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The Reduced Energy-Momentum Method

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This work mainly reviews the construction of the reduced energy-momentum method used to solve the non-linear stability of relative equilibria of a pseudo-rigid body, following the paper by Simo, Lewis & Marsden [1]. Due to the time limitation, we will only discussed the material up to the construction of the reduced Hamiltonian. Some of the modern geometric concepts are needed to develop the method and we will elaborate about these shortly.

Introduction

It is well-known that linear instability of equilibrium solutions of a dynamical system determines the non-linear instability of the solutions. However, this is not the case for the stable counterparts due to the effects of, for instance, resonance and Arnold diffusion [2]. General stability criterion, called the Energy-Momentum-Casimir method, had been developed based on the classical Lagrange-Dirichlet criterion. This method uses the Casimir functions (i.e. functions on the reduced space which Poisson-commute with the reduced Hamiltonian) and the momentum map. However, there is no guarantee that a Casimir function can be found. This lead to the development of the Reduced Energy-Momentum method (which basically is the variation of the Energy-Momentum method). Applying this method, the stability criterion can be obtained from the amended potential which depends only on the configuration space, by constraining the value of the momentum map to a constant (which is, in our case for SO(3) symmetry group, the total angular momentum).

The Reduced Energy-Momentum Method

Suppose that the configuration of a body is given in a Q space. We can then write the phase space as the symplectic cotangent bundle, $P = T^*Q$, equipped with the symplectic 2-form, Ω, on TP. Further, suppose that our system describing the dynamics of the body is Hamiltonian, $H : P \to \mathbb{R}$, and possesses a symmetry induced by a Lie group G, that is the Hamiltonian is invariant under the group action. This action on P is symplectic since it is just the cotangent lift of the (diffeomorphic) action of (Lie group) G (with Lie algebra $\mathfrak{g}$) on Q. Associated with this group action is a (cotangent lift induced) momentum map $J : P \to \mathfrak{g}^*$ which is defined in terms of the infinitesimal generator of the action of G on Q,

$$J(z) \cdot \xi = \langle p, \xi_Q(q) \rangle, \tag{1}$$

for $\xi \in \mathfrak{g}$, where a dot and $\langle \cdot, \cdot \rangle$ denote the duality pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ and a non-degenerate pairing between T_qQ and T_q^*Q respectively.

The Hamiltonian function and the momentum map defined above are used in the following theorem [1].

Theorem 1. *$z_e \in P$ is a relative equilibrium of a mechanical system with Hamiltonian H and momentum map J for the symplectic action of a Lie group G on the phase space $P = T^*Q$ iff there exists a $\xi_e \in \mathfrak{g}$ such that (z_e, ξ_e) is a critical point of the energy-momentum functional $H_{\mu_e} : P \times \mathfrak{g} \to \mathbb{R}$, defined as*

$$H_{\mu_e}(z, \xi) := H(z) - (J(z) - \mu_e) \cdot \xi, \tag{2}$$

where $\mu_e = J(z_e)$ is the value of the momentum map at z_e. (μ_e is assumed to be a regular value of J.)

z_e is a relative equilibrium of the system if the trajectory of the Hamilton's equations coincides with the one of the action of $exp[t\xi_e]$, i.e. the symmetry group orbit in the direction of ξ_e. Eqn. 2 is just an optimization problem, that is z_e is just the stationary solution of H restricted to subset $J^{-1}(\mu_e) = \{z \in P \mid J(z) = \mu_e\}$ (in our case later on, μ_e is just a constant total angular momentum), with ξ acts as the Lagrange multiplier. Here we assume the Hamiltonian H to be the sum of kinetic and potential energies, $K : P \to \mathbb{R}$ and $V : Q \to \mathbb{R}$, resp. Also assume that K is a positive-definite quadratic form, which is given as

$$K = \frac{1}{2}\langle p, p\rangle_{g^{-1}} := \langle \mathbf{F}L^{-1}(p), \mathbf{F}L^{-1}(p)\rangle_g, \tag{3}$$

where $\langle \cdot, \cdot \rangle_g$ is the Riemannian metric on Q (which takes values from TQ) and $\mathbf{F}L : TQ \to T^*Q$ the Legendre transformation. Finally, assume that V is leftinvariant under the action of G on Q.

