Laplace-Runge-Lenz Vector

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The central inverse square law force problem is an interesting one in physics. It is interesting not only because of its applicability to a great deal of situations ranging from the orbits of the planets to the spectrum of the hydrogen atom, but also because it exhibits a great deal of symmetry. In fact, in addition to the usual conservations of energy E and angular momentum $\boldsymbol{L}$, the Kepler problem exhibits a hidden symmetry. There exists an additional conservation law that does not correspond to a cyclic coordinate. This conserved quantity is associated with the so called Laplace-Runge-Lenz (LRL) vector $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - mk\hat{\boldsymbol{r}} \tag{LRL Vector}$$

The nature of this hidden symmetry is an interesting one. Below is an attempt to introduce the LRL vector and begin to discuss some of its peculiarities.

A Some History

The LRL vector has an interesting and unique history. Being a conservation for a general problem, it appears as though it was discovered independently a number of times. In fact, the proper name to attribute to the vector is an open question. The modern popularity of the use of the vector can be traced back to Lenz's use of the vector to calculate the perturbed energy levels of the Kepler problem using old quantum theory [1]. In his paper, Lenz describes the vector as "little known" and refers to a popular text by Runge on vector analysis.

In Runge's text, he makes no claims of originality [1]. In his text, he uses the vector to illustrate the derivation of the orbit equation from the additional symmetry. A similar use of the vector will be illustrated below.

Similar to Lenz's use, in his 1926 paper, Pauli used the LRL vector to derive the energy levels of the Hydrogen atom without use of the Schrödinger equation . Here Pauli refers to the vector as "previously utilized by Lenz" [1]. Since then, it has become popular to refer to the vector as the Runge-Lenz vector, as it is named in Shankar [7] and other popular quantum contexts.

Unfortunately, and earlier appearance of the vector can be found in Laplace's *Traté de mécanique celeste* [1], in this work, Laplace not only discovers the vector but goes on to demonstrate its relation to the energy E and angular momentum $\boldsymbol{L}$, as will be demonstrated below.

Because of the earlier utilization by Laplace, it has become customary to describe the vector as the Laplace-Runge-Lenz vector as is done here. Unfortunately, the history goes back further.

As Goldstein notes in his examination of the history [1], some others attribute the vector as originating with Hamilton. He relates such and asks for others to help in his search for the appearance in Hamilton's papers.

In a followup note [2], Goldstein relates that reference to the vector can be found in Hamilton's July 1845 paper, *Applications of Quaternions to Some Dynamical Questions*, where we derives its existence, where he refers to it as the "eccentricity vector", which is closely related to the vector $\boldsymbol{A}$ presented here.

He goes on to describe how this vector determines the evolution of the momentum with his "hodograph", to be illustrated below.

Now, the connection to the hodograph allows us to trace the history further back as Goldstein notes [2]. In fact, Gibb's in his *Vector Analysis* looks at the hodographs for the Kepler problem and presents it in modern vector notation. This predates Runge's analysis by about 20 years.

Goldstein goes on along with other authors to then search for evidence of the LRL vector as early as Newton's *Principia*, but little evidence is found.

Finally, in an added note to his second investigation, Goldstein relates the work done by Professor Otto Volk in tracing the history of the LRL vector, where the magnitude of the LRL vector appears as conserved in a work by Jakob Hermann. In a letter between Hermann and Johnan I. Bernoulli, Bernoulli goes on to generalize Hermann's result, in affect giving the direction of the LRL vector as well as its magnitude.

Goldstein concludes by suggesting an appropriate name might be the Hermann-Bernoulli-Laplace vector. And in fact, at least one author [8] goes on to call it the Hermann-Bernoulli-Laplace-Hamilton-Runge-Lenz vector. In this paper we will stick with the more traditional Laplace-Runge-Lenz vector name.

On the Quantum side of things, we should note a bit about the history as well. Following the important 1926 paper by Pauli, where he derives the hydrogen spectrum purely algebraically (the approach recounted in english by Valent [9]), Rogers recounts some of the history [6].

