The Duality in the Formulation of a Classical Particle in a Yang–Mills Field

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1 Introduction [2], [4]

1. Given a manifold X and a closed 2-form B on it, we can consider a new symplectic form on T^*X, $\Omega = \Omega^{st} - B$ (where Ω^{st} is the canonical symplectic form, and we identify B with its lift $\pi^* B$ to T^*X). Such a form is always non-degenerate by Proposition 6.6.2 in Marsden [2], and hence gives a non-canonical symplectic structure on T^*X. If B is in fact an exact form, $B = dA$ for some 1-form A on X, then the equations of motion given by Ω and some Hamiltonian H are exactly the same as those given by Ω^{st} and a new Hamiltonian $t_A^* H$, where t_A is the fiber translation by A (Proposition 6.6.1 in [2]).

2. There is another way of producing these same equations of motion on T^*X, which is in a certain sense more natural as well as more general. Namely, we start with a principal G-bundle $P^\sharp$ over T^*X and a connection A on $P^\sharp$. The correspondence with the previous discussion comes from the fact that the curvature of A is a pullback of a 2-form on the base manifold T^*X. If $P^\sharp$ is a pullback of a principal bundle P over X, then this 2-form can be considered as a 2-form on X, and we are in the situation 1. The advantages of this approach are as follows:

1)It immediately generalizes to the Yang–Mills situation, i.e. when G is the internal symmetry group (Sternberg [4]).

2)Further, given P and an internal symmetry space F, there is a symplectic manifold $(T^*P \times F)_0$ with a natural symplectic structure, and the non-canonical equations of motion on T^*X are the projections of the canonical equations of motion on this manifold given a choice of a connection on P (Weinstein [5]).

3)Even in the electromagnetic case, i.e. when F ={point} and $G = S^1 = U(1)$, a 2-form on X which is not exact will arise as a certain curviture (Kummer [1]). The following is true for circle bundles (possibly for any abelian G ??):

For any compact manifold X, the first Chern class homomorphism from the set of the equivalence classes of circle bundles over X to $H^2(X : Z)$ is an isomorphism. That is , for any 2-form B on X that represents a cohomology class *with integer coefficients* there is a circle bundle P over X such that the curvature of any connection on P is cohomologous to B. Thus for some such connection B is exactly its curvature. (The bundle is constructed as a certain quotient of the path space of X, and is not very explicit).

2 The Setup [5]

Let X be a manifold, G a Lie group, $P \xrightarrow{\pi} X$ a principal G-bundle, and F a Hamiltonian G-space. Then T^*X, T^*P are symplectic manifolds with canonical 1-forms Θ_X, Θ_P, respectively, and F is equipped with a symplectic form Ω_F, G-action $\phi : G \rightarrow Diff_{can}(F)$, and a momentum map $\Phi : F \rightarrow \mathcal{G}^*$, where $\mathcal{G}^*$ is the dual of the Lie algebra $\mathcal{G}$ of G. Note that Φ is G-equivariant, i.e. the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\phi_g} & F \\ \downarrow{\scriptstyle \Phi} & & {\scriptstyle \Phi}\downarrow \\ \mathcal{G}^* & \xrightarrow{Ad^*_g} & \mathcal{G}^* \end{array} \quad \forall g \in G$$

where Ad^* is the coadjoint representation of G on $\mathcal{G}^*$.

3 The Geometry [3], [5]

The right action of G on P lifts in a natural way to a symplectic right action on T^*P with a momentum map $\rho : T^*P \rightarrow \mathcal{G}^*$. This map can be described as follows: for $p \in P$, $x = \pi(p) \in X$, we have an exact sequence (the inclusion of the vertical subspace):

$$0 \longrightarrow \mathcal{G} \longrightarrow T_pP \xrightarrow{d\pi} T_xX \longrightarrow 0 \tag{1}$$

and its dual:

$$0 \longleftarrow \mathcal{G}^* \xleftarrow{\rho} T_p^*P \xleftarrow{(d\pi)^*} T_x^*X \longleftarrow 0 \tag{2}$$

Putting together such maps on the fibers gives $\rho : T^*P \rightarrow \mathcal{G}^*$.

It will be convenient to work with left rather than right actions. Thus define a left action of G on P by $(g \cdot p) = p \cdot g^{-1}$. The lift of this action to T^*P will have the momentum map $-\rho$.

Therefore we have a G-equivariant map $(-\rho + \Phi) : T^*P \times F \rightarrow \mathcal{G}^*$. We can apply the standard reduction procedure to this map, reducing at the zero element of $\mathcal{G}^*$. Specifically, we take the inverse image $(-\rho + \Phi)^{-1}(0)$ and divide by G to obtain the 'reduced manifold' $(T^*P \times F)_0$. Since 0 is a regular value of ρ, it is also a regular value of $(-\rho + \Phi)$, so $(-\rho + \Phi)^{-1}(0)$ is a submanifold; it is preserved by the action of G since the maps are equivariant and 0 is preserved by the coadjoint action. Also, $\mathcal{G}$ acts freely on P, hence also on $T^*P \times F$. Consequently, from the general theory $(T^*P \times F)_0$ is a manifold with the natural symplectic structure (that inherited from $T^*P \times F$).

