The falling cat. Ideas in control theory from geometrical mechanics.

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§ Introduction:

The geometrical concepts play an important role in unifying under the same umbrella phenomena which apparently have no
relation and belong to completely different domains of science.

The phase shift is one of the examples which shows that
complicating things, instead of accepting easy explanations, can lead
to the discovery of unbelivable invariant understructures (geometrical or
and this emphasizes once more the necessity of cross-calibration of
different fields. There might be engineers saying that a compact
and dynamical explanation of the falling cat phenomenon can
be found in Kane and Sobie's paper [1]. But here we wouldn't know
that the cat has the same trajectory to a particle in Young-dills
potentials (for whatever good it will do to the cat), so, geometrizing things the good things.

On the other hand, the formalism which I'm trying to outline
here helped building control theories used in space maneuvers of
satellites,..., possibly in medicine (human robots), etc. I will give
more influnces later.

The central idea is that when one variable of the system
moves in a periodic way (internal variable) then an overall
motion is observed. Speaking about the falling cat example:
it moves the joints creating an overall change in position of
its body even if it moves under the constraint of total
angular velocity being constant.

Thus are many other examples as the neurodynamics (moving
through fluids with high density), the insects, the fish,...
this only to speak about the animal world.
According to Berry himself, one of the first published papers related to geometric phase went out the press in 1941 (the year when he was born). The author is Victor Vladimirovich Starshinov. This was about the polarization of rotation of an outgoing ray relative to a parallel incident ray.

A further result, by 1942, came closer remarking that two circular polarizations have different phase velocities.

In 1956, in Bangalore (India), S. Pancharatnam, in investigating the interference patterns produced by plates of anisotropic crystal, found the existing theories inadequate to explain his observations. To calculate the phase change he rephrased states of polarization as points on the "Poincaré sphere." The poles, representing left and right-handed circular polarization, saturate... binary polarizations... seems clear that Pancharatnam discovered the geometric phase.

Unfortunately he died at 35 years old, and the official discovery of the geometric phase remained unknown for 30 more years.

There were some other scientists, but touching the problem and have a glimpse one in a while in what it might have been but none of them came as close as the two physicists discussed above.
5.3. The falling cat: The dynamical explanation

How can the cat actually succeed in falling without falling? A simple dynamical reasoning leads to the following argument:

- Due to the cat cannot change its zero angular momentum \( \Pi = 0 \). But \( \Pi = I \omega \), and by changing the shape the cat changes \( I \), the amount of inertia. \( \Pi \) has to change consequently. The forces exerted by the muscles in changing the shape are internal to the system so \( \Pi \) remains 0.

In their paper [3], Kove and Solar elaborated a model for the cat consisting from cylinders moving in some prescribed manner around joints. Probably the authors spent a lot of time modeling cats, by the way in which they set up the constraints for the motion of the joints. The restrictions are meant to play the role of the cat's muscles, or control forces applied in the joints.

Approximating the cat by ellipsoids and imposing the following 3 restrictions, the authors prove that the motion is possible by making use of fact that the angular momentum is conserved.

The three restrictions are:
- The loss of the cat heeds, but does not twist.
- At the instant of release, the spine is bent forward.
- Subsequent to this instant, the spine is bent first to one side, then backward, then to the other side and finally forward again.
- The backward bend that occurs during the movements is far less pronounced than the initial and terminal forward bend.

It is shown that the motion is possible if \( A_1, B \), are centroidal principal axes of inertia, where \( A_1, B \), are the front and rear halves of the cat and \( A_1, B \), the spine.
Before moving on to the formalization of rigid body motion, I will make an overview of the formalization of rigid body motion. This is because our cat will be treated as a bundle of rigid bodies interconnected by joints, able to be operated by control devices.

§4. Rigid Bodies

The actual configuration of a rigid body is considered to be a mapping from the reference configuration. Each position of the rigid body is specified by a Euclidean motion $T$. Any rigid motion consists in a translation and a rotation. The translation we take care of by imposing the motion function with respect to a frame passing through the center of mass. Every rotation of the body corresponds to an orthogonal tensor $Q(t) \in SO_3$ ($Q$ is proper orthogonal).

**Remark:** It is to be remarked that there is a phylogenetic difference and the forces which appear in one case and not in another between a change in observer and a proper rigid body motion, another technical one, that $\det Q = 1$ in the case of rigid body motion, but can be $\det Q = 1$ in a change in observer.

