$ is a hyperbolic set.

Definition: hyperbolic sets

for all $p \in I$, there are two independent lines L_p^+, L_p^- in TI_p = tangent space of I at p, such that L_p^+, L_p^- vary continuously with p, that $D\phi \, L_p^{\pm} = L_{\phi(p)}^{\pm}$ and with $\lambda > 1$, and the norm $|x| = \max(|\xi|, |\eta|)$, where $x = (\xi, \eta)$, we have:

$$\text{if } x \in L_p^+, \text{ then } |D\phi(x)| \geq \lambda |x|$$

$$\text{if } x \in L_p^- \text{ then } |D\phi^{-1}(x)| \geq \lambda |x|$$

L_p^+ is the unstable manifold at p, and L_p^- is the stable manifold at p, as defined here.

Let $D\phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ at a point p, and $\Delta = ad - bc$.

theorem: hyperbolic sets

if $\Delta, \Delta^{-1} \leq \frac{1}{2}\mu^{-2}$, then I is hyperbolic.

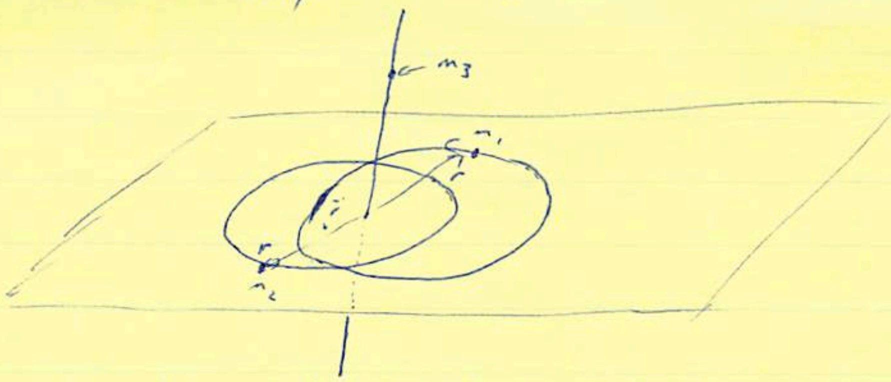
Moser proves this with a short argument involving the contraction principle on line bundles, that I didn't understand completely. The next theorem is proved almost identically.

theorem hyperbolic structure

under the joining assumption, the differential iteration assumption, and ~~the~~ $0 < \mu \leq \frac{1}{2}\min(|\Delta|^{\frac{1}{2}}, |\Delta|^{-1/2})$ then $u(s), v(s)$ are continuously differentiable curves whose tangents at points of I agree with the lines of the hyperbolic structure.

In this theorem, the points of I are viewed as the intersection of a horizontal curve $u(s)$ and a vertical curve ~~$v(s)$~~ $v(s)$.

The Three Body Problem.



The three body problem considered here is pictured above. There are two primary masses revolving about their center of mass in the plane. A third mass travels on the line perpendicular to the plane. We assume $m_1 = m_2$, $m_3 = 0$ and that these are all mass points. With these restrictions, m_3 will stay on the center line. So this problem becomes finding solutions to $\ddot{z} + \dfrac{z}{(z^2 + r^2)^{3/2}}$ where $r(t)$ = the distance from one of the primaries to the center of mass and $r(t) = \frac{1}{2}(1 - \epsilon \cos t) + O(\epsilon^2)$. The fact that $\epsilon > 0$ is what gives us chaos.

We need to define a mapping that will have chaos. Suppose $z(t)$ is a solution to the above equation. $z(t)$ can be described by giving the time $t_0 \pmod{2\pi}$ at which $z(t_0) = 0$, where $z = 0$ ~~means~~ is the intersection of the center line with the plane, and the velocity $v(t_0) = |\dot{z}(t_0)|$ as the ~~mass~~ third mass goes through $z = 0$. This is symmetric with respect to z, so the sign of $v(t_0)$ doesn't matter. So a solution can be given by $(v(t_0), t_0)$, or $v(t_0)e^{it_0}$, since only the magnitude of $v(t_0)$ matters, and t is mod 2π. So this means that polar coordinates are the natural coordinates to use here.