Hulthén and Klein showed that the six constants of the Kepler problem form a Lie algebra isomorphic to O(4), as is hopefully demonstrated below. Then Fock went on to explicitly demonstrate the degeneracy of the wave functions under such transformations, and in 1936 Bargmann made the connection between Pauli's approach and the group theory clear.

Since then, the LRL vector has been quite popular as a simple example of hidden symmetries. People have gone on to make generalizations of it to apply to cases with electric and magnetic fields, and relativistic versions as well. For an introduction to the generalizations, see the review by Leach and Flessas [4].

B Kepler Problem

The Laplace-Runge-Lenz (LRL) vector has its origins in the peculiarities of the Kepler problem. The vector itself

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - mk\hat{\boldsymbol{r}} \tag{LRL Vector}$$

is an additional conserved quantities for central force problems with an inverse square potential. Here $\boldsymbol{p}$ is the momentum, $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ the angular momentum, m the mass (or for the two body problem, replace $\mu = m_1 m_2/(m_1 + m_2$ the reduced mass). And k is a constant characterizing the strength of the potential.

We are interesting in the general Kepler Problem, i.e. the Hamiltonian

$$H = \frac{|\boldsymbol{p}|^2}{2m} - \frac{k}{r}$$

This problem is a very interesting one, marked by a great deal of symmetry. Besides the usual conserved energy E, since the Hamiltonian is rotationally invariant, we know to expect conservation of the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$.

(a) Derivation

In order to derive the conservation of the LRL vector, we can follow the development in Goldstein [3]. Given that our force is central, we can restrict Newton's law to the form

$$\dot{\boldsymbol{p}} = f(r)\frac{\boldsymbol{r}}{r}$$

I.e. the force should point in the radial direction and depend only on r. This allows us to find a form for

$$\begin{aligned}
\dot{\boldsymbol{p}} \times \boldsymbol{L} &= f(r)\frac{\boldsymbol{r}}{r} \times (\boldsymbol{r} \times \boldsymbol{p}) = f(r)\frac{\boldsymbol{r}}{r} \times (\boldsymbol{r} \times m\dot{\boldsymbol{r}}) \\
&= \frac{mf(r)}{r}\left[\boldsymbol{r} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}})\right] = \frac{mf(r)}{r}\left[\boldsymbol{r}(\boldsymbol{r} \cdot \dot{\boldsymbol{r}}) - r^2\dot{\boldsymbol{r}}\right]
\end{aligned}$$

where we have used Laplace's identity in the last equality. Next we need to be clever and further simplify the expression by noticing

$$\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = \frac{1}{2}\left(\dot{\boldsymbol{r}} \cdot \boldsymbol{r} + \boldsymbol{r} \cdot \dot{\boldsymbol{r}}\right) = \frac{1}{2}\frac{d}{dt}(\boldsymbol{r} \cdot \boldsymbol{r}) = \frac{1}{2}\frac{dr^2}{dt} = r\dot{r}$$

i.e. the radial component of the velocity is just $\dot{r}$. Using this, we obtain

$$\dot{\boldsymbol{p}} \times \boldsymbol{L} = \frac{mf(r)}{r}\left[r\dot{r}\boldsymbol{r} - r^2\dot{\boldsymbol{r}}\right] = mf(r)r^2\left[\frac{\dot{r}\boldsymbol{r}}{r^2} - \frac{\dot{\boldsymbol{r}}}{r}\right]$$

Since $\boldsymbol{L}$ is a conserved quantity, we can write

$$\frac{d}{dt}\left(\boldsymbol{p} \times \boldsymbol{L}\right) = -mf(r)r^2\left[\frac{\dot{\boldsymbol{r}}}{r} - \frac{\dot{r}\boldsymbol{r}}{r^2}\right] = -mf(r)r^2\frac{d}{dt}\left[\frac{\boldsymbol{r}}{r}\right]$$