Let $P^\sharp$ be the pullback of P to T^*X, i.e.

defined by the diagram: $\begin{array}{ccc} P^\natural & \longrightarrow & P \\ \downarrow & & \downarrow \pi \\ T^*X & \longrightarrow & X \end{array}$

We will also consider the manifold $P^\natural \times_G F$ (which, since $P^\natural$ is a principal G-bundle over T^*X, is the associated F-bundle over T^*X).

We show that a connection on P gives an isomorphism of $(T^*P \times F)_0$ and $P^\natural \times_G F$. First notice that we may also consider $P^\natural$ as a pullback of T^*X to P. Therefore the exact sequence (2) gives exactly

$$0 \rightarrow P^\natural \rightarrow T^*P \rightarrow \mathcal{G}^* \rightarrow 0 \tag{3}$$

where the first two manifolds are considered as bundles over P. A connection A on P gives by definition a splitting of (1) and therefore of (2) and (3). Therefore it gives an isomorphism $T^*P \overset{A}{\cong} P^\natural \times \mathcal{G}^*$ (fiberwise isomorphism). Denote by $\sigma : T^*P \rightarrow P^\natural$ the projection onto the first factor. Note that the projection onto the second factor is exactly ρ.

For $(v, f) \in T^* \times F$, $(v, f) \in (-\rho + \Phi)^{-1}(0)$ we have $\rho(v) = \Phi(f)$, and so (v, f) is uniquely determined by $(\sigma(v), f) \in P^\natural \times F$. Quotienting out by G now gives the isomorphism $(T^*P \times F)_0 \cong P^\natural \times_G F$.

Note that $P^\natural$ is by definition a bundle over T^*X, hence so is $P^\natural \times_G F$ as well as, through the above direct product representation, T^*P. Thus we obtain the diagram

$$\begin{array}{ccccc} T^*P \times F & \rightsquigarrow & (T^* \times F)_0 & \overset{A}{\cong} & P^\natural \times_G F \\ \downarrow A & & & & \downarrow \\ T^*X & & = & & T^*X \end{array}$$

4 The Symplectic Structure [1], [4]

Suppose we are given a Hamiltonian H on X (for example, X is a Riemannian manifold and H is the kinetic energy). We can produce the equations of motion for X governed by the connection A in two ways. We can canonically lift H to $P^\natural \times_G F$, but consider on this manifold the noncanonical symplectic structure obtained from the isomorphism with $(T^*P \times F)_0$. Or we can consider the canonical symplectic structure on $(T^*P \times F)_0$, but with the noncanonical Hamiltonian obtained by pulling H back to T^*P. This is exactly the duality noted in the **Introduction** for the electromagnetic case (there $F = \{\text{point}\}$, and so both manifolds are in fact isomorphic to T^*X).

The advantage of the first approach is that the dependence on the connection A is only through a differential of a certain 1-form depending on A (to be made precise later). In particular for $F = \{e\}$ the dependence is only on the curvature of A, and not on A itself. The advantage of the second approach is (to quote Weinstein) that the symplectic structure on phase space is determined by the geometry of the system, and the Hamiltonian is determined by the 'physics' (represented in this example by the function on T^*X and the connection on P).

We now calculate the difference between the canonical form on $T^*P \times F$ and the lift of the canonical form on T^*X. In what follows, in places where no confusion should arise we will denote forms and their lifts by the same letter. First we introduce some additional notation. Recall that we have the projection $\pi : P \to X$, and let $\tau_X : T^*X \to X$, $\tau_P : T^*P \to P$ be the canonical projections. Recall also that from the sequence (2) $P^\flat$ (as a subbundle of T^*P) is put together from the images of the maps $T_x^* \xrightarrow{(d\pi)^*} T_p^*P$. Thus $P^\flat$ is naturally identified with $(d\pi)^*(T^*X) \subset T^*P$ (and so is the horizontal subbundle of T^*P) and the projection $P^\flat \to T^*X$ is $(d\pi^*)^{-1}$ (resticted to $P^\flat$).