From here, if we follow the original energy-momentum method to obtain the stability of the relative equilibria, we would have written the energy-momentum functional in eqn. 2 as

$$H_{\mu_e}(z,\xi) = K_\xi(z) + V_\xi(q) + \mu_e \cdot \xi, \tag{4}$$

where $V_\xi(q) := V(q) - \frac{1}{2}\xi \cdot \mathcal{I}(q)\xi$ and $K_\xi(z) := \frac{1}{2}|p - \mathbf{F}L(\xi_Q(q))|^2_{g^{-1}}$. $\mathcal{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ is just the locked inertia tensor, that is the inertia tensor at a fixed body configuration (so at this configuration it is analoque to rigid body inertia tensor). The optimization process will yield $\xi_e = \mathcal{I}^{-1}(q_e)J(z_e)$, which is just the relation of body angular velocity and angular momentum, obtained under the Lie-Poisson reduction of the rigid body dynamics [2]. However, as described in [1], the stability of the equilibria obtained from the second variation of H_{μ_e} is non-optimal and this method only provides the sufficient condition for stability.

We can avoid the difficulty above by using the reduced energy-momentum method. Based on the form of ξ_e obtained from the optimization process above, we write the Lagrange multiplier as

$$\xi(z) := \mathcal{I}^{-1}(q)J(z). \tag{5}$$

This will define a mapping to a locked velocity field

$$z = (q,p) \mapsto (q, p_J(z)), \tag{6}$$

where $p_J(z) := \mathbf{F}L[\xi(z)_Q(q)] \in T^*_qQ$, that is p_J is just the momentum induced from the group orbit in the direction ξ. The Hamiltonian then takes the form

$$H(z) = V(q) + \frac{1}{2}J(z) \cdot \mathcal{I}^{-1}(q)J(z) + \frac{1}{2}|p - p_J(z)|^2_{g^{-1}}. \tag{7}$$

Since the value of $\mu_e = J(z_e)$ is conserved under the dynamic orbit passing through z_e if z_e is a relative equilibrium solution, the energy momentum functional in eqn. 2 then can be written as $H_{\mu_e}(z, \xi) = V_{\mu_e}(q) + \frac{1}{2}|\bar{p}|^2_{g^{-1}}$ where $\bar{p} := p - p_J(z)$ and $V_{\mu_e}(q) := V(q) + \frac{1}{2}\mu_e \cdot \mathcal{I}^{-1}(q)\mu_e$. The stability analysis can now be applied only to the Smale's amended potential, $V_{\mu_e}(q)$, which depends only on Q. The shift from p to $\bar{p}$ can be interpreted as the shift of $z \in P$ onto $z \in J^{-1}(0)$, that is $\Sigma(P) = J^{-1}(0)$ where $\Sigma(q,p) := (q, p - p_J(z))$. The energy momentum functional can then be viewed as a reduced Hamiltonian,

$$h_{\mu_e}(\bar{z}) := V_{\mu_e} + \frac{1}{2}|\bar{p}|^2_{g^{-1}}. \tag{8}$$

Therefore, this reduced energy momentum method can be regarded as a reduction method since on the level set of $z \in J^{-1}(\mu_e)$,

$$H(z) = H_{\mu_e}(z, \xi) = h_{\mu_e}(\bar{z}). \tag{9}$$

The optimization process to determine the stability of the relative equilibria is described further in [1]. We now turn into applying this method to pseudo-rigid body dynamics.

Pseudo-rigid body

Pseudo-rigid body dynamics concerns with the dynamics of a deformable "rigid" body. Here we imagine an almost rigid body with a smooth boundary which can be deformed under stress applied to the body. In contrast to the fluid system, the theory of pseudo-rigid bodies deals with large-scale deformation of the body as a whole. The subject belongs in the scope of theory of elasticity. The following brief introduction follows the exposition given in [3].