So we've made some progress, found and interesting expression. Unfortunately, at this point we are stuck, unless of course $f(r) \propto r^{-2}$ such as for the Kepler problem. This peculiarity enables us to take $f(r) = -k/r^2$ and we obtain

$$\frac{d}{dt}(\boldsymbol{p} \times \boldsymbol{L}) = \frac{d}{dt}\left(\frac{mk\boldsymbol{r}}{r}\right)$$

which is to say we have

$$\frac{d}{dt}\left[\boldsymbol{p} \times \boldsymbol{L} - mk\frac{\boldsymbol{r}}{r}\right] = 0$$

i.e. we've found that

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - mk\frac{\boldsymbol{r}}{r}$$

is conserved.

(b) Some Discussion

Now that we have found a new conserved vector for the Kepler problem, the question remains as to its utility. In particular, we should already be a little concerned. The one body Kepler problem has 6 degrees of freedom, but we already have E and $\boldsymbol{L}$ and now $\boldsymbol{A}$, suggesting $1+3+3=7$ conserved quantities. Surely this cannot be the case. In particular, we know that for the Kepler problem, there is nothing that should tell us the initial time of our motion, leaving, at most, 5 degrees of freedom that can be conserved. This implies that there should be some relations between $\boldsymbol{A}$ and $E, \boldsymbol{L}$. In particular, we notice that if we take the dot product

$$\boldsymbol{A} \cdot \boldsymbol{L} = \left[\boldsymbol{p} \times \boldsymbol{L} - mk\frac{\boldsymbol{r}}{r} \right] \times \boldsymbol{L} = (\boldsymbol{p} \times \boldsymbol{L}) \times \boldsymbol{L} - mk\frac{\boldsymbol{r}}{r} \times (\boldsymbol{r} \times \boldsymbol{p}) = 0$$

i.e. $\boldsymbol{A}$ is perpendicular to $\boldsymbol{L}$ at all points of the motion, i.e. it lies in the plane of motion.

In particular, we can compute the LRL vector at various points in the orbit. Borrowing from Goldstein [3] (Figure **??**) we notice that $\boldsymbol{A}$ lies in the plane of the orbit, as discussed, is a constant of the motion, and appears to point in the direction of the symmetry axis of our ellipse.

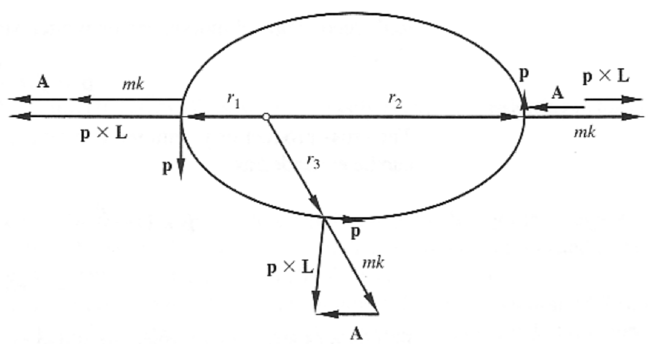

FIGURE 3.18 The vectors **p**, **L**, and **A** at three positions in a Keplerian orbit. At perihelion (extreme left) $|\mathbf{p} \times \mathbf{L}| = mk(1+e)$ and at aphelion (extreme right) $|\mathbf{p} \times \mathbf{L}| = mk(1-e)$. The vector **A** always points in the same direction with a magnitude mke.

Figure 1: LBL vector at various points in the orbit. Borrowed from Goldstein [3].