The map f we define is the return map.

$$f(v_0, t_0) = (v_1, t_1)$$

where v_1 is the velocity at time t_1

where t_1 is the next time after t_0 with $z(t_1) = 0$

Moser now states a sequence of 5 lemmas that help him to prove that the map f has chaos.

I will state and discuss his lemmas but will omit the proofs; this paper's getting a little long.

Lemma 1: domain of ϕ

there exists a real analytic simple closed curve in $\mathbb{R}^2$ in whose interior D_0 the mapping ϕ is defined.

if ~~v_0, t_0~~ $\phi(v_0, t_0)$ is outside D_0, the solutions escape.

It makes sense that D_0 is connected, since for each time t_0, ~~there should be a fixed~~ the potential energy at $z=0$ is uniquely defined, so the escape velocity required to reach infinity is also uniquely defined. So if you go out along ~~each~~ the ray corresponding to t_0, there's a certain radius $v_e(t_0)$, beyond which the solution escapes. Changing t_0 a little should change $v_e(t_0)$ just a little, so the map $t_0 \to v_e(t_0)$ should be continuous. The set of $v_e(t_0)$ should be a simple closed curve in $\mathbb{R}^2$. In the case $\epsilon = 0$, we get $v_e(t_0) = 2$, for this choice of units. If $\epsilon = 0$, then the primaries are always at a distance $\frac{1}{2}$ from the center of mass, so here D_0 is just a circle.

Lemma 2. ~~image~~ image of D_0 and reflections.

if $D_1 = \phi(D_0)$, then $D_1 = \rho(D_0)$, where $\rho(v,t) = (v, -t)$ and $\phi^{-1} = \rho^{-1}\phi\rho$, and ϕ preserves the area element $v\,dv\,dt$.

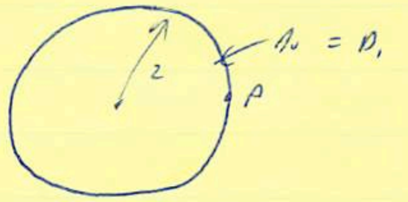
The reflection property just says that if you run time backwards, you get the same solutions. The original equation is second order in time, and the perturbation is $\epsilon \cos t$, which is symmetric with respect to time, so inverting time, going to the next zero, and inverting back is the same as going to the previous zero in normal time. So for $D_1 = \phi(D_0) = \rho(D_0)$ we have ϕ^{-1} defined in D_1.

Lemma 3 transversality.

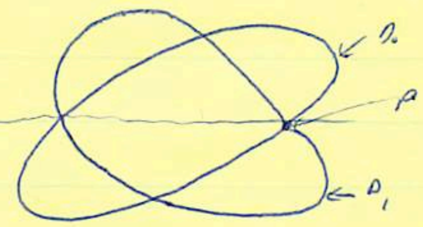
if ϵ is small enough and $\epsilon > 0$, then $D_0 \neq D_1$ and ∂D_0, ∂D_1 intersect non tangentially.

This is what results in chaos.

for $\epsilon = 0$ we have



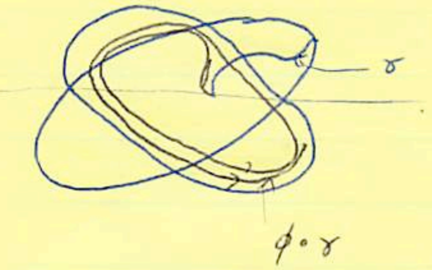
for $\epsilon > 0$ we have



It turns out that P is a homoclinic point, and so ~~th~~ this lemma gives us transversality at the intersections. It would be nice if we could interpret ∂D_0 and ∂D_1 as the stable and unstable manifolds for the equilibrium orbit P represents. The problem is that which solution is represented by the intersection depends on your choice of time. The time coordinate in this map tells us the positions of the primaries at the time the 3^{rd} mass goes through $z=0$. P corresponds to a solution where $t=0$. Our choice of $t=0$ is purely arbitrary, so which orbit is represented by the intersection is also arbitrary. This reflects a certain degeneracy in the system, in that all orbits on ∂D_0 represent getting to ∞ with zero velocity, and all orbits on ∂D_1 represent coming from $-\infty$ with no initial velocity. This problem is Moser's Fig 22 (below). W^+ is the stable manifold (solutions going to ∞) W^- is the unstable manifold (solutions coming from ∞) and Γ (all orbits that end up at ∞ with no velocity) is a hyperbolic periodic orbit. So then W^+ and W^- ~~transversally on the plane,~~ intersect transversally on the plane, giving rise to chaos. So ∂D_0 can be interpreted as part of a stable manifold, and ∂D_1 as part of an unstable manifold.