- Euler's equations for a RB are: $\dot{\theta} = \Omega \times \omega$ where $\Omega$ is the angular momentum (the Lagrangian one) and $\omega$ is the angular velocity, the kinetic energy: $K = \frac{1}{2} \Omega^T \Omega$ where $I$ is the inertia tensor in $3 \times 3$ matrix (if the body is ungrounded).
- Euler's equations in terms of $\omega$: $\omega \times I \omega = -I \ddot{\omega}$.
- $K$ is a quadratic form and its eigenvalues are the principal ones of inertia.
- For the RB, $K$ is taken to be the Lagrangian.
Waking up of the 

\( \mathbf{R}^3 \), it can be

shown that Rota's equation are Hamiltonian equivalent to the

\( \mathbf{R}^3 \) Poisson structure. The configuration space is \( \mathbf{S}^2 \( r \), the

\( \mathbf{R}^3 \) - algebra \( \mathbf{R}^3 \times \mathbf{R}^3 \) and also \( g \sim \alpha \).

Then the \( \mathbf{R}^3 \) Poisson structure on \( \mathbf{R}^3 \) to the rigid body bracket:

\[ \{ R, \mathbf{R} \} = -\mathbf{I} \cdot \mathbf{R} \times \mathbf{D} \mathbf{R} \]

The Poisson map going from carrying the canonical bracket

into \( \mathbf{R}^3 \) Poisson bracket is ensured by Euler's angles.

We have two parallel descriptions: one in terms of the

Lagrangian on \( T \mathbf{S}^2 \), and another one in terms of Hamilton.

descriptions on \( T^* \mathbf{S}^2 \). (Via Legendre transform.) The Hamiltonian:

\[ H = \frac{1}{2} \mathbf{I} \mathbf{I} \] (which is an ellipsoid.

But we have a constraint on the motion: the angular

momentum, \( \mathbf{I} = \) constant.

In our Poisson formalism, a way of obtaining constants of

motion are by Casimir's functions. Our Poisson manifold,

Casimir's functions are always \( q \mathbf{q} \) for which \( \mathbf{q} \mathbf{q} = 0 \) for all \( \mathbf{q} \).

For the rigid body, \( e = \frac{1}{2} \mathbf{I} \mathbf{I} \) (for any Hamiltonian).

In the space \( \mathbf{I}, \mathbf{I}, \mathbf{I} \), \( e \) describes a sphere. The intersection

between \( e \) (this sphere) and the ellipsoid from 0 gives the trajectory

which are on the cover of the book.

We will all later that the energy uncertainty method used

Casimir's functions are used successfully by Bloch, Kruskopf,

Mandev and Lambrecht in [3] in an attempt to study the

stability of \( \mathbf{R}^3 \) with internal and external torques.
§ 5. Connections, Geometric Phase, Reduction.

The mathematical definition of a connection $A$ is

1. $A - g_{\mu} \text{-valued one-form on } \mathbb{S}(\mu) \subset P$
2. $A_{\mu} : \mathbb{S}(\mu) \to \mathfrak{g}$ for $\mu \in \mathfrak{g}$
3. $\mathcal{L}_{\xi} A = A_{\xi} \cdot A$ for $\xi \in \mathfrak{g}$

A principal bundle with a connection $A$ is a G-bundle $E$ with a $G$-valued one-form $A$.

The geometric phase is the change of the path $\gamma$ with respect to the connection $A$ (parameterization independent).

For the R3, $P = T^* SO(3)$. The connection map $\gamma : \mathbb{R}^3$.

The lift reduction of $T^* SO(3)$ by $SO(3)$ on the angular momentum

coordinate map. The reduced spaces $\mathbb{S}(\mu)/\mathfrak{g}_{\mu}$ are spaces in the $\mathbb{R}^3$ of Euclidean space radius $\mu$ (by rotation about $\mu$ axis).

The trajectories of the reduced dynamics system are the ones obtained in § 4.

There are strict definitions. But especially for connections (which is a curvilinear object) there is not only one point of view, they can be seen in terms of horizontal lift operators.

But explaining in simple words what a connection is, I'm afraid it is not easy. As far as I understand a connection is a prescription of horizontal directions at each point

In a horizontal slice with respect to which the space is divided into a vertical and a horizontal direction. An example: Christoffel symbols, connection, parallel transport etc. Somewhat similar to an of projectors. Somehow, in the sense that the direction which project to zero are natural directions. Also, one is thought of characterizing the curvature of the space.
The notion of parallel transport is best understood as a motion in the horizontal direction defined by a connection.

In the case of the sphere, the horizontal directions are given by the great circles.

The general procedure of Hamilton, Montgomery, and Levi 543 is to reduce the equations of a dynamical system to the sphere (12)

Here it and then by reconstruction, we back to the original trajectory.

By & cylindrical phase dissapear. What remains is the geometric phase. I reproduce one of their figures here:

\[\text{true traj.}
\]

\[\text{traj.}
\]

\[\text{reduced trajectory}
\]

Without going into details, this technique should be ear to important to work on reduced trajectories. Well, because the path on the sphere can be controlled (by suitable controls)

and so, too, by the procedure outlined above, one can have the true trajectory modified at his will.