In order to find the other relation amongst $E, \boldsymbol{L}, \boldsymbol{A}$ we need to look at the direction $\boldsymbol{A}$ points in detail. We can take the dot product with the radius vector

$$\boldsymbol{A} \cdot \boldsymbol{r} = Ar\cos\theta = \boldsymbol{r} \cdot (\boldsymbol{p} \times \boldsymbol{L}) - mkr$$

and permuting the triple product we get the form

$$\boldsymbol{A} \cdot \boldsymbol{r} = (\boldsymbol{r} \times \boldsymbol{p}) \cdot \boldsymbol{L} = L^2$$

and we can obtain the equation

$$Ar\cos\theta = L^2 - mkr$$

$$\frac{1}{r} = \frac{mk}{L^2}\left(1 + \frac{A}{mk}\cos\theta\right)$$

which has the form of a conic section. Interestingly enough, we have found the orbits of the Kepler problem in terms of the LRL vector.

Comparing this with the standard solution for the orbit [3]

$$\frac{1}{r} = \frac{mk}{L^2}\left(1 + \sqrt{1 + \frac{2EL^2}{mk^2}}\cos(\theta - \theta')\right)$$

we find that we can write

$$\frac{A^2}{m^2k^2} = 1 + \frac{2EL^2}{mk^2}$$

which gives us

$$A^2 = m^2k^2 + 2mEL^2$$

Which is precisely the other relation we were seeking.

Sanity returned we discovery that among our supposed 7 conserved quantities we have 2 relations, bringing the number back down to 5, as is to be expected. So, if we take our energy E and angular momentum $\boldsymbol{L}$ as our primary conserved quantities, then the addition of $\boldsymbol{A}$ only gives us a single additional conserved degree of freedom? Where does this peculiar symmetry arise?

We are still left with the peculiarity of the Kepler problem, given that it is completely integrable. The appearance of the unique conserved vector depended on our force law being inverse square. There is another peculiarity associated with inverse square laws, in particular, we know them to admit closed orbits according to Bertran's Theorem [3].

So, it would appear as though our peculiar symmetry, the LRL vector seems to be tied with the fact that our orbits are closed.

With this realization, we might expect something akin to the LRL vector for different problems, in particular Bertand's Theorem would suggest an analog to the LRL vector for the harmonic oscillator in particular, the other potential that admits closed orbits.

We will leave the investigation of the corresponding harmonic oscillator case until later.

(c) Hodographs

So, from our pictorial investigation, we are lead to believe that our LRL vector points in the direction of the symmetry axis of our orbit. In fact, as Goldstein recounts [2], Hamiltonian may have been the first to utilize the LRL conservation. He referred to it as the eccentricity vector, where if we rearrange our expression as

$$mk\frac{\boldsymbol{r}}{r} = \boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{A}$$

and take the dot product of it with itself, we obtain

$$(mk)^2 = A^2 + p^2 L^2 + 2\boldsymbol{L} \cdot (\boldsymbol{p} \times \boldsymbol{A})$$

and now choosing $\boldsymbol{L}$ to be along the z axis and the major semiaxis along the x we obtain [2]

$$p_x^2 + (p_y - A/L)^2 = (mk/L)^2$$

but this gives us an interesting representation for the momentum vector. In fact it illustrates that the momentum should just travel around a circle of radius mk/L centered A/L away from the center of force perpendicular to $\boldsymbol{A}$.

C Hydrogen Atom

Another interesting problem that fits the bill for the above description (1) central force, and (2) inverse square force, is the Hydrogen atom. Just as before we have the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{e^2}{r}$$

which has the same form. As we know, borrowing from Shankar, we have energy levels [7].

$$E_{nlm} = -\frac{me^4}{2\hbar^2}\frac{1}{n^2}$$

which we notice depends only on n and not l or m. We have a great deal of degeneracy in the orbitals of the hydrogen atom. Not only do the energy levels not depend on m, corresponding to the rotational symmetry, but neither do they depend on l, a much deeper symmetry, which we might suspect to be linked to our LRL vector.