lemma 4 spiraling.

let γ be a C^1 arc, ~~with~~ with γ intersecting ∂D_0 nontangentially at $\gamma(0)$, and for $s \in]0,1]$ have $\gamma(s) \in D_0$. Then the image curve $\phi \circ \gamma$ approaches ∂D_1 spiralling.



Intuitively, this means that as you get closer to the escape velocity (∂D_0), the longer it takes to come back. This lemma will be used to build up the horizontal and vertical strips needed for Moser's assumptions for chaos.

Sector bundles.

For $\delta > 0$ and small, Define $D_0(\delta)$ to be points within δ of ∂D_0. If $p \in D_0(\delta)$, there is a unique closest point $q \in \partial D_0$. We can define two directions at $p \in D_0(\delta)$, by letting the tangent direction be parallel to the tangent to ∂D_0 at q, and the normal direction is perpendicular to that. Let the bundle $\Sigma_0 = \Sigma_0(\delta^{\frac{1}{3}})$ be all lines which ~~form an angle $\le \delta^{\frac{1}{3}}$ with~~ are within an angle of $\delta^{\frac{1}{3}}$ of the tangent direction. Let Σ_0' be all the other lines through p. Σ_1 and Σ_1' are similar bundles defined in D_1 by reflection. All this will allow us to state lemma 5, which will give us the iteration assumption.

lemma 5 sector bundles

There exists $\beta \in]0,1[$, such that for $\delta > 0$, small

1) ϕ sends $D_0(\delta) \to D_1(\delta^\beta)$
2) $d\phi$ sends $\Sigma_0'(\delta^{\frac{1}{3}}) \to \Sigma_1(\delta^{\frac{\beta}{3}})$ such that if $\xi_0 \in \Sigma_0'(\delta^{\frac{1}{3}})$, $\xi_1 = d\phi\, \xi_0$, $\eta_0 =$ projection of ξ_0 ~~along~~ in the tangent direction, and likewise for η_1 then
$$|\eta_1| \le \delta^{-\frac{1}{3}} |\eta_0|$$

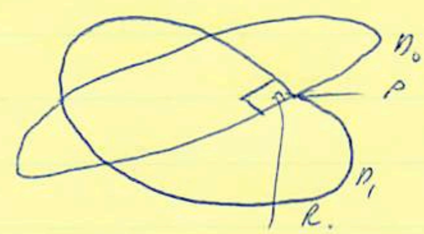
① says that ϕ sends stuff near the boundary of D_0 to stuff near the boundary of D_1, and ② says that vectors that are not close, as measured by $\delta^{\frac{1}{2}}$, to the tangent direction get sent to vectors that are close to the tangent direction. So this supports lemma 4, as ~~&~~ the derivative maps non tangential vectors to tangential ones, then if a curve comes into $\partial\Omega$ non tangentially ~~it~~ its image must come into $\partial\Omega$, tangentially. ~~Proof lemma~~

~~Pbs~~ Chaos in the 3-body problem.

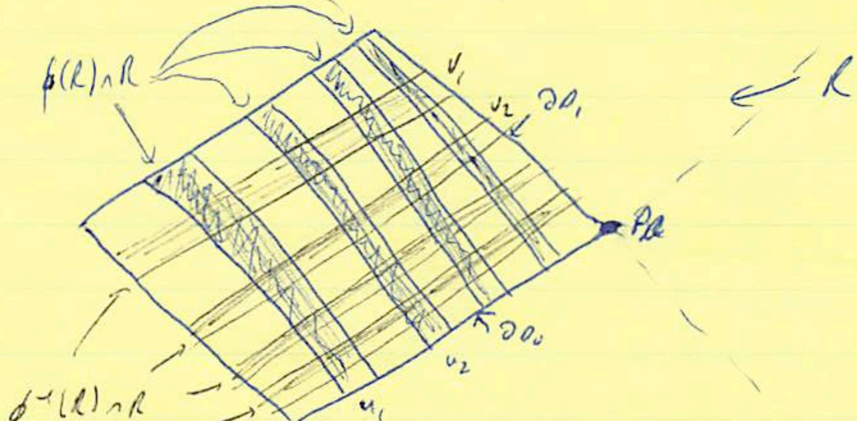
Theorem:

ϕ on D_0 has the shift σ on $O(\sigma)$ as a subsystem, and there is a hyperbolic invariant set I homeomorphic to S on which ϕ is equivalent to σ.