Finally, we specify the connection on P by the $\mathcal{G}$-valued connection 1-form $T_pP \xrightarrow{\omega} \mathcal{G}$. Its dual $T_p^*P \xleftarrow{\omega_p^*} \mathcal{G}^*$, $\forall p \in P$ gives the inclusion of the vertical subspace of T_p^*P (where by analogy with vertical forms vertical elements of T_p^*P are those that annihilate the horizontal subspace of T_pP). Therefore we have a split exact sequence

$$0 \to P^\flat \underset{}{\overset{\sigma}{\rightleftharpoons}} T^*P \underset{\rho}{\overset{\omega^*}{\rightleftharpoons}} \mathcal{G}^* \to 0 \tag{4}$$

and so for $\alpha \in T_p^*P$

$$\sigma(\alpha) = \alpha - \omega_p^*\rho(\alpha)$$

Thus the projection $\tilde{\pi}$ in $\begin{array}{ccc} T^*P & \xrightarrow{\tilde{\pi}} & T^*X \\ \downarrow \tau_P & & \tau_X \downarrow \\ P & \xrightarrow{\pi} & X \end{array}$ is

$$\tilde{\pi}(\alpha) = (d\pi^*)^{-1}\sigma(\alpha) = (d\pi^*)^{-1}(\alpha - \omega_p^*\rho(\alpha)) \text{ for } \alpha \in T_p^*P$$

Therefore for $v \in T_\alpha T^*P$, θ_X, θ_P canonical 1-forms on X, P

$$\begin{aligned} (\tilde{\pi}^*\theta_X)_\alpha(v) &= \theta_{X\,\tilde{\pi}(\alpha)}(d\tilde{\pi}(\alpha)v) \\ &= \langle \tilde{\pi}(\alpha), d\tau_X d\tilde{\pi}(\alpha)v \rangle \\ &= \langle \tilde{\pi}(\alpha), d\pi(p) d\tau_P v \rangle \\ &= \langle \alpha - \omega_p^*\rho(\alpha), d\tau_P v \rangle \\ &= \theta_{P\,\alpha}(v) - \langle \omega_p^*\rho(\alpha), d\tau_P v \rangle \end{aligned}$$

For $(\alpha, \beta) \in T_p^*P \times F$, $(v, u) \in T_{(\alpha,\beta)}(T^*P \times F) = T_\alpha T^*P \times T_\beta F$, still denoting $\tilde{\pi} : T^*P \times F \to T^*X$

$$(\tilde{\pi}^*\theta_X)_{(\alpha,\beta)}(v, u) = \theta_{P\,\alpha,\beta}(v) - \langle \omega_p^*\rho(\alpha), d\tau_P v \rangle$$

For $(\alpha, \beta) \in (-\rho + \Phi)^{-1}(0)$, i.e. when $\rho(\alpha) = \Phi(\beta)$

$$(\theta_P - \tilde{\pi}^*\theta_X)_{(\alpha,\beta)}(v, u) = \langle \omega_p^*\Phi(\beta), d\tau_P v \rangle = \langle \Phi(\beta), \omega \circ d\tau_P v \rangle$$

Note that $(\omega \circ d\tau_P) = \tau_P^*\omega$ is the connection on the principal G-bundle $P^\flat \to T^*X$ corresponding to ω.

Φ is a $\mathcal{G}^*$-valued function on F, and $d\tau_P^*\omega$ is a $\mathcal{G}$-valued 1-form on $P^\sharp$. Therefore we can define a real-valued one form Ψ on $T^*P \times F$ (or on $P^\sharp \times F$) by

$$\Psi(v,u) = \langle \Phi(\beta), \omega \circ d\tau_P v \rangle, \text{ for } (v,u) \in T_\alpha T^*P \times T_\beta F$$

Then

$$\theta_P = \theta_X + \Psi$$

$$d\theta_P = d\theta_X + d\Psi$$

$$\Omega_P + \Omega_F = \Omega_X - (d\Psi - \Omega_F)$$

Since both $(\Omega_P + \Omega_F)$ and Ω_X project down when we quotient $P^\sharp \times F$ out by G, $(d\Psi - \Omega_F)$ can be considered as a form on $P^\sharp \times_G F$.

5 Examples [1], [3], [4]

1)For $G = U(1)$ (or any abelian Lie group) and $F = \{e\}$ we obtain the situation in the **Introduction** (the minimal coupling in electrodynamics). $P^* \times_G \{e\} \cong P^\sharp / G \cong T^*X$, $\Psi(v) = e \cdot \tau_P^*\omega(v)$, $\Omega_F = 0$. Thus $\Omega_P - \Omega_X = e \cdot d(\tau_P^*\omega)$ is the curvature of the connection on $P^\sharp$, considered as a 2-form on T^*X. Since $\tau_P^*\omega$ itself is a horizontal 1-form on $P^\sharp$ considered as a bundle over X, $d\Psi = \Omega_P - \Omega_X$ is in fact a lift of a 2-form of X. Note that Ψ itself need not even be a form on $P^\sharp \times_G F$.

2)More generally, for F a coadjoint orbit of G acting on $\mathcal{G}^*$, we obtain the interaction af a Yang–Mills field with a particle arising from the irreducible representation of G.

3)For $F = \mathcal{G}^*$, we obtain a more complete description of the motion of such a particle, provided by Wong's equations. Since this F has the coadjoint orbits as symplectic leaves, it turns out that the reduced phase space in this case has the reduced phase spaces of the previous paragraph as symplectic leaves.

References

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