(a) Quantum LRL

So, we suspect the high level of symmetry should derive from the LRL vector just as it did in the classical case. As such, we begin by attempting to look for a quantum analog to:

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - mk\frac{\boldsymbol{r}}{r}$$

or in this case we will use a slight modification

$$\boldsymbol{N} = \frac{\boldsymbol{p} \times \boldsymbol{L}}{m} - \frac{e^2}{r}\boldsymbol{r}$$

which suggests a particular quantum form, taking care to ensure Hermiticity

$$\hat{\boldsymbol{N}} = \frac{1}{2m}\left[\hat{\boldsymbol{P}} \times \hat{\boldsymbol{L}} - \hat{\boldsymbol{L}} \times \hat{\boldsymbol{P}}\right] - \frac{e^2}{r}\hat{\boldsymbol{R}}$$

using hats to denote the quantum operators.

Lucky for us, this operator manages to commute with the Hamiltonian. With a conservative force $\hat{\boldsymbol{P}}$ we know commutes, and the rotational symmetry ensures that so do $\hat{\boldsymbol{R}}, \hat{\boldsymbol{L}}$. So we have $[\hat{\boldsymbol{N}}, \hat{\boldsymbol{H}}] = 0$. This implies that it should act as the generator of some symmetry.

Having found the corresponding operator, it would be nice to see how it generates the degeneracy. This can be accomplished following the procedure in Shankar [7]. We need to express $\hat{\boldsymbol{N}}$ in spherical form

$$\hat{N}_1^{\pm 1} = \mp \frac{N_x \pm iN_y}{\sqrt{2}} \qquad \hat{N}_1^0 = N_z$$

Having done this we can consider having our $\hat{N}_1^1$ operator act on a state $| \, nll\rangle$. We know that this should produce a state with the same energy since our operator commutes with the Hamiltonian $[\hat{N}_1^1, H] = 0$. So what does our operator due to our state? Well, we know that $\hat{N}_1^1 \mid nll\rangle$ should behave like $| \, 11\rangle \otimes | \, ll\rangle = | \, l+1, l+1\rangle$ i.e. we have

$$\hat{N}_1^1 \mid n, l, l\rangle = c \mid n, l+1, l+1\rangle$$

And we've done it. We can use the quantum analogy to the LRL vector to demonstrate the degeneracy in the different l levels.

D Poisson Brackets

Drawing from our experience with the Quantum application, it might be useful to look into the algebraic structure of the LRL vector. First lets revisit the Poisson brackets for the angular momentum vector. Again drawing from Goldstein [3] we know:

$$\{L_i, L_j\} = \epsilon_{ijk} L_k$$

But this structure is the same as for the generators of rotation in three dimensional space. I.e. the group of transformations generated by L_i can be identified with SO(3).

Or, to be clear, lets follow the notation of Rogers [6] and write

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

$$\boldsymbol{D} = \frac{1}{\sqrt{-2mE}}\left[\frac{\boldsymbol{p} \times \boldsymbol{L}}{mk} - \frac{\boldsymbol{r}}{r}\right]$$

In order to discover the SO(4) structure that these imply, consider looking at the two linear combinations

$$\boldsymbol{M} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{D}) \qquad \boldsymbol{N} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{D})$$

These new guys have the following Poisson structure

$$\{M_i, M_j\} = \epsilon_{ijk} M_k$$

$$\{N_i, N_j\} = \epsilon_{ijk} N_k$$

$$\{N_i, M_j\} = 0$$

But at this point, it should be clear that the structure created by $\boldsymbol{L}, \boldsymbol{D}$ is that of $SO(3) \otimes SO(3)$ or in other words $SO(4)$.

So we've done it. We have investigated the structure defined by the angular momentum and LRL vector in concert and now see that we should expect a great deal of symmetry in the Kepler problem, namely $SO(4)$ or a 4 dimensional rotational symmetry.

Unfortunately for us, this deeper symmetry is not clear in our usual coordinates [6]. In order to better appreciate its appearance

E Geometric Treatment

The difficulty of finding coordinates that represent the symmetry was handled in a rather nice geometric way by Rogers. Here, he suggests considering the approach used by Fock when he investigated the Hydrogen Atom. Here our aim would be to project the momentum space stereo-graphically onto the 3-sphere. Fortunately for the case of negative energy bound orbits, we saw that the motion in momentum space was particularly simple, in the form of the hodographs, i.e. the momentum traces out a circle. Under a stereographic projection, we expect our circles to be mapped to circles on the 3-sphere. So, if we wish to describe a symmetry between the orbits in momentum space, we are interesting in mapping circles to circles on the 3-sphere.

