The Foucault Pendulum
Analyzed with an
Eye Towards Berry's Phase

Very good

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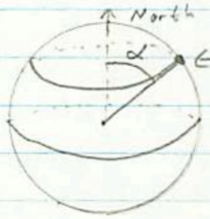
The Foucault pendulum is a classic physics experiment with which we are all familiar. Most of us as children saw at the Smithsonian or some local science musuem the long string with a heavy pendulum bob slowly knocking down pins sitting on a large circle. We were told that this experiment is evidence of the earth's rotation. Those of us who were lucky enough to take some advanced physics learned in a chapter on noninertial frames that the motion of the Foucault pendulum can be explained by the Coriolis force. The rate at which the pendulum rotates around the circle can be calculated easily enough once one makes the reasonable assumption that the Coriolis force is much stronger than the centrifugal force. This rate is $2\pi(1-\cos\alpha)$ radians per day where α is the colatitude of the pendulum.



The quantity $2\pi(1-\cos\alpha)$ appearing in a problem involving motion along a fixed latitude should be significant to those who have seen some differential geometry. Recall that vectors parallel translated along a fixed latitude rotate by an angle $2\pi(1-\cos\alpha)$. In the early 1980's, M.V. Berry noticed the relation between phase shifts in certain systems, more specifically systems with slowly-varying parameters such that for each choice of parameters the system is integrable, and angles arising from parallel translations along the curves traced in parameter space. These phase shifts are now popularly referred to as Berry's phases.

Given this rather vague description of Berry's phase, I shall now calculate the phase as stated above for the Foucault pendulum.

Assume the radius of the earth is R, $\frac{2\pi}{\varepsilon}$ equals 24 hours, r is the length of the pendulum, and the pendulum is at colatitude α. Let us perform our calculations with respect to a coordinate system with origin at the center of the earth and z-axis pointing through the North Pole. We assume the earth rotates with respect to this coordinate system about the z-axis with rate ε.

The position of the pendulum bob can be described as

$$\vec{\ell}(t) = A(\varepsilon t)\left[\vec{x}(\theta(t), \phi(t)) + R\hat{k}\right]$$

where $A(t) = \begin{pmatrix} -\sin t & -\cos\alpha\cos t & \sin\alpha\cos t \\ \cos t & -\cos\alpha\sin t & \sin\alpha\sin t \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$

and

$$x(t) = \begin{pmatrix} r\sin\phi(t)\cos\theta(t) \\ r\sin\phi(t)\sin\theta(t) \\ r\cos\phi(t) \end{pmatrix}.$$

Notice that $A(t)$ moves the pendulum with the rotation of the earth.

The Lagrangian can be written:

$$L = \tfrac{1}{2}m\,\dot{\ell}^T\dot{\ell} - mg\,\ell^T(t)(A(\varepsilon t)\hat{k}).$$

Now

$$\dot{\ell} = A(\varepsilon t)(\dot{x}(t)) + \varepsilon\dot{A}(\varepsilon t)(x(t) + R\hat{k})$$

$\Rightarrow$

$$L = \tfrac{1}{2}m\,\dot{x}^TA^TA\dot{x} + \varepsilon m(x+R\hat{k})^T\dot{A}^TA\dot{x} + \tfrac{1}{2}m\varepsilon^2(x+R\hat{k})^T\dot{A}^T\dot{A}(x+R\hat{k}) - mg(x+R\hat{k})^TA^TA\hat{k}$$

We must now calculate A^TA, $\dot{A}^TA$, and $\dot{A}^T\dot{A}$.

① $A^TA = \begin{pmatrix} -\sin t & \cos t & 0 \\ -\cos\alpha\cos t & -\cos\alpha\sin t & \sin\alpha \\ \sin\alpha\cos t & \sin\alpha\sin t & \cos\alpha \end{pmatrix}\begin{pmatrix} -\sin t & -\cos\alpha\cos t & \sin\alpha\cos t \\ \cos t & -\cos\alpha\sin t & \sin\alpha\sin t \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (which we already knew by construction of A)

② $\dot{A}^T A = \begin{pmatrix} -\cos t & -\sin t & 0 \\ \cos\alpha \sin t & -\cos\alpha \cos t & 0 \\ -\sin\alpha \sin t & \sin\alpha \cos t & 0 \end{pmatrix} \begin{pmatrix} -\sin t & -\cos\alpha \cos t & \sin\alpha \cos t \\ \cancel{\cos\alpha}\ \cos t & -\cos\alpha \sin t & \sin\alpha \sin t \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$

$$= \begin{pmatrix} 0 & \cos\alpha & -\sin\alpha \\ -\cos\alpha & 0 & 0 \\ \sin\alpha & 0 & 0 \end{pmatrix}$$