To prove this we will define a region where we have the joining and differential iteration assumptions satisfied. Look at $R = D_0(\delta) \cap D_1(\delta)_c$ containing P in its closure



R is symmetric, and two of the borders of R touch ∂D_0 non tangentially. The image under ϕ of these two borders, and therefor the image of R itself, approach the ∂D_1 tangentially. So $R \cap \phi(R)$ consists of a bunch of strips that connect two of the sides of R

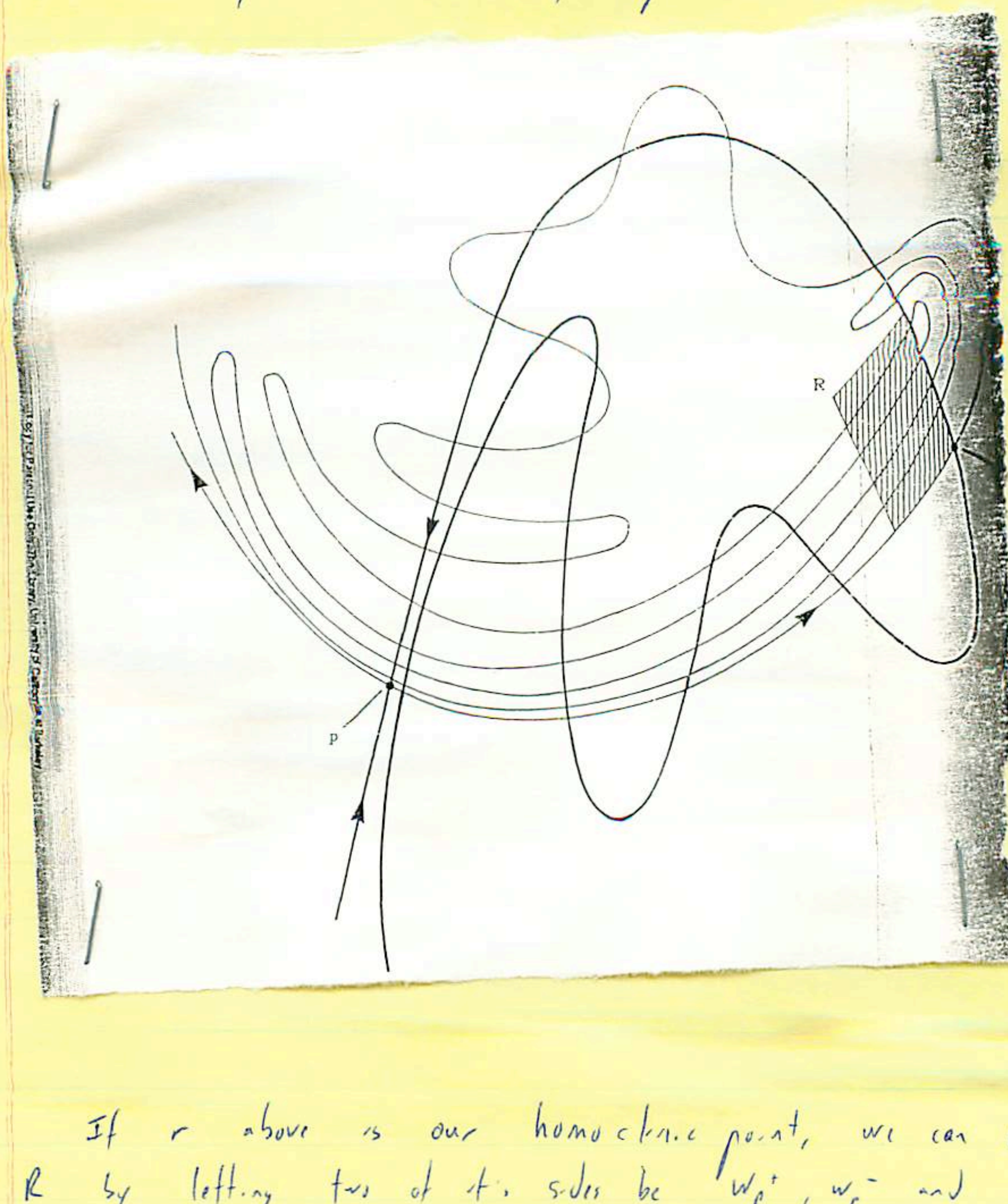


These strips will play the role of horizontal strips, with ∂D_1 as U_∞. Since R is symmetric, we get $\phi^{-1}(R)\cap R = \phi^{-1}\rho(R) \cap \rho(R) = \rho(\phi(R)\cap R)$. ~~So on R, φ is reflection.~~ Define $V_k = \rho U_k$. These are our vertical strips with ∂D_0 as V_∞. Moser has this picture on page 94, ~~p~~ but his labeling of the strips is reversed. He says $\phi(V_k) = U_k$, which is the joining assumption, but doesn't really prove it. ~~I think it can be shown by $U_k = \phi\rho^{-1}\phi\rho(U_k) = \phi$~~ I think he makes a mistake in notation on page 95, on the underlined

part. I think it should be "... the tangent of $\partial\Omega_0$ at p".
~~with this modification, this part~~
The iteration assumption on these strips comes mostly from lemma 5, for if we let $S^+ = \Sigma_1(S^{\frac{1}{3}})$, we find $D\phi(S^+) \subset S^+$, and if we let $S^- = \Sigma_0(S^{\frac{1}{3}})$ we get $D\phi^{-1}(S^-) \subset S^-$. So we have chaos in our system. Note that the entire structure hinges upon the transversality of $\partial\Omega_0$ and $\partial\Omega_1$. If they intersected tangentially, it's hard to see how to even define R.

Moser's treatment of homoclinic points.