At this point, Rogers suggests the mapping [6]

$$\boldsymbol{P} = \frac{2p_o\boldsymbol{p}}{p_0^2 + p^2} \qquad P_4 = \frac{p^2 - p_0^2}{p_0^2 + p^2}$$

where we identify

$$p_0 = \sqrt{-2mE}$$

These four coordinates satisfy the condition

$$P^2 + P_4^2 = 1$$

And in addition to a mapping for the momentum, we wish to also find a mapping for the coordinates in configuration space. Here orbits are ellipses, which themselves could be parallel projections on the sphere. We consider (again following Rogers [6]).

$$\boldsymbol{R} = \frac{\boldsymbol{r}}{r} - \frac{\boldsymbol{r}\cdot\boldsymbol{p}}{mk}\boldsymbol{p} \qquad R_4 = \frac{p_0}{mk}\boldsymbol{r}\cdot\boldsymbol{p}$$

These two are not independent and satisfy

$$R^2 + R_4^2 = 1$$

and in particular these coordinates are themselves orthogonal to one another

$$\boldsymbol{R}\cdot\boldsymbol{P} + R_4 P_4 = 0$$

So we have in total 8 coordinates with 3 constraints or only 5 specifications. We are missing one dimension in our 6 degree of freedom system, and we will take it to be E.

We can obtain the inverse relations:

$$\boldsymbol{r} = -\frac{k}{2E}[(1-P_4)\boldsymbol{R} + R_4\boldsymbol{P}] \qquad \boldsymbol{p} = \sqrt{-2mE}\frac{\boldsymbol{P}}{1-P_4}$$

So at this point, hopefully what we have accomplished is a simplification of our equations, thereby highlighting the high degree of symmetry.

At this point, Rogers goes on to demonstrate how we can properly understand these new coordinates in terms of quaternions [6]. Once he reviews the quaternion algebra, he is in a position to give a proper geometrical interpretation of the result. Unfortunately, I do not feel comfortable enough with his approach to recount it here.

An alternative way to see the O(4) symmetry is given by Moser [5]. In his account, the transformations are more direct and less geometric.

F Conclusion

Having demonstrating the appearance and utility of the LRL vector to some physical problems, hopefully an appreciation for the depth of the corresponding symmetry was conveyed.

References

[1] H. Goldstein. Prehistory of the"Runge–Lenz"vector. *American Journal of Physics*, 43:737, 1975.

[2] H. Goldstein. More on the prehistory of the Laplace or Runge–Lenz vector. *American Journal of Physics*, 44:1123, 1976.

[3] H. Goldstein, C. Poole, J. Safko, and S.R. Addison. Classical mechanics. *American Journal of Physics*, 70:782, 2002.

[4] PGL Leach and GP Flessas. Generalisations of the Laplace-Runge-Lenz vector. *Journal reference: J. Nonlinear Math. Phys*, 10(3):340–423, 2003.

[5] J. Moser. Regularization of Kepler's problem and the averaging method on a manifold. *Communications on Pure and Applied Mathematics*, 23(4), 1970.

[6] H.H. Rogers. Symmetry transformations of the classical Kepler problem. *Journal of Mathematical Physics*, 14:1125, 1973.

[7] R. Shankar. *Principles of quantum mechanics*. Springer, 1994.

[8] PR Subramanian. Hermann-Bernoulli-Laplace-Hamilton-Runge-Lenz Vector. *Physics Education*, 7(4):323–27, 1991.

[9] G. Valent. The hydrogen atom in electric and magnetic fields: Pauli's 1926 article. *American Journal of Physics*, 71:171, 2003.