③ $\dot{A}^T \dot{A} = \begin{pmatrix} -\cos t & -\sin t & 0 \\ \cos\alpha \sin t & -\cos\alpha \cos t & 0 \\ -\sin\alpha \sin t & \sin\alpha \cos t & 0 \end{pmatrix} \begin{pmatrix} -\cos t & \cos\alpha \sin t & -\sin\alpha \sin t \\ -\sin t & -\cos\alpha \cos t & \sin\alpha \cos t \\ 0 & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\alpha & -\cos\alpha \sin\alpha \\ 0 & -\cos\alpha \sin\alpha & \sin^2\alpha \end{pmatrix}.$$

Also, $\dot{x} \equiv \begin{pmatrix} r\cos\phi \cos\theta\, \dot{\phi} - r\sin\phi \sin\theta\, \dot{\theta} \\ r\cos\phi \sin\theta\, \dot{\phi} + r\sin\phi \cos\theta\, \dot{\theta} \\ -r\sin\dot{\phi} \end{pmatrix}$

Plugging all these into L gives:

$$L = \tfrac{1}{2} m r^2 \dot{\phi}^2 + \tfrac{1}{2} m r^2 \sin^2\phi\, \dot{\theta}^2 - mgr\cos\phi$$
$$+ \varepsilon m \big(r^2 \cos\alpha \sin^2\phi\, \dot{\theta} - r^2 \sin\alpha \cos\phi \sin\phi \sin\theta\, \dot{\theta}$$
$$- Rr \sin\alpha \sin\phi \sin\theta\, \dot{\theta} + Rr \sin\alpha \cos\phi \cos\theta\, \dot{\phi}$$
$$+ r^2 \sin\alpha \cos\theta\, \dot{\phi} \big)$$

$$+ \varepsilon^2 \tfrac{1}{2} m \big(r^2 \sin^2\phi \cos^2\theta + r^2 \cos^2\alpha \sin^2\phi \sin^2\theta$$
$$- 2r^2 \cos\alpha \sin\alpha \cos\phi \sin\phi \sin\theta - 2Rr \cos\alpha \sin\alpha \sin\phi \sin\theta$$
$$- r^2 \cos^2\phi \sin^2\alpha + 2Rr \sin^2\alpha \cos\phi + R^2 \sin^2\alpha \big)$$

Note that $\frac{\partial L}{\partial t} \equiv 0$.

Now, let's calculate p_θ and p_ϕ.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \sin^2\phi\, \dot{\theta} + \varepsilon m \big(r^2 \cos\alpha \sin^2\phi - r^2 \sin\alpha \cos\phi \sin\phi \sin\theta$$
$$- Rr \sin\alpha \sin\phi \sin\theta \big)$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = mr^2\dot\phi + \varepsilon m\,(Rr\sin\alpha\cos\phi\cos\theta + r^2\sin\alpha\cos\theta).$$

$$\Rightarrow \begin{cases} \dot\theta = \dfrac{p_\theta}{mr^2\sin^2\phi} - \varepsilon\left(\cos\alpha - \sin\alpha\cot\phi\sin\theta - \dfrac{R}{r}\sin\alpha\csc\phi\sin\theta\right) \\[2ex] \dot\phi = \dfrac{p_\phi}{mr^2} - \varepsilon\left(\dfrac{R}{r}\sin\alpha\cos\phi\cos\theta + \sin\alpha\cos\theta\right). \end{cases}$$

Then $H = p_\theta\dot\theta + p_\phi\dot\phi - L(\theta, \phi, p_\theta, p_\phi)$

$$= \frac{p_\phi^2}{2mr^2} + \frac{p_\theta^2}{2mr^2\sin^2\phi} + mgr\cos\phi$$

$$- \varepsilon\, p_\theta\left(\cos\alpha - \sin\alpha\cot\phi\sin\theta - \frac{R}{r}\sin\alpha\csc\phi\sin\theta\right)$$

$$- \varepsilon\, p_\phi\left(\frac{R}{r}\sin\alpha\cos\phi\cos\theta + \sin\alpha\cos\theta\right)$$

$$- \varepsilon^2\,\frac{1}{2}mr^2\Big[\frac{R^2}{r^2}\sin^2\alpha\left(\cos^2\phi\cos^2\theta + \sin^2\theta + 1\right)$$

$$+ \frac{2R}{r}\left(2\sin^2\alpha\cos\phi - 2\sin\alpha\cos\alpha\sin\phi\sin\theta\right)$$

$$+ \sin^2\alpha\cos^2\phi\sin\theta + \cos^2\alpha\left(\sin^2\phi\sin^2\theta + \sin^2\phi\right)$$

$$- 4\sin\alpha\cos\alpha\cos\phi\sin\phi\sin\theta + \sin^2\phi\cos^2\theta\Big].$$

Note that again $\frac{dH}{dt} = \frac{\partial H}{\partial t} \equiv 0$ as in the unperturbed $\varepsilon = 0$ system.