Consider a 3-dimensional body with total mass m. The motion of the body in 3-space can be decomposed into the motion of the center of mass and the one of the matter surrounding the center of mass. Without losing generality we can assume that the center of mass is fixed at the origin (hence, we won't have a contribution to the kinetic energy from the motion of the center of mass in $\mathbb{R}^3$). Assume at time $t = 0$, the body is in a reference configuration $\mathcal{B}$ (with boundary $\partial\mathcal{B}$) and let the position of each point on the body be given by $\mathbf{X} \in \mathbb{R}^3$ relative to a fixed frame of reference (with the center of mass at the origin). The vectors $\mathbf{X}$ are usually called directors. Now the motion of point $\mathbf{X}$ in time can be represented by $x(t) = \mathbf{F}(t) \cdot \mathbf{X}$ where $\mathbf{F}(t)$ is a matrix representation of deformation element belonging to orientation preserving group $GL^+(3, \mathbb{R})$ (for fixed frame, $x(t)$ can be written in terms of $\mathbf{F}(t)$). $\mathbf{F}(t)$ can also be written as $\mathbf{F}(t) = \mathbf{R}(t)\mathbf{U}(t)$ where $\mathbf{R}(t) \in SO(3)$ and $\mathbf{U}(t)$ is positive-definite, symmetric matrix corresponding to translation motion. If $\mathbf{U}(t) = \mathbf{I}$, we obtain the rigid body motion.

The total energy of the system consists of the kinetic and potential energy. The potential energy can be decomposed into the one resulting from external forces and the one from internal. The external forces itself consist of body and surface forces. Let $\mathbf{b}(x, \mathbf{X}, t)$ (where $x(t) = \mathbf{F}(t)\mathbf{X}$) be the body force per unit mass, where $\mathbf{b} : \mathcal{B} \times [0,T] \to \mathbb{R}^3$, and $\mathbf{t}(x, \mathbf{X}, t)$ be the nominal traction vector on the boundary, where $\mathbf{t} : \partial\mathcal{B} \times [0,T] \to \mathbb{R}^3$. The moment force or astatic load tensor is given as

$$vol[\mathcal{B}] \cdot \mathbf{A}(t) := \int_{\mathcal{B}} \varrho_0 \mathbf{b} \otimes \mathbf{X}\, d\mathcal{B} + \int_{\partial\mathcal{B}} \mathbf{t} \otimes \mathbf{X}\, d\mathcal{B}, \tag{10}$$

where $\varrho_0 : \mathcal{B} \to \mathbb{R}$ is the reference density. However, in the rest of the paper we assume that there are not any external forces, hence $\mathbf{A}(t) = 0$. Assuming that the body is elastic, the internal forces come from the the potential created by the deformation. For a fixed frame of reference, we can represent the deformation as a function of $\mathbf{F}(t)$ and hence the potential energy or stored energy function is given as $W : GL^+(3,\mathbb{R}) \to \mathbb{R}$. If we further assume that the body is homogeneous and frame-invariant, then homogeneity implies $W(\mathbf{F}) = W(\Lambda\mathbf{F})$ where $\Lambda \in SO(3)$ (i.e. $\mathbf{F}(t)$ does not depend on $\mathbf{X}$) and frame-invariance implies $W(\mathbf{F}) = W(\mathbf{F^T F})$ since by the definition of frame-invariance $Q(\mathbf{F^T F})Q^T = (Q\mathbf{F})^T(Q\mathbf{F}) = \mathbf{F}^T\mathbf{F}$ for $Q \in SO(3)$. Now let define reference inertia dyadic by

$$\mathbb{E} := \int_{\mathcal{B}} \mathbf{X} \otimes \mathbf{X} \varrho_0(\mathbf{X})\, d\mathcal{B}, \tag{11}$$

and the spatial inertia dyadic by $\mathbf{e}(t) := \mathbf{F}(t)\mathbb{E}\mathbf{F}(t)^T$. Let $\Pi = \dot{\mathbf{F}}\mathbb{E} \in L(3,\mathbb{R})$ be the momentum conjugate to $\mathbf{F}$. Hence, the kinetic energy can be written as

$$\begin{aligned} K &:= \frac{1}{2}\int_{\mathcal{B}} \dot{x} \cdot \dot{x} \varrho_0(\mathbf{X}) d\mathcal{B} = \frac{1}{2}\int_{\mathcal{B}} \dot{\mathbf{F}}\mathbf{X} \cdot \dot{\mathbf{F}}\mathbf{X} \varrho_0(\mathbf{X}) d\mathcal{B} && (12)\\ &= \frac{1}{2}\dot{F}[\int_{\mathcal{B}} \mathbf{X} \cdot \mathbf{X} \varrho_0(\mathbf{X})\, d\mathcal{B}]\dot{F}^T = \frac{1}{2}tr[\dot{\mathbf{F}}\mathbb{E}\dot{\mathbf{F}}^T] = \frac{1}{2}tr[\dot{\Pi}\mathbb{E}^{-1}\dot{\Pi}^T] \\ &= \frac{1}{2}\langle \Pi, \Pi \rangle_{\mathbb{E}^{-1}}, && (13) \end{aligned}$$