If p is a hyperbolic fixed point for a diffeomorphism ϕ, then p has a stable manifold W_p^+, and an unstable manifold, W_p^-. So if $q \in W_p^+$, then $\phi^k(q) \to p$ as $k \to \infty$, and for $r \in W_p^-$, either $(\phi^{-1})^k(r) \to p$ as $k \to \infty$, or $\phi^k(r)$ goes away from p; these are the same. If $r \in W_p^- \cap W_p^+$, then r is a homoclinic point and $\cancel{\phi^k(r)}$ $\lim_{k\to\infty} \phi^k(r) = \lim_{k\to-\infty} \phi^k(r) = p$. Near homoclinic points where W_p^+ and W_p^- intersect transversally, we can go through essentially the same construction as was done for the 3-body map.



If r above is our homoclinic point, we can define R by letting two of its sides be W_p^+, W_p^- and and the other two sides are parallel to these. If $q \in R$, define $\kappa(q)$ = smallest positive integer with $\phi^{\kappa(q)} \in R$. Define $\hat{\phi}(q) = \phi^{\kappa(q)}(q)$, and this is defined on some set $D(\hat{\phi})$. This gives us

Th 3.7 if ϕ is a C^∞ diffeomorphism, with homoclinic point r, where W_p^+, W_p^- intersect transversally, then Φ has the shift as a subsystem.

The proof is basically the same idea, but involves translating R along the manifolds to the fixed point to build the horizontal and vertical strips.

Moser's last theorem is:

Any diffeomorphism ϕ satisfying the joining and differential iteration assumptions, and satisfying the restriction $0 < \mu \le \frac{1}{2} \min(|\sigma|^{\frac{1}{2}}, |\sigma|^{-\frac{1}{2}})$ does not have a real analytic integral. (σ = determinant of Jacobian, μ is the ~~Lipschitz constant~~ constant defined in the statement of the differential iteration assumption).

The assumptions here mean that ϕ has an invariant set I, where ϕ is basically the shift, and the last assumption says that if $p \in I$, then $p = u(s) \cap v(s)$, which are continuously differentiable lines that agree with the hyperbolic structure.