Coming back to the cat: controlling the trajectory on the sphere, it's like running the cat's tail and muscle, and the geometric phase is the modified motion of the whole body.
From the following formula for $\theta$:

$$\theta = \frac{-\lambda + \sqrt{4\mu T + 16\mu T}}{4\mu T}, \quad \lambda = \frac{2\mu T}{\mu T + 1}$$

As it can be seen from (2), the relative angle $\theta$ of the rod about the axis is split into 2 parts. One is dependent exclusively of the geometry (area), the other is dependent only on energy and period of motion $\frac{2\mu T}{\mu T + 1}$.

The first one is the geometrical phase. The other one is the dynamical phase. Moreover, dividing the area enclosed by the reduced trajectory:

$$\lambda = \frac{2\mu T}{\mu T + 1}$$

It is conceivable, I think, that $\lambda$ depends only on the area. So $\lambda$ is the solid angle to the center of the sphere which encloses the trajectory.

I want to remark that the following: the constraints are very closely related to the constraints. Actually the constraints are somehow determined by the constraints.

In this case, the existence of angular momentum determines the nature of “liminal” (off on the great circles).

It is somewhat intuitively to think that the geometric phase (2) depends only on the area because it the constraint acts as a “projector” and changing the projection plane, the area changes.

But in the same time and surprisingly enough, it doesn’t matter which sphere the rod lies on or the solid angle because (2) is independent (27) and the solid angles at the center of the rod add up to $\frac{4\pi}{3}$.
Also another remark will be that periodical to reduced trajectory does not mean periodical true trajectory. On the sphere, all the trajectories are periodical except 2: the Lissajous orbits. This is another way of saying that the 2 orbits (reduced and true) differ by something (so).

Reduction: The reduced trajectory is something like an "Eulerian" description of the motion. Some supplementary forces are emerging which are not real forces. That's why it is needed a procedure to restore back the true trajectory among all possible ones.

Remains à vous montres (or to our cats) to the cat's problem. The reductive space is the shape space, which means the totality of the shapes of its body, conceived as a bundle of cylinders, and leaving some fixed angles, one to another. Of course the original space is the shape space modulo a retranslation.

In other words: reductive takes advantages of the symmetries to simplify the problem (reduce the number of unknowns essentially) but then, the solution inherits the simplifications and by "reconstructive" methods it is cleaned of the arbitrariness.
Optimization and controls:

To permit a few words about W. von der Lubbe's approach to the cat's problem.

Now we are not interested anymore in the possibility of accomplishing a single motion, but due to the new geometrical insight gained we can ask ourselves (would about cat's energy) which is the optimal way in which she can do it.

So, in general, the problem is stated as follows:

"Given a deformable body in free fall with initial angular momentum \( \mathbf{J}_0 \), find the most effective way to deform it so that to achieve a desired orientation."

Killing the translational \( \to \) set \( \zeta \), the center of mass at zero. (This is a question when there is friction?)

As we said before, the angular momentum being conserved \( \Rightarrow \) characteristic a corollary. Of tangent vector would be \( \zeta \) with.

The angular momentum conserves \( \Rightarrow \) zero.

The conclusion is that: "it turns \& \zeta \& \& is momentum to the total problem \( \mathbf{J} \) \( \Rightarrow \) principle covariant \& \( \mathbf{J} \) \& \( \lambda(t) \) satisfies Hamilton's differential equation for \( \mathbf{J}_0 \). When \( \mathbf{J}_0 \to \) the horizontal kinetic energy.

\[ \mathbf{J}_0 = \frac{1}{2} \dot{ll}^2 + \frac{1}{4} c^2 \phi \]

One can realize that the phenomenon of the phase shift raises questions about control theory. The idea that it can be possible to change the trajectory of a body (in some let's say) without influencing the mechanical laws (conservation of linear momentum) is quite surprising.
The most important application for this phenomenon is at the stability of RB's.

In §35, the authors study the idea of stability of coupled rigid and flexible bodies. The method used is energy-Lagrange method. They show that three internal rotors can realize any external torque feedback for the rigid body.

One of the conclusions is that mechanical systems subject to some external forces determined by some feedback laws, can be modeled by as Hamiltonian systems. These are good news because all the arsenal of tools on energy-conservation Lagrange function et can be used to design.

Finally, not insisting any more on the application of the geometric plane (which are a lot, there to clear), I just want to briefly acknowledge the paper of Krüller, J [1993] in which he treats non-holonomic constraints.

The new thing about such a problem is the fact that the forces which assure the constraint break the symmetry of zero angular momentum.

And now, in the final, I should explain the motto of the paper. It was said by H. Princen in his review of Shore's book, as a reaction at his ideas that one can replace the idea of force by equivalent velocity constraints. More exactly he stated that the geometric curvature of the path is always a minimum, subjected to the constraints.
References:


[11] Class Notes: (Ch. 1 + Ch. 13)