where the second line in the equation results from the relation $a_i \cdot a_j = tr[a_i \otimes a_j]$. The total energy is then given by

$$H(\mathbf{F}, \Pi) := \frac{1}{2}\langle \Pi, \Pi \rangle_{\mathbb{E}^{-1}} + W(\mathbf{F}^T\mathbf{F}), \tag{14}$$

with equations of motion given as

$$\begin{aligned} \dot{\mathbf{F}} &= \Pi\mathbb{E}^{-1} \\ \dot{\Pi} &= -\mathbf{F}[2\partial_{\mathbf{C}} W(\mathbf{F}^T\mathbf{F})], \end{aligned} \tag{15}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. The Hamiltonian function in eqn. 14 then induces hamiltonian structure with configuration space $Q = GL^+(3, \mathbb{R})$ and phase space $P = T^*GL^+(3, \mathbb{R})$ with pairing between $T^*GL^+(3, \mathbb{R})$ and $TGL^+(3, \mathbb{R})$ given by $\langle \Pi, \mathbf{V} \rangle := tr[\Pi^T\mathbf{V}]$.

As mentioned before, $W(\mathbf{F}^T\mathbf{F})$ is invariant under the action of symmetry group $SO(3)$. It is also clear that the kinetic energy is also invariant under this group since it was formed by inner product (and the body is homogeneous). Hence $SO(3)$ is the symmetry group for our hamiltonian system. This group action generates an infinitesimal generator which itself induces a momentum map as was explained in detail in [2]. Here we can identify an element of $so(3)$ with one of $\mathbb{R}^3$ under the isomorphism

$$\hat{\xi} \in so(3) \mapsto \xi \in \mathbb{R}^3 \Leftrightarrow \hat{\xi}\alpha = \xi \times \alpha \quad \forall \alpha \in \mathbb{R}^3, \tag{16}$$

and the infinitesimal generator of the action of $SO(3)$ on Q, $\xi_Q(\mathbf{F}) = \hat{\xi}\mathbf{F} = \xi \times \mathbf{F}$, induces cotangent lift momentum map (cf. eqn. 1),

$$J(\mathbf{F}, \Pi) \cdot \xi = \langle \Pi, \hat{\xi}\mathbf{F} \rangle = \langle \Pi\mathbf{F}^T, \hat{\xi} \rangle = \langle skew[\Pi\mathbf{F}^T], \hat{\xi} \rangle \quad \forall \xi \in \mathbb{R}^3, \tag{17}$$

where the skewness results from the skewness of the elements of $so^*(3) = J(P)$. Hence the total angular momentum matrix in $so^*(3)$ is given by $\hat{J}(\mathbf{F},\Pi) = skew[\Pi\mathbf{F}^T]$. We will use this expression later on in the amended potential.

By the definition of the locked inertia tensor in the preceding section, we have in this case $\mathcal{I}(\mathbf{F}) : \mathbb{R}^3 \to \mathbb{R}^3$ which is obtained as follows,

$$\begin{aligned} \xi \cdot \mathcal{I}(\mathbf{F})\eta &= \xi \cdot J(\mathbf{F}, \mathbb{F}L(\eta_Q(\mathbf{F}))) = \langle \hat{\xi}\mathbf{F}, \mathbb{F}L(\eta_Q(\mathbf{F}))\rangle \\ &= \langle \hat{\xi}\mathbf{F}, \hat{\eta}\mathbf{F}\rangle_{\mathbf{E}} = tr[\hat{\xi}\mathbf{F}\mathbb{E}(\hat{\eta}\mathbf{F})^T] = -tr[\hat{\xi}\mathbf{F}\mathbb{E}\mathbf{F}^T\hat{\eta}] \\ &= -tr[\hat{\eta}\hat{\xi}(\mathbf{F}\mathbb{E}\mathbf{F}^T)] = -tr[\hat{\eta}\hat{\xi}(\mathrm{e})] = tr[(\eta\cdot\xi)\mathrm{e} - (\eta\otimes\xi)\mathrm{e}] \\ &= tr[(\eta\cdot\xi)\mathrm{e}] - tr[(\eta\otimes\xi)\mathrm{e}] = (\eta\cdot\xi)tr[\mathrm{e}] - tr[\eta\otimes(\mathrm{e}\xi)] \\ &= \xi\cdot(tr[\mathrm{e}]\mathbf{I})\eta - \xi\cdot\mathrm{e}\eta = \xi\cdot[tr[\mathrm{e}]\mathbf{I} - \mathrm{e}]\eta, \end{aligned} \tag{18}$$