The phase diagram for Moser's 3-body problem

...amounts to introduction of a different metric.

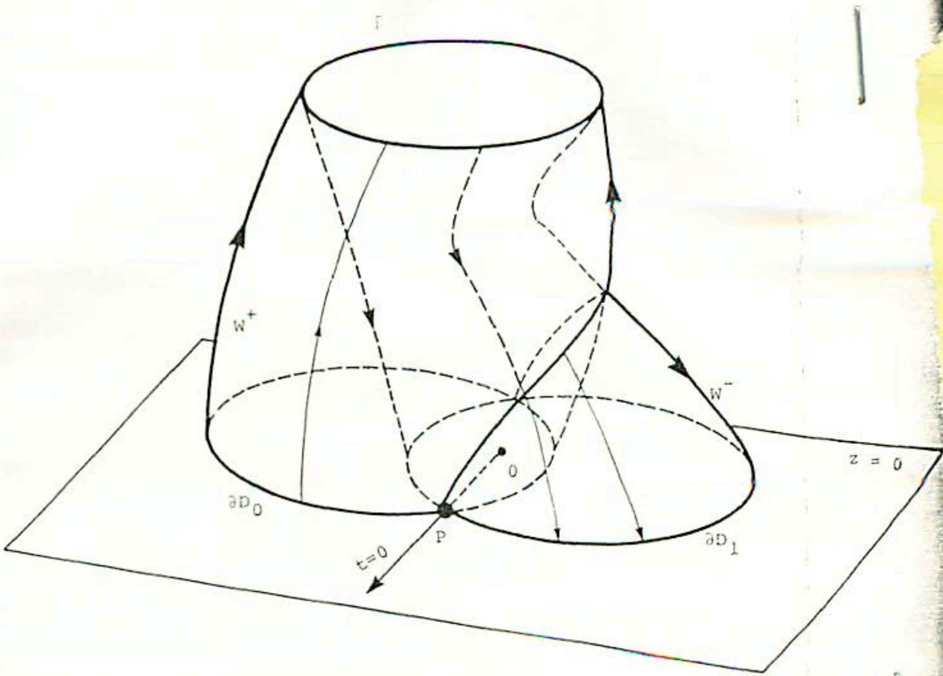
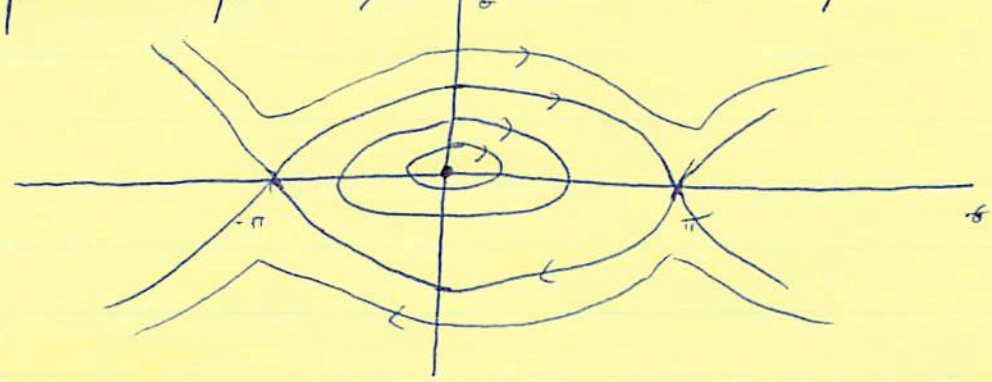