Tracing through the above calculations, we find that the terms of order 1 in ε are the coriolis terms (having come from the $\dot A^T A$ part of the Lagrangian) and that the ε^2 terms are the centrifugal terms (having come from the $\dot A^T \dot A$ part of the Lagrangian.)

So, we may ignore centrifugal terms without generating much error and examine

$$H = \frac{p_\phi^2}{2mr^2} + \frac{p_\theta^2}{2mr^2\sin^2\phi} + mgr\cos\phi$$
$$- \varepsilon\, p_\theta\left(\cos\alpha - \sin\alpha\cot\phi\sin\theta - \frac{R}{r}\sin\alpha\csc\phi\sin\theta\right)$$
$$- \varepsilon\, p_\phi\left(\frac{R}{r}\sin\alpha\cos\phi\cos\theta + \sin\alpha\cos\theta\right)$$

$$\Rightarrow (1)\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2 \sin^2\phi} - \varepsilon\left(\cos\alpha - \sin\alpha \cot\phi \sin\theta - \frac{R}{r}\sin\alpha \csc\phi \sin\theta\right)$$

$$(2)\ \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} - \varepsilon\left(\frac{R}{r}\sin\alpha\cos\phi\cos\theta + \sin\alpha\cos\theta\right)$$

as before.

And

$$(3)\ \dot{p_\theta} = -\frac{\partial H}{\partial \theta} = \varepsilon\Big(-p_\theta \sin\alpha\cot\phi\cos\theta - p_\theta \frac{R}{r}\sin\alpha\csc\phi\cos\theta - p_\phi \sin\alpha\cos\phi\sin\theta - p_\phi\sin\alpha\sin\theta\Big)$$

$$(4)\ \dot{p_\phi} = -\frac{\partial H}{\partial \phi} = \frac{p_\theta^2\cos\phi}{mr^2\sin^3\phi} + mgr\sin\phi + \varepsilon\Big(p_\theta\sin\alpha\sin\theta\csc^2\phi + p_\theta\frac{R}{r}\sin\alpha\sin\theta\cot\phi\csc\phi - p_\phi\sin\alpha\sin\phi\cos\theta\Big)$$

To take our next step, we must discuss the phase space for the $\varepsilon = 0$ case. When $\varepsilon = 0$, we have action-angle variables $H, \psi, p_\theta, \theta$ such that

$$\dot{H} = 0 \qquad \dot{\psi} = \omega_1(H, p_\theta)$$
$$\dot{p_\theta} = 0 \qquad \dot{\theta} = \omega_2(H, p_\theta).$$

Thus, our ~~traj~~ trajectories lie on a two-dimensional torus in $\mathbb{R}^4$.

Now, we may view the equations above (1) - (4) as describing motions close to the original torus. So, by the standard averaging argument, we may average the functions linear in ε over the original torus to obtain equations whose solutions are close to the solutions of the equations above. (Essentially, we are neglecting the small oscillations of the system.)

To integrate f over the torus, we have

$$\langle f \rangle = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} f(\theta, \psi)\, d\theta\, d\psi.$$

So, integrating the ε terms in (1) - (4) gives

$$(1)' \qquad \dot{\theta} = \frac{p_\theta}{mr^2\sin^2\phi} - \varepsilon\cos\alpha$$

$$(2)' \qquad \dot{\phi} = \frac{p_\phi}{mr^2}$$

$(3)'$ $\quad \dot{p}_\theta = 0$

$(4)'$ $\quad \dot{p}_\phi = \dfrac{p_\theta^2 \cos\phi}{mr^2 \sin^3\phi} + mgr\sin\phi$

since all the ε terms (except for a special one) ~~cont~~ are linear in $\cos\theta$ or $\sin\theta$.
So, we have recovered three equations that are the same as in the $\varepsilon = 0$ case. Importantly, these equations $(2)'$, $(3)'$, $(4)'$ are independent of θ.
Thus

$$\Delta\theta = \int_0^{2\pi/\varepsilon} \left(\frac{p_\theta}{mr^2 \sin^2\phi} - \varepsilon\cos\alpha \right) dt$$

$$= (\Delta\theta)_{\varepsilon=0} - 2\pi\cos\alpha .$$

$-2\pi\cos\alpha$ is also known as Hannay's angle.
Using the convention in Berry (1985), this angle may be written

$$-2\pi + 2\pi(1 - \cos\alpha)$$

where the -2π is the change in θ for one revolution around the torus in phase space.

$\therefore$ We have the change in radians per day given by $2\pi(1-\cos\alpha)$ as stated in the introduction.

$\square$

Final Remark: I am still curious as to how ~~to~~ to go about this problem using connections and vector bundles. Hopefully, we may discuss it this fall.

References

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3) Marsden, J.E. classnotes for Math 514 4/26/88.

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