where the first and third equalities in the second line are the consequence of eqn. 3 and the skewness of $\hat{\eta}$, respectively. Hence, we obtain the spatial locked inertia tensor (it's spatial since it's in the form $\mathbf{F}s\mathbf{F}^T$)

$$\begin{aligned} \mathcal{I}(\mathbf{F}) &= [tr[\mathrm{e}]\mathbf{I} - \mathrm{e}] = \mathbf{F}[tr[\mathrm{e}]\mathbf{C}^{-1} - \mathbb{E}]\mathbf{F}^T \\ &= \mathbf{F}[tr[\mathbf{F}\mathbb{E}\mathbf{F}^T]\mathbf{C}^{-1} - \mathbb{E}]\mathbf{F}^T = \mathbf{F}[tr[\mathbb{E}\mathbf{C}]\mathbf{C}^{-1} - \mathbb{E}]\mathbf{F}^T, \end{aligned} \tag{19}$$

where the last equality comes from the definition of $\mathbb{E}$ in eqn. 11.

From the spatial locked inertia tensor obtained in the previous paragraph, the amended potential is given as

$$V_\mu(\mathbf{F}) := W(\mathbf{F}^T\mathbf{F}) + \frac{1}{2}\mu\cdot\mathcal{I}^{-1}(\mathbf{F})\mu. \tag{20}$$

In order to obtain the shifted momenta, we need to define the space of variations at a configuration $\mathbf{F} \in Q$. Let $\delta\mathbf{F}$ be the material variation and $\delta\mathbf{f}$ be the spatial variation at a configuration $\mathbf{F}$. Then the tangent space of the material variation is given as $T_F Q = \{\delta\mathbf{F} = \delta\mathbf{f} \cdot \mathbf{F} \mid \delta\mathbf{f} \in L(3, \mathbb{R})\}$. Since $\Pi = \dot{\mathbf{F}}\mathbb{E}$,

$$\mathbb{F}L(\delta\mathbf{F}) = \delta\mathbf{F}\mathbb{E} = \delta\mathbf{f}\mathbf{F}\mathbf{F}^{-1}\mathbf{e}\mathbf{F}^{-T} = [\delta\mathbf{f}\,\mathbf{e}]\mathbf{F}^{-T}, \tag{21}$$

and hence $\Pi_{\mu_e} := [(\mathcal{I}(\mathbf{F})^{-1}\mu_e)\mathbf{e}]\mathbf{F}^{-T}$ which gives the shifted momenta as

$$\tilde{\Pi} := \Pi - \Pi_{\mu_e} = [\Pi\mathbf{F}^T - (\mathcal{I}(\mathbf{F})^{-1}\mu_e)\mathbf{e}]\mathbf{F}^{-T}. \tag{22}$$

By choosing $[\Pi\mathbf{F}^T - (\mathcal{I}(\mathbf{F})^{-1}\mu_e)\mathbf{e}]$ to be symmetric, we get $J(\mathbf{F}, \tilde{\Pi}) = skew[\tilde{\Pi}\mathbf{F}^T] = skew[\Pi\mathbf{F}^T - (\mathcal{I}(\mathbf{F})^{-1}\mu_e)\mathbf{e}] = 0$ as required from the previous section. The reduced hamiltonian is given as

$$h_{\mu_e}(\mathbf{F}, \tilde{\Pi}) = V_{\mu_e} + \frac{1}{2}|\tilde{\Pi}|^2_{\mathbb{E}^{-1}}. \tag{23}$$

The further detail on the stability analysis of relative equilibria of pseudo-rigid body is also given in [1].

References

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