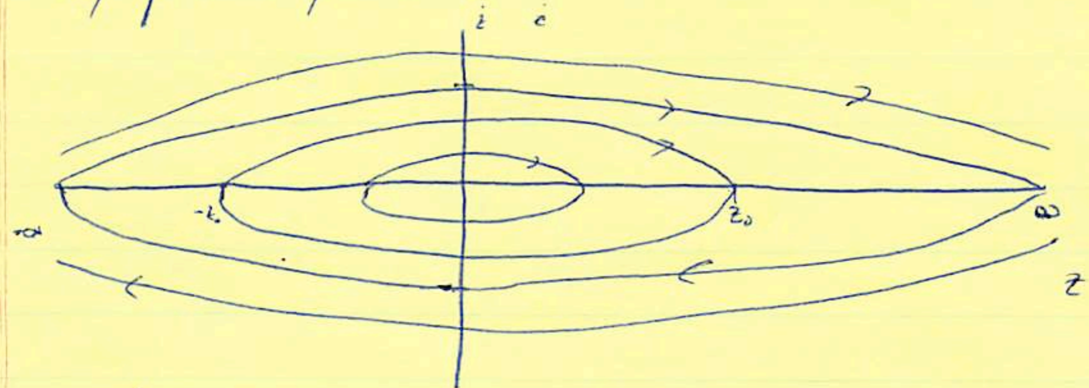
Fig. 22

Pendulum; 3-body problem, Melnikov

pendulum phase diagram unforced pendulum



3-body problem for $\epsilon = 0$



The pendulum and the 3-body problem have qualitatively the same phase diagram, so we would expect ~~from~~ that we would be able to get chaos in the 3-body problem. I tried to apply Melnikov's method to the 3-body problem, but finding the homoclinic orbit involved an integral I couldn't evaluate. What Moser does, however, is build the chaos at the hyperbolic point at ∞.

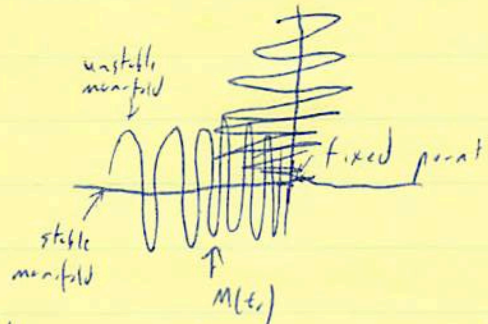
An interesting question is the relationship between the phase diagram and the return map. In the diagram above, ~~each orbit that returns will return~~ each orbit that returns has period 2, ie $z_0 \mapsto -z_0 \mapsto z_0$. For $\epsilon \neq 0$, you get exactly the same diagram, but you have to specify at what time you are considering it. For each t_0, you get a well defined homoclinic orbit, ie there is a well defined escape velocity. So the third dimension that's needed to define the phase space for the perturbed system is t. Then the return map is defined in the $z=0$ plane, ~~the~~ ~~Dem~~ This is essential Moser's diagram.

Melnikov's theorem states that if

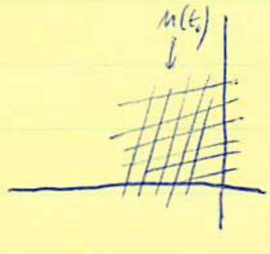
$$M(t_0) = \int_{-\infty}^{\infty} \{H_0, H_1\}(\bar{x}(t-t_0),t)\,dt$$

has simple zeros, then the system has the shift as a subsystem.

The interpretation given in class is that $M(t_0)$ represents one of the stable or unstable manifolds in terms of the other.



The homoclinic points we've seen have an orbit coming in and an orbit coming out, so I think if you reverse time, you switch the directions of the arrows, and hence flip the stable and unstable orbits. So if $M(t_0)$ ~~should~~ describes the unstable manifold in terms of the stable manifold with time running forward, than it should also describe the stable manifold in terms of the unstable manifold with time running backward. So $M(t_0)$ gives us the picture above, ~~in a sense~~ So in a small neighborhood of the fixed point, since $M(t_0)$ has simple zeros, ~~then~~ $M(t_0)$ is almost straight lines. This gives us the structure Moser built. ~~The~~



Hyperion.

One way to look at the chaos in the pendulum and 3-body orbits is that there has to be a small forcing function that makes the velocity required to ~~an~~ reach the homoclinic point ~~to~~ dependent upon time. In the three body problem, the forcing term is the ~~time depe~~ eccentricity in the orbits of the primaries. In the case of Hyperion, the driving function turns out to be resonances. I don't entirely understand the physical processes, but the math is fairly straightforward. Wisdom gives us the equation

$$\frac{d^2 \eta}{dt^2} + \frac{\omega_0^2}{2} \sum_{m=-\infty}^{\infty} H\left(\left(\frac{m}{2}\right), e\right) \sin\left(2\eta - mt\right) = 0.$$

The $H\left(\frac{m}{2}, e\right)$ are constants dependent on m and the eccentricity e. η is something called "the orientation of the satellites long axis", I'm not sure with respect to what. Our satellite is assumed to be ~~a~~ triaxial ellipsoid with moments of inertia $A < B < C$. $\omega_0^2 = \frac{3(B-A)}{C}$ is a measure of the asymmetry of the planet in the plane perpendicular to the long axis. This is what gives us chaos.

~~If we average this,~~

Wisdom averages this by noting if $\left|\frac{d\eta}{dt} - p\right| \ll \frac{1}{2}$, where $p = \frac{m}{2}$, then we can let $\gamma_p = \eta - pt$, so $\left|\frac{d\gamma_p}{dt}\right| \ll \frac{1}{2}$ then we get $\frac{d^2\gamma_p}{dt^2} + \frac{\omega_0^2}{2} H(p, e) \sin 2\gamma_p + \frac{\omega_0^2}{2} \sum H\left(p + \frac{n}{2}, e\right) \sin(2\gamma_p - nt) =$

The last sum, $\sum H\left(p + \frac{n}{2}, e\right) \sin(2\gamma_p - nt)$ will average to zero, leaving us with $\frac{d^2\gamma_p}{dt^2} + \frac{\omega_0^2}{2} H(p, e) \sin 2\gamma_p = 0$

This is the pendulum equation. If there is no interaction between resonance states, then when we plot the phase diagram ~~η~~ for $(\gamma_p, \dot{\gamma}_p)$ ~~$(\eta, \dot{\eta})$~~ we get a stack of pendulums. For each p, the phase diagram is the pendulum diagram. ~~A~~ Interactions between resonances becomes the driving terms. Wisdom shows that the width of the chaotic bands ~~produced are~~ increase with ω_0. Also, the ~~strength~~ resonance widths also increase with ω_0. So for large enough ω_0, not only are the chaotic bands wide, but they are also ~~pro~~ start overlapping.

THE CHAOTIC ROT

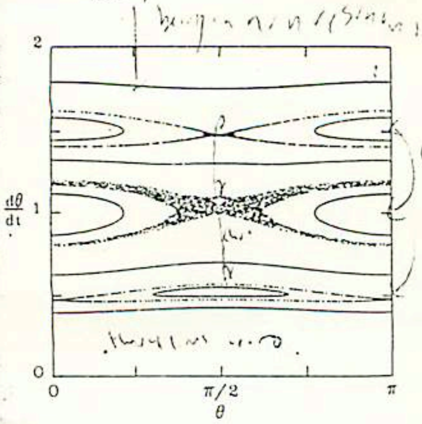


FIG. 1. Surface of section for $\omega_0 = 0.2$ and $e = 0.1$ $d\vartheta/dt$ versus ϑ at successive periapse passages for te[n] separate trajectories: three illustrating quasiperiodi[c] libration, three illustrating the surrounding chaotic lay[ers] and four illustrating that quasiperiodic rotatio[n] separates the chaotic zones.

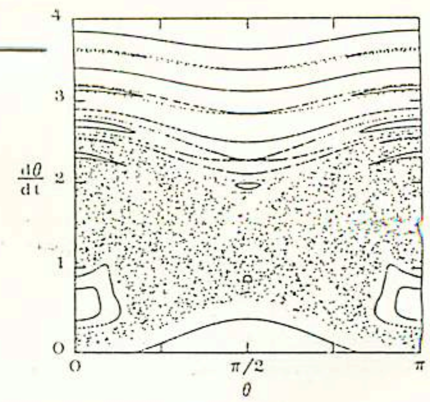


FIG. 2. Surface of section for $\omega_0 = 0.89$ and $e = 0.1$. Hyperion's spin–orbit phase space is dominated by a chaotic zone which is so large that even the $p = 1/2$ and $p = 2$ states are surrounded by it.

These are the phase diagrams for Hyperion. On the left is ω_0 small. There is chaos in the pendulums, but not a whole lot. Each pendulum represents one resonance state, and they are seperated by a quasiperiodic rotation. On the right is ω_0 large. Here the chaos zone has just over taken ~~the~~ most of the phase space illustrated, because two of the resonances overlapped. Of course, when resonances overlap the averaging method doesn